

Skill-Assessment Exercise 8.3

PROBLEM: Sketch the root locus and its asymptotes for a unity feedback system that has the forward transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

ANSWER: The complete solution is at www.wiley.com/college/nise.

8.5 Refining the Sketch

The rules covered in the previous section permit us to sketch a root locus rapidly. If we want more detail, we must be able to accurately find important points on the root locus along with their associated gain. Points on the real axis where the root locus enters or leaves the complex plane—real-axis breakaway and break-in points—and the $j\omega$ -axis crossings are candidates. We can also derive a better picture of the root locus by finding the angles of departure and arrival from complex poles and zeros, respectively.

In this section, we discuss the calculations required to obtain specific points on the root locus. Some of these calculations can be made using the basic root locus relationship that the sum of the zero angles minus the sum of the pole angles equals an odd multiple of 180° , and the gain at a point on the root locus is found as the ratio of (1) the product of pole lengths drawn to that point to (2) the product of zero lengths drawn to that point. We have yet to address how to implement this task. In the past, an inexpensive tool called a SpiruleTM added the angles together rapidly and then quickly multiplied and divided the lengths to obtain the gain. Today we can rely on hand-held or programmable calculators as well as personal computers.

Students pursuing MATLAB will learn how to apply it to the root locus at the end of Section 8.6. Other alternatives are discussed in Appendix H.2 at www.wiley.com/college/nise. The discussion can be adapted to programmable hand-held calculators. All readers are encouraged to select a computational aid at this point. Root locus calculations can be labor intensive if hand calculations are used.

We now discuss how to refine our root locus sketch by calculating real-axis breakaway and break-in points, $j\omega$ -axis crossings, angles of departure from complex poles, and angles of arrival to complex zeros. We conclude by showing how to find accurately any point on the root locus and calculate the gain.

Real-Axis Breakaway and Break-In Points

Numerous root loci appear to break away from the real axis as the system poles move from the real axis to the complex plane. At other times the loci appear to return to the real axis as a pair of complex poles becomes real. We illustrate this in Figure 8.13. This locus is sketched using the first four rules: (1) number of branches, (2) symmetry, (3) real-axis segments, and (4) starting and ending points. The figure shows a root locus leaving the real axis between -1 and -2 and returning to the real axis between $+3$ and $+5$. The point where the locus leaves the real axis, $-\sigma_1$, is called the *breakaway point*, and the point where the locus returns to the real axis, σ_2 , is called the *break-in point*.

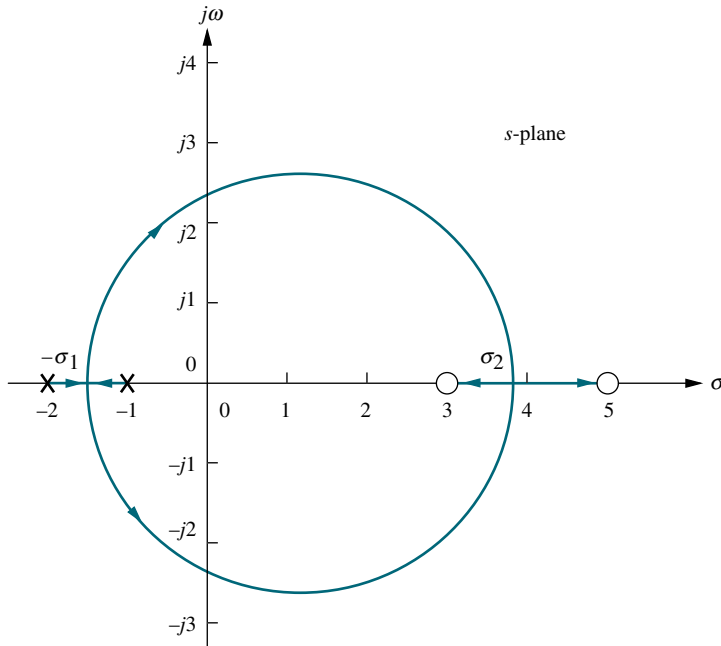


FIGURE 8.13 Root locus example showing real-axis breakaway ($-\sigma_1$) and break-in points (σ_2)

At the breakaway or break-in point, the branches of the root locus form an angle of $180^\circ/n$ with the real axis, where n is the number of closed-loop poles arriving at or departing from the single breakaway or break-in point on the real axis (Kuo, 1991). Thus, for the two poles shown in Figure 8.13, the branches at the breakaway point form 90° angles with the real axis.

