Hamilton's Principle—Lagrangian and Hamiltonian Dynamics

7.1 Introduction

Experience has shown that a particle's motion in an inertial reference frame is correctly described by the Newtonian equation $\mathbf{F} = \dot{\mathbf{p}}$. If the particle is not required to move in some complicated manner and if rectangular coordinates are used to describe the motion, then usually the equations of motion are relatively simple. But if either of these restrictions is removed, the equations can become quite complex and difficult to manipulate. For example, if a particle is constrained to move on the surface of a sphere, the equations of motion result from the projection of the Newtonian vector equation onto that surface. The representation of the acceleration vector in spherical coordinates is a formidable expression, as the reader who has worked Problem 1-25 can readily testify.

Moreover, if a particle is constrained to move on a given surface, certain forces must exist (called **forces of constraint**) that maintain the particle in contact with the specified surface. For a particle moving on a smooth horizontal surface, the force of constraint is simply $\mathbf{F}_c = -m\mathbf{g}$. But, if the particle is, say, a bead sliding down a curved wire, the force of constraint can be quite complicated. Indeed, in particular situations it may be difficult or even impossible to obtain explicit expressions for the forces of constraint. But in solving a problem by using the Newtonian procedure, we must know *all* the forces, because the quantity $\mathbf{F}$ that appears in the fundamental equation is the *total* force acting on a body.

To circumvent some of the practical difficulties that arise in attempts to apply Newton's equations to particular problems, alternate procedures may be developed. All such approaches are in essence *a posteriori*, because we know beforehand that a result equivalent to the Newtonian equations must be obtained. Thus, to effect a simplification we need not formulate a *new* theory of mechanics—the Newtonian theory is quite correct—but only devise an alternate method of dealing with complicated problems in a general manner. Such a method is contained in **Hamilton's Principle,** and the equations of motion resulting from the application of this principle are called **Lagrange's equations.**

If Lagrange's equations are to constitute a proper description of the dynamics of particles, they must be equivalent to Newton's equations. On the other hand, Hamilton's Principle can be applied to a wide range of physical phenomena (particularly those involving *fields*) not usually associated with Newton's equations. To be sure, each of the results that can be obtained from Hamilton's Principle was *first* obtained, as were Newton's equations, by the correlation of experimental facts. Hamilton's Principle has not provided us with any new physical theories, but it has allowed a satisfying unification of many individual theories by a single basic postulate. This is not an idle exercise in hindsight, because it is the goal of physical theory not only to give precise mathematical formulation to observed phenomena but also to describe these effects with an economy of fundamental postulates and in the most unified manner possible. Indeed, Hamilton's Principle is one of the most elegant and far-reaching principles of physical theory.

In view of its wide range of applicability (even though this is an after-the-fact discovery), it is not unreasonable to assert that Hamilton's Principle is more "fundamental" than Newton's equations. Therefore, we proceed by first postulating Hamilton's Principle; we then obtain Lagrange's equations and show that these are equivalent to Newton's equations.

Because we have already discussed (in Chapters 2, 3, and 4) dissipative phenomena at some length, we henceforth confine our attention to *conservative* systems. Consequently, we do not discuss the more general set of Lagrange's equations, which take into account the effects of nonconservative forces. The reader is referred to the literature for these details.*

7.2 Hamilton's Principle

Minimal principles in physics have a long and interesting history. The search for such principles is predicated on the notion that nature always minimizes certain important quantities when a physical process takes place. The first such minimum principles were developed in the field of optics. Hero of Alexandria, in the second century B.C., found that the law governing the reflection of light could be obtained by asserting that a light ray, traveling from one point to another by a reflection from a plane mirror, always takes the shortest possible path. A simple geometric construction verifies that this minimum principle does indeed lead to the equality of the angles of incidence and reflection for a light ray reflected from a plane mirror. Hero's principle of the *shortest path* cannot, however, yield a correct law for *refraction*. In 1657, Fermat reformulated the principle by postulating that a light ray always travels from one point to another in a medium by a path that requires the least time.* Fermat's principle of *least time* leads immediately, not only to the correct law of reflection, but also to Snell's law of refraction (see Problem 6-7).†

Minimum principles continued to be sought, and in the latter part of the seventeenth century the beginnings of the calculus of variations were developed by Newton, Leibniz, and the Bernoullis when such problems as the brachistochrone (see Example 6.2) and the shape of a hanging chain (a catenary) were solved.

