

Since the settling time is inversely proportional to the real part of the complex poles for this second-order system, the conclusion is that regardless of the value of gain, the settling time for the system remains the same under all conditions of underdamped responses.

Also, as we increase the gain, the damping ratio diminishes, and the percent overshoot increases. The damped frequency of oscillation, which is equal to the imaginary part of the pole, also increases with an increase in gain, resulting in a reduction of the peak time. Finally, since the root locus never crosses over into the right half-plane, the system is always stable, regardless of the value of gain, and can never break into a sinusoidal oscillation.

These conclusions for such a simple system may appear to be trivial. What we are about to see is that the analysis is applicable to systems of order higher than 2. For these systems, it is difficult to tie transient response characteristics to the pole location. The root locus will allow us to make that association and will become an important technique in the analysis and design of higher-order systems.

8.3 Properties of the Root Locus

In Section 8.2, we arrived at the root locus by factoring the second-order polynomial in the denominator of the transfer function. Consider what would happen if that polynomial were of fifth or tenth order. Without a computer, factoring the polynomial would be quite a problem for numerous values of gain.

We are about to examine the properties of the root locus. From these properties we will be able to make a rapid *sketch* of the root locus for higher-order systems without having to factor the denominator of the closed-loop transfer function.

The properties of the root locus can be derived from the general control system of Figure 8.1(a). The closed-loop transfer function for the system is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \quad (8.12)$$

From Eq. (8.12), a pole, s , exists when the characteristic polynomial in the denominator becomes zero, or

$$KG(s)H(s) = -1 = 1\angle(2k + 1)180^\circ \quad k = 0, \pm 1, \pm 2, \pm 3, \dots \quad (8.13)$$

where -1 is represented in polar form as $1\angle(2k + 1)180^\circ$. Alternately, a value of s is a closed-loop pole if

$$|KG(s)H(s)| = 1 \quad (8.14)$$

and

$$\angle KG(s)H(s) = (2k + 1)180^\circ \quad (8.15)$$

Equation (8.13) implies that if a value of s is substituted into the function $KG(s)H(s)$, a complex number results. If the angle of the complex number is an odd multiple of 180° , that value of s is a system pole for some particular value of K . What

value of K ? Since the angle criterion of Eq. (8.15) is satisfied, all that remains is to satisfy the magnitude criterion, Eq. (8.14). Thus,

$$K = \frac{1}{|G(s)||H(s)|} \quad (8.16)$$

We have just found that a pole of the closed-loop system causes the angle of $KG(s)H(s)$, or simply $G(s)H(s)$ since K is a scalar, to be an odd multiple of 180° . Furthermore, the magnitude of $KG(s)H(s)$ must be unity, implying that the value of K is the reciprocal of the magnitude of $G(s)H(s)$ when the pole value is substituted for s .

Let us demonstrate this relationship for the second-order system of Figure 8.4. The fact that closed-loop poles exist at -9.47 and -0.53 when the gain is 5 has already been established in Table 8.1. For this system,

$$KG(s)H(s) = \frac{K}{s(s+10)} \quad (8.17)$$

Substituting the pole at -9.47 for s and 5 for K yields $KG(s)H(s) = -1$. The student can repeat the exercise for other points in Table 8.1 and show that each case yields $KG(s)H(s) = -1$.

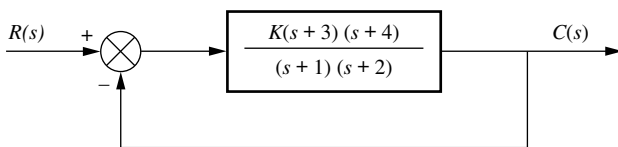
It is helpful to visualize graphically the meaning of Eq. (8.15). Let us apply the complex number concepts reviewed in Section 8.1 to the root locus of the system shown in Figure 8.6. For this system the open-loop transfer function is

$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)} \quad (8.18)$$

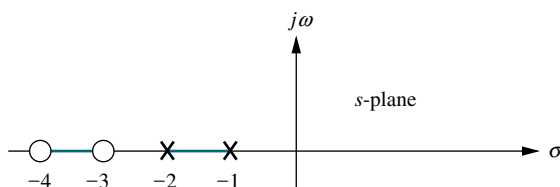
The closed-loop transfer function, $T(s)$, is

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)} \quad (8.19)$$

If point s is a closed-loop system pole for some value of gain, K , then s must satisfy Eqs. (8.14) and (8.15).



(a)



(b)

FIGURE 8.6 **a.** Example system; **b.** pole-zero plot of $G(s)$

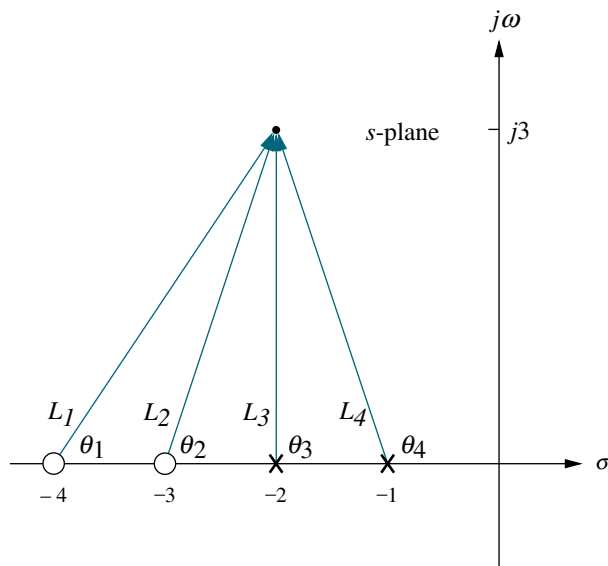


FIGURE 8.7 Vector representation of $G(s)$ from Figure 8.6(a) at $-2 + j3$

Consider the point $-2 + j3$. If this point is a closed-loop pole for some value of gain, then the angles of the zeros minus the angles of the poles must equal an odd multiple of 180° . From Figure 8.7,

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ \quad (8.20)$$

Therefore, $-2 + j3$ is not a point on the root locus, or alternatively, $-2 + j3$ is not a closed-loop pole for any gain.