We now show how to find the breakaway and break-in points. As the two closed-loop poles, which are at -1 and -2 when $K = 0$, move toward each other, the gain increases from a value of zero. We conclude that the gain must be maximum along the real axis at the point where the breakaway occurs, somewhere between -1 and -2 . Naturally, the gain increases above this value as the poles move into the complex plane. We conclude that the breakaway point occurs at a point of maximum gain on the real axis between the open-loop poles.

Now let us turn our attention to the break-in point somewhere between $+3$ and $+5$ on the real axis. When the closed-loop complex pair returns to the real axis, the gain will continue to increase to infinity as the closed-loop poles move toward the open-loop zeros. It must be true, then, that the gain at the break-in point is the minimum gain found along the real axis between the two zeros.

The sketch in Figure 8.14 shows the variation of real-axis gain. The breakaway point is found at the maximum gain between -1 and -2 , and the break-in point is found at the minimum gain between $+3$ and $+5$.

There are three methods for finding the points at which the root locus breaks away from and breaks into the real axis. The first method is to maximize and minimize the gain, K , using differential calculus. For all points on the root locus, Eq. (8.13) yields

$$K = -\frac{1}{G(s)H(s)} \quad (8.31)$$

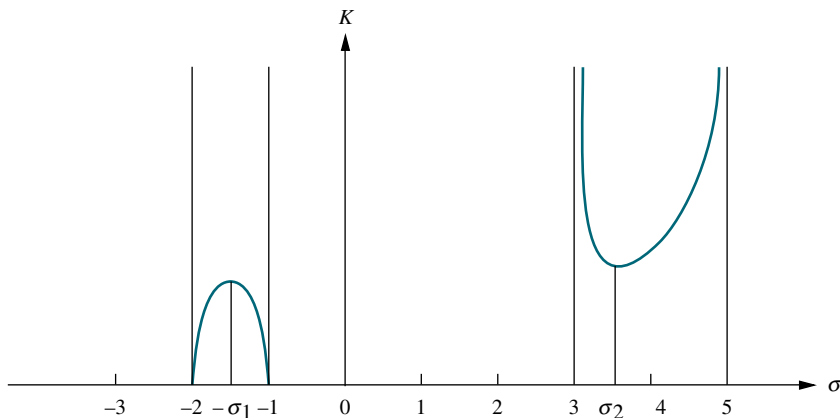


FIGURE 8.14 Variation of gain along the real axis for the root locus of Figure 8.13

For points along the real-axis segment of the root locus where breakaway and break-in points could exist, $s = \sigma$. Hence, along the real axis Eq. (8.31) becomes

$$K = -\frac{1}{G(\sigma)H(\sigma)} \quad (8.32)$$

This equation then represents a curve of K versus σ similar to that shown in Figure 8.14. Hence, if we differentiate Eq. (8.32) with respect to σ and set the derivative equal to zero, we can find the points of maximum and minimum gain and hence the breakaway and break-in points. Let us demonstrate.

Example 8.3

Breakaway and Break-in Points via Differentiation

PROBLEM: Find the breakaway and break-in points for the root locus of Figure 8.13, using differential calculus.

SOLUTION: Using the open-loop poles and zeros, we represent the open-loop system whose root locus is shown in Figure 8.13 as follows:

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2 - 8s + 15)}{(s^2 + 3s + 2)} \quad (8.33)$$

But for all points along the root locus, $KG(s)H(s) = -1$, and along the real axis, $s = \sigma$. Hence,

$$\frac{K(\sigma^2 - 8\sigma + 15)}{(\sigma^2 + 3\sigma + 2)} = -1 \quad (8.34)$$

Solving for K , we find

$$K = \frac{-(\sigma^2 + 3\sigma + 2)}{(\sigma^2 - 8\sigma + 15)} \quad (8.35)$$

Differentiating K with respect to σ and setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2 - 26\sigma - 61)}{(\sigma^2 - 8\sigma + 15)^2} = 0 \quad (8.36)$$

Solving for σ , we find $\sigma = -1.45$ and 3.82 , which are the breakaway and break-in points.