The first application of a general minimum principle in mechanics was made in 1747 by Maupertuis, who asserted that dynamical motion takes place with minimum action.‡ Maupertuis's **principle of least action** was based on theological grounds (action is minimized through the "wisdom of God"), and his concept of "action" was rather vague. (Recall that *action* is a quantity with the dimensions of *length* × *momentum* or *energy* × *time*.) Only later was a firm mathematic foundation of the principle given by Lagrange (1760). Although it is a useful form from which to make the transition from classical mechanics to optics and to quantum mechanics, the principle of least action is less general than Hamilton's Principle and, indeed, can be derived from it. We forego a detailed discussion here.§

In 1828, Gauss developed a method of treating mechanics by his principle of **least constraint;** a modification was later made by Hertz and embodied in his principle of **least curvature.** These principles‖ are closely related to Hamilton's Principle and add nothing to the content of Hamilton's more general formulation; their mention only emphasizes the continual concern with minimal principles in physics.

In two papers published in 1834 and 1835, Hamilton¶ announced the dynamical principle on which it is possible to base all of mechanics and, indeed, most of classical physics. Hamilton's Principle may be stated as follows**:

> *Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.*

In terms of the calculus of variations, Hamilton's Principle becomes

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0 \qquad \textbf{(7.1)}$$

where the symbol δ is a shorthand notation to describe the variation discussed in Sections 6.3 and 6.7. This variational statement of the principle requires only that the integral of $T - U$ be an *extremum*, not necessarily a *minimum*. But in almost all important applications in dynamics, the minimum condition occurs.

The kinetic energy of a particle expressed in fixed, rectangular coordinates is a function only of the $\dot{x}_i$, and if the particle moves in a conservative force field, the potential energy is a function only of the x_i:

$$T = T(\dot{x}_i), \quad U = U(x_i)$$

If we define the difference of these quantities to be

$$L \equiv T - U = L(x_i, \dot{x}_i) \tag{7.2}$$

then Equation 7.1 becomes

$$\boxed{\delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i)\, dt = 0} \tag{7.3}$$

The function L appearing in this expression may be identified with the function f of the variational integral (see Section 6.5),

$$\delta \int_{x_1}^{x_2} f\{y_i(x), y_i'(x); x\}\, dx$$

if we make the transformations

$$\begin{aligned} x &\rightarrow t \\ y_i(x) &\rightarrow x_i(t) \\ y_i'(x) &\rightarrow \dot{x}_i(t) \\ f\{y_i(x), y_i'(x); x\} &\rightarrow L(x_i, \dot{x}_i) \end{aligned}$$

The Euler-Lagrange equations (Equation 6.57) corresponding to Equation 7.3 are therefore

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1, 2, 3} \quad \text{Lagrange equations of motion} \tag{7.4}$$

These are the **Lagrange equations of motion** for the particle, and the quantity L is called the **Lagrange function** or **Lagrangian** for the particle.

By way of example, let us obtain the Lagrange equation of motion for the one-dimensional harmonic oscillator. With the usual expressions for the kinetic and potential energies, we have

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

Substituting these results into Equation 7.4 leads to

$$m\ddot{x} + kx = 0$$

which is identical with the equation of motion obtained using Newtonian mechanics.

$$L = \frac{1}{2}\, ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

We now treat θ *as if it were a rectangular coordinate* and apply the operations specified in Equation 7.4; we obtain

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

which again is identical with the Newtonian result (Equation 4.21). This is a remarkable result; it has been obtained by calculating the kinetic and potential energies in terms of θ rather than x and then applying a set of operations designed for use with rectangular rather than angular coordinates. We are therefore led to suspect that the Lagrange equations are more general and useful than the form of Equation 7.4 would indicate. We pursue this matter in Section 7.4.

Another important characteristic of the method used in the two preceding simple examples is that nowhere in the calculations did there enter any statement regarding *force*. The equations of motion were obtained only by specifying certain properties associated *with the particle* (the kinetic and potential energies), and without the necessity of explicitly taking into account the fact that there was an external agency acting *on the particle* (the force). Therefore, insofar as *energy* can be defined independently of Newtonian concepts, Hamilton's Principle allows us to calculate the equations of motion of a body completely without recourse to Newtonian theory. We shall return to this important point in Sections 7.5 and 7.7.