The second method is a variation on the differential calculus method. Called the *transition method*, it eliminates the step of differentiation (Franklin, 1991). This method, derived in Appendix M.2 at www.wiley.com/college/nise, is now stated:

Breakaway and break-in points satisfy the relationship

$$\sum_1^m \frac{1}{\sigma + z_i} = \sum_1^n \frac{1}{\sigma + p_i} \quad (8.37)$$

where z_i and p_i are the negative of the zero and pole values, respectively, of $G(s)H(s)$.

Solving Eq. (8.37) for σ , the real-axis values that minimize or maximize K , yields the breakaway and break-in points without differentiating. Let us look at an example.

Example 8.4

Breakaway and Break-in Points Without Differentiation

PROBLEM: Repeat Example 8.3 without differentiating.

SOLUTION: Using Eq. (8.37),

$$\frac{1}{\sigma - 3} + \frac{1}{\sigma - 5} = \frac{1}{\sigma + 1} + \frac{1}{\sigma + 2} \quad (8.38)$$

Simplifying,

$$11\sigma^2 - 26\sigma - 61 = 0 \quad (8.39)$$

Hence, $\sigma = -1.45$ and 3.82 , which agrees with Example 8.3.

For the third method, the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise can be used to find the breakaway and break-in points. Simply use the program to search for the point of maximum gain between -1 and -2 and to search for the point of minimum gain between $+3$ and $+5$. Table 8.2 shows the results of the search. The locus leaves the axis at -1.45 , the point of maximum gain between -1 and -2 , and reenters the real axis at $+3.8$, the point of minimum gain between $+3$ and $+5$. These results are the same as those obtained using the first two methods. MATLAB also has the capability of finding breakaway and break-in points.

The $j\omega$ -Axis Crossings

We now further refine the root locus by finding the imaginary-axis crossings. The importance of the $j\omega$ -axis crossings should be readily apparent. Looking at Figure 8.12, we see that the system's poles are in the left half-plane up to a particular value of gain. Above this value of gain, two of the closed-loop system's poles move into the right half-plane, signifying that the system is unstable. The $j\omega$ -axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation. The value of ω at the axis crossing yields the frequency of oscillation, while the gain at the $j\omega$ -axis crossing yields, for this example, the maximum positive gain for system stability. We should note here that other examples

TABLE 8.2 Data for breakaway and break-in points for the root locus of Figure 8.13

Real-axis value	Gain		Comment
-1.41	0.008557		
-1.42	0.008585		
-1.43	0.008605		
-1.44	0.008617		
-1.45	0.008623	←	Max. gain: breakaway
-1.46	0.008622		
3.3	44.686		
3.4	37.125		
3.5	33.000		
3.6	30.667		
3.7	29.440		
3.8	29.000	←	Min. gain: break-in
3.9	29.202		

illustrate instability at small values of gain and stability at large values of gain. These systems have a root locus starting in the right-half-plane (unstable at small values of gain) and ending in the left-half-plane (stable for high values of gain).

To find the $j\omega$ -axis crossing, we can use the Routh-Hurwitz criterion, covered in Chapter 6, as follows: Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

Example 8.5

Frequency and Gain at Imaginary-Axis Crossing

PROBLEM: For the system of Figure 8.11, find the frequency and gain, K , for which the root locus crosses the imaginary axis. For what range of K is the system stable?

SOLUTION: The closed-loop transfer function for the system of Figure 8.11 is

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K} \quad (8.40)$$

Using the denominator and simplifying some of the entries by multiplying any row by a constant, we obtain the Routh array shown in Table 8.3.

A complete row of zeros yields the possibility for imaginary axis roots. For positive values of gain, those for which the root locus is plotted, only the s^1 row can yield a row of zeros. Thus,

$$-K^2 - 65K + 720 = 0 \quad (8.41)$$

From this equation K is evaluated as

$$K = 9.65 \quad (8.42)$$

TABLE 8.3 Routh table for Eq. (8.40)

s^4	1	14	$3K$
s^3	7	$8 + K$	
s^2	$90 - K$	$21K$	
s^1	$\frac{-K^2 - 65K + 720}{90 - K}$		
s^0	$21K$		

Forming the even polynomial by using the s^2 row with $K = 9.65$, we obtain

$$(90 - K)s^2 + 21K = 80.35s^2 + 202.7 = 0 \quad (8.43)$$

and s is found to be equal to $\pm j1.59$. Thus the root locus crosses the $j\omega$ -axis at $\pm j1.59$ at a gain of 9.65. We conclude that the system is stable for $0 \leq K < 9.65$.

Another method for finding the $j\omega$ -axis crossing (or any point on the root locus, for that matter) uses the fact that at the $j\omega$ -axis crossing, the sum of angles from the finite open-loop poles and zeros must add to $(2k + 1)180^\circ$. Thus, we can search $j\omega$ -axis until we find the point that meets this angle condition. A computer program, such as the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise or MATLAB, can be used for this purpose. Subsequent examples in this chapter use this method to determine the $j\omega$ -axis crossing.

Angles of Departure and Arrival

In this subsection, we further refine our sketch of the root locus by finding angles of departure and arrival from complex poles and zeros. Consider Figure 8.15, which shows the open-loop poles and zeros, some of which are complex. The root locus starts at the open-loop poles and ends at the open-loop zeros. In order to sketch the root locus more accurately, we want to calculate the root locus departure angle from the complex poles and the arrival angle to the complex zeros.

If we assume a point on the root locus ϵ close to a complex pole, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180° . Except for the pole that is ϵ close to the point, we assume all angles drawn from all other poles and zeros are drawn directly to the pole that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the pole that is ϵ close. We can solve for this unknown angle, which is also the angle of departure from this complex pole. Hence, from Figure 8.15(a),

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k + 1)180^\circ \quad (8.44a)$$

or

$$\theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k + 1)180^\circ \quad (8.44b)$$

If we assume a point on the root locus ϵ close to a complex zero, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180° . Except for the zero that is ϵ close to the point, we can assume all angles drawn from all other poles and zeros are drawn directly to the zero that is near the point. Thus,

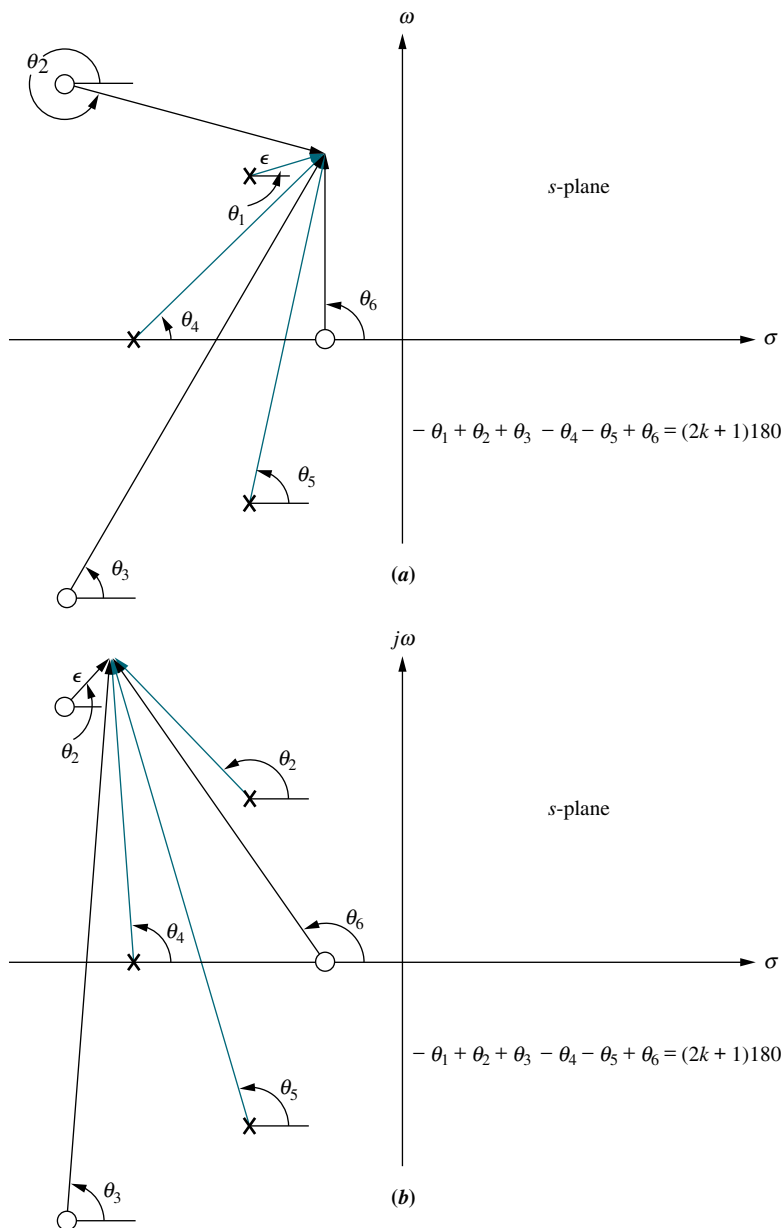


FIGURE 8.15 Open-loop poles and zeros and calculation of **a.** angle of departure; **b.** angle of arrival

the only unknown angle in the sum is the angle drawn from the zero that is ϵ close. We can solve for this unknown angle, which is also the angle of arrival to this complex zero. Hence, from Figure 8.15(b),

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k + 1)180^\circ \tag{8.45a}$$

or

$$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + (2k + 1)180^\circ \tag{8.45b}$$

Let us look at an example.

Example 8.6

Angle of Departure from a Complex Pole

PROBLEM: Given the unity feedback system of Figure 8.16, find the angle of departure from the complex poles and sketch the root locus.

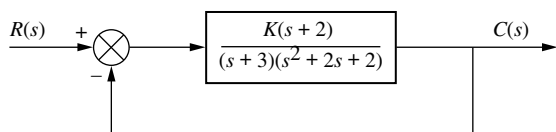


FIGURE 8.16 Unity feedback system with complex poles

SOLUTION: Using the poles and zeros of $G(s) = (s + 2)/[(s + 3)(s^2 + 2s + 2)]$ as plotted in Figure 8.17, we calculate the sum of angles drawn to a point ϵ close to the complex pole, $-1 + j1$, in the second quadrant. Thus,

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = -\theta_1 - 90^\circ + \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^\circ \quad (8.46)$$

from which $\theta = -251.6^\circ = 108.4^\circ$. A sketch of the root locus is shown in Figure 8.17. Notice how the departure angle from the complex poles helps us to refine the shape.

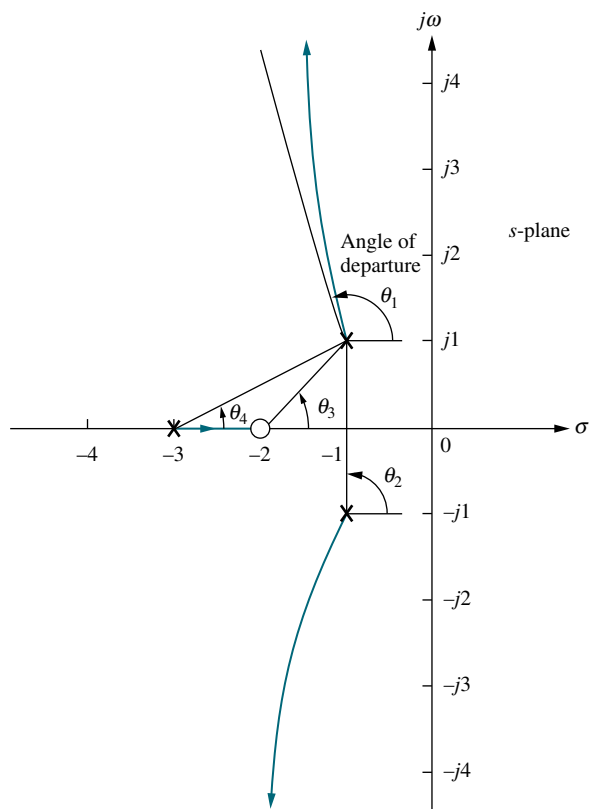


FIGURE 8.17 Root locus for system of Figure 8.16 showing angle of departure

Plotting and Calibrating the Root Locus

Once we sketch the root locus using the rules from Section 8.4, we may want to accurately locate points on the root locus as well as find their associated gain. For example, we might want to know the exact coordinates of the root locus as it crosses the radial line representing 20% overshoot. Further, we also may want the value of gain at that point.

Consider the root locus shown in Figure 8.12. Let us assume we want to find the exact point at which the locus crosses the 0.45 damping ratio line and the gain at that point. Figure 8.18 shows the system's open-loop poles and zeros along with the $\zeta = 0.45$ line. If a few test points along the $\zeta = 0.45$ line are selected, we can evaluate their angular sum and locate that point where the angles add up to an odd multiple of 180° . It is at this point that the root locus exists. Equation (8.20) can then be used to evaluate the gain, K , at that point.

Selecting the point at radius 2 ($r = 2$) on the $\zeta = 0.45$ line, we add the angles of the zeros and subtract the angles of the poles, obtaining

$$\theta_2 - \theta_1 - \theta_3 - \theta_4 - \theta_5 = -251.5^\circ \quad (8.47)$$

Since the sum is not equal to an odd multiple of 180° , the point at radius = 2 is not on the root locus. Proceeding similarly for the points at radius = 1.5, 1, 0.747, and 0.5, we obtain the table shown in Figure 8.18. This table lists the points, giving their radius, r , and the sum of angles indicated by the symbol \angle . From the table we see that the point at radius 0.747 is on the root locus, since the angles add up to -180° . Using Eq. (8.21), the gain, K , at this point is

$$K = \frac{|A||C||D||E|}{|B|} = 1.71 \quad (8.48)$$

In summary, we search a given line for the point yielding a summation of angles (zero angles–pole angles) equal to an odd multiple of 180° . We conclude that the point is on the root locus. The gain at that point is then found by multiplying the pole lengths drawn to that point and dividing by the product of the zero lengths drawn to that point. A computer program, such as that discussed in Appendix H.2 at www.wiley.com/college/nise or MATLAB, can be used.

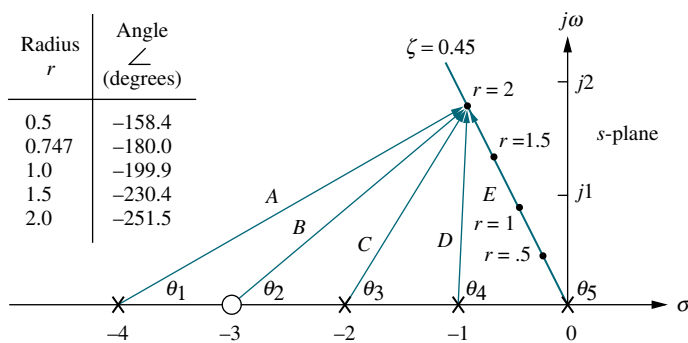


FIGURE 8.18 Finding and calibrating exact points on the root locus of Figure 8.12

Skill-Assessment Exercise 8.4

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2 - 4s + 13)}$$

do the following:

- a. Sketch the root locus.
- b. Find the imaginary-axis crossing.
- c. Find the gain, K , at the $j\omega$ -axis crossing.
- d. Find the break-in point.
- e. Find the angle of departure from the complex poles.

ANSWERS:

- a. See solution at www.wiley.com/college/nise.
- b. $s = \pm j\sqrt{21}$
- c. $K = 4$
- d. Break-in point = -7
- e. Angle of departure = -233.1°

The complete solution is at www.wiley.com/college/nise.

8.6 An Example

We now review the rules for sketching and finding points on the root locus, as well as present an example. The root locus is the path of the closed-loop poles of a system as a parameter of the system is varied. Each point on the root locus satisfies the angle condition, $\angle G(s)H(s) = (2k + 1)180^\circ$. Using this relationship, rules for sketching and finding points on the root locus were developed and are now summarized:

Basic Rules for Sketching the Root Locus

Number of branches The number of branches of the root locus equals the number of closed-loop poles.

Symmetry The root locus is symmetrical about the real axis.

Real-axis segments On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

Starting and ending points The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

Behavior at infinity The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes are given by

the real-axis intercept and angle in radians as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.49)$$

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.50)$$

where $k = 0, \pm 1, \pm 2, \pm 3, \dots$

Additional Rules for Refining the Sketch

Real-axis breakaway and break-in points The root locus breaks away from the real axis at a point where the gain is maximum and breaks into the real axis at a point where the gain is minimum.

Calculation of $j\omega$ -axis crossings The root locus crosses the $j\omega$ -axis at the point where $\angle G(s)H(s) = (2k + 1)180^\circ$. Routh-Hurwitz or a search of the $j\omega$ -axis for $(2k + 1)180^\circ$ can be used to find the $j\omega$ -axis crossing.

Angles of departure and arrival The root locus departs from complex, open-loop poles and arrives at complex, open-loop zeros at angles that can be calculated as follows. Assume a point ϵ close to the complex pole or zero. Add all angles drawn from all open-loop poles and zeros to this point. The sum equals $(2k + 1)180^\circ$. The only unknown angle is that drawn from the ϵ close pole or zero, since the vectors drawn from all other poles and zeros can be considered drawn to the complex pole or zero that is ϵ close to the point. Solving for the unknown angle yields the angle of departure or arrival.

Plotting and calibrating the root locus All points on the root locus satisfy the relationship $\angle G(s)H(s) = (2k + 1)180^\circ$. The gain, K , at any point on the root locus is given by

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} - \frac{\prod \text{finite pole lengths}}{\prod \text{finite zero lengths}} \quad (8.51)$$

Let us now look at a summary example.

Example 8.7

Sketching a Root Locus and Finding Critical Points

PROBLEM: Sketch the root locus for the system shown in Figure 8.19(a) and find the following:

- The exact point and gain where the locus crosses the 0.45 damping ratio line
- The exact point and gain where the locus crosses the $j\omega$ -axis
- The breakaway point on the real axis
- The range of K within which the system is stable

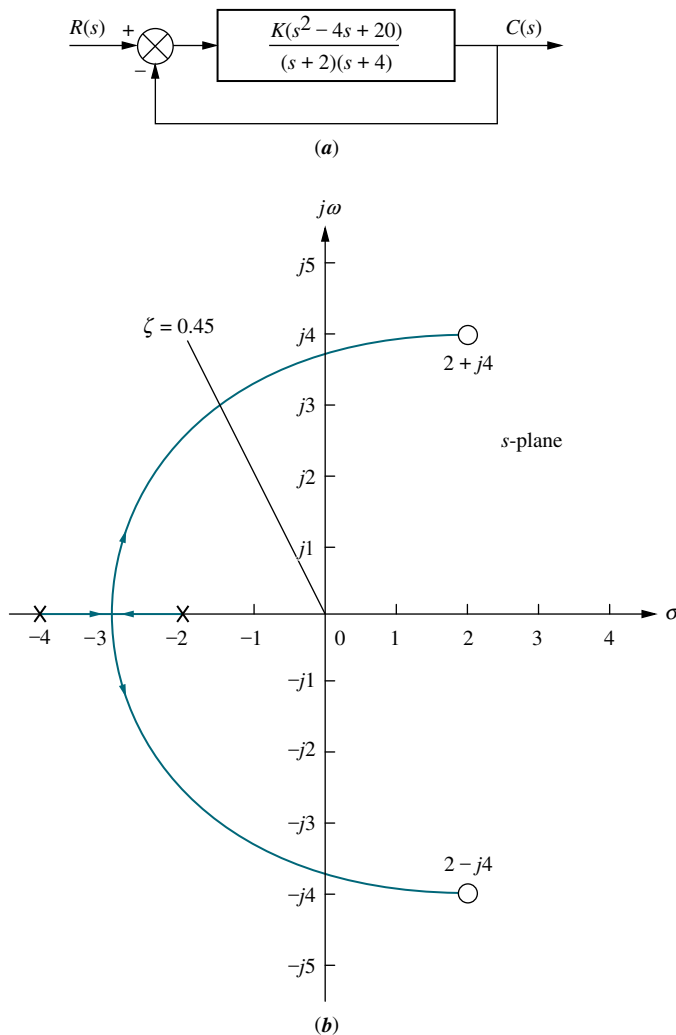


FIGURE 8.19 a. System for Example 8.7; b. root locus sketch.

SOLUTION: The problem solution is shown, in part, in Figure 8.19(b). First sketch the root locus. Using Rule 3, the real-axis segment is found to be between -2 and -4 . Rule 4 tells us that the root locus starts at the open-loop poles and ends at the open-loop zeros. These two rules alone give us the general shape of the root locus.

- a.** To find the exact point where the locus crosses the $\zeta = 0.45$ line, we can use the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise to search along the line

$$\theta = 180^\circ - \cos^{-1} 0.45 = 116.7^\circ \quad (8.52)$$

for the point where the angles add up to an odd multiple of 180° . Searching in polar coordinates, we find that the root locus crosses the $\zeta = 0.45$ line at $3.4 \angle 116.7^\circ$ with a gain, K , of 0.417.

- b.** To find the exact point where the locus crosses the $j\omega$ -axis, use the root locus program to search along the line

$$\theta = 90^\circ \quad (8.53)$$

for the point where the angles add up to an odd multiple of 180° . Searching in polar coordinates, we find that the root locus crosses the $j\omega$ -axis at $\pm j3.9$ with a gain of $K = 1.5$.

- c. To find the breakaway point, use the root locus program to search the real axis between -2 and -4 for the point that yields maximum gain. Naturally, all points will have the sum of their angles equal to an odd multiple of 180° . A maximum gain of 0.0248 is found at the point -2.88 . Therefore, the breakaway point is between the open-loop poles on the real axis at -2.88 .
- d. From the answer to **b**, the system is stable for K between 0 and 1.5 .

MATLAB

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Students who are using MATLAB should now run `ch8pl` in Appendix B. You will learn how to use MATLAB to plot and title a root locus, overlay constant ζ and ω_n curves, zoom into and zoom out from a root locus, and interact with the root locus to find critical points as well as gains at those points. This exercise solves Example 8.7 using MATLAB.

Skill-Assessment Exercise 8.5

WileyPLUS

WPCS

Control Solutions

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s-2)(s-4)}{(s^2+6s+25)}$$

do the following:

- a. Sketch the root locus.
- b. Find the imaginary-axis crossing.
- c. Find the gain, K , at the $j\omega$ -axis crossing.
- d. Find the break-in point.
- e. Find the point where the locus crosses the 0.5 damping ratio line.
- f. Find the gain at the point where the locus crosses the 0.5 damping ratio line.
- g. Find the range of gain, K , for which the system is stable.

ANSWERS:

- a. See solution at www.wiley.com/college/nise.
- b. $s = \pm j4.06$
- c. $K = 1$
- d. Break-in point = $+2.89$
- e. $s = -2.42 + j4.18$
- f. $K = 0.108$
- g. $K < 1$

The complete solution is at www.wiley.com/college/nise.

TryIt 8.3

Use MATLAB, the Control System Toolbox, and the following statements to plot the root locus for Skill-Assessment Exercise 8.5.

Solve the remaining parts of the problem by clicking on the appropriate points on the plotted root locus.

```
numg=poly([2 4]);
deng=[1 6 25];
G=tf(numg,deng)
rlocus(G)
z=0.5
sgrid(z,0)
```