If these calculations are repeated for the point $-2 + j(\sqrt{2}/2)$, the angles do add up to 180° . That is, $-2 + j(\sqrt{2}/2)$ is a point on the root locus for some value of gain. We now proceed to evaluate that value of gain.

From Eqs. (8.5) and (8.16),

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}} \quad (8.21)$$

Looking at Figure 8.7 with the point $-2 + j3$ replaced by $-2 + j(\sqrt{2}/2)$, the gain, K , is calculated as

$$K = \frac{L_3 L_4}{L_1 L_2} = \frac{\frac{\sqrt{2}}{2}(1.22)}{(2.12)(1.22)} = 0.33 \quad (8.22)$$

Thus, the point $-2 + j(\sqrt{2}/2)$ is a point on the root locus for a gain of 0.33.

We summarize what we have found as follows: Given the poles and zeros of the open-loop transfer function, $KG(s)H(s)$, a point in the s -plane is on the root locus for a particular value of gain, K , if the angles of the zeros minus the angles of the poles, all drawn to the selected point on the s -plane, add up to $(2k + 1)180^\circ$. Furthermore, gain K at that point for which the angles add up to $(2k + 1)180^\circ$ is found by dividing the product of the pole lengths by the product of the zero lengths.

Skill-Assessment Exercise 8.2

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2+4s+13)}$$

do the following:

- a. Calculate the angle of $G(s)$ at the point $(-3 + j0)$ by finding the algebraic sum of angles of the vectors drawn from the zeros and poles of $G(s)$ to the given point.
- b. Determine if the point specified in **a** is on the root locus.
- c. If the point specified in **a** is on the root locus, find the gain, K , using the lengths of the vectors.

ANSWERS:

- a. Sum of angles = 180°
- b. Point is on the root locus
- c. $K = 10$

The complete solution is at www.wiley.com/college/nise.

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TryIt 8.2

Use MATLAB and the following statements to solve Skill-Assessment Exercise 8.2.

```
s=-3+0j;
G=(s+2)/(s^2+4*s+13);
Theta=(180/pi)*...
    angle(G)
M=abs(G);
K=1/M
```

8.4 Sketching the Root Locus

It appears from our previous discussion that the root locus can be obtained by sweeping through every point in the s -plane to locate those points for which the angles, as previously described, add up to an odd multiple of 180° . Although this task is tedious without the aid of a computer, the concept can be used to develop rules that can be used to *sketch* the root locus without the effort required to *plot* the locus. Once a sketch is obtained, it is possible to accurately plot just those points that are of interest to us for a particular problem.

The following five rules allow us to sketch the root locus using minimal calculations. The rules yield a sketch that gives intuitive insight into the behavior of a control system. In the next section, we refine the sketch by finding actual points or angles on the root locus. These refinements, however, require some calculations or the use of computer programs, such as MATLAB.

- 1. Number of branches.** Each closed-loop pole moves as the gain is varied. If we define a *branch* as the path that one pole traverses, then there will be one branch for each closed-loop pole. Our first rule, then, defines the number of branches of the root locus:

The number of branches of the root locus equals the number of closed-loop poles.

As an example, look at Figure 8.5(b), where the two branches are shown. One originates at the origin, the other at -10 .

- 2. Symmetry.** If complex closed-loop poles do not exist in conjugate pairs, the resulting polynomial, formed by multiplying the factors containing the closed-loop poles,

would have complex coefficients. Physically realizable systems cannot have complex coefficients in their transfer functions. Thus, we conclude:

The root locus is symmetrical about the real axis.

An example of symmetry about the real axis is shown in Figure 8.5(b).

- 3. Real-axis segments.** Let us make use of the angle property, Eq. (8.15), of the points on the root locus to determine where the real-axis segments of the root locus exist. Figure 8.8 shows the poles and zeros of a general open-loop system. If an attempt is made to calculate the angular contribution of the poles and zeros at each point, P_1 , P_2 , P_3 , and P_4 , along the real axis, we observe the following: (1) At each point the angular contribution of a pair of open-loop complex poles or zeros is zero, and (2) the contribution of the open-loop poles and open-loop zeros to the left of the respective point is zero. The conclusion is that the only contribution to the angle at any of the points comes from the open-loop, real-axis poles and zeros that exist to the right of the respective point. If we calculate the angle at each point using only the open-loop, real-axis poles and zeros to the right of each point, we note the following: (1) The angles on the real axis alternate between 0° and 180° , and (2) the angle is 180° for regions of the real axis that exist to the left of an odd number of poles and/or zeros. The following rule summarizes the findings:

On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

Examine Figure 8.6(b). According to the rule just developed, the real-axis segments of the root locus are between -1 and -2 and between -3 and -4 as shown in Figure 8.9.

- 4. Starting and ending points.** Where does the root locus begin (zero gain) and end (infinite gain)? The answer to this question will enable us to expand the sketch of the root locus beyond the real-axis segments. Consider the closed-loop transfer function, $T(s)$, described by Eq. (8.3). $T(s)$ can now be evaluated for both large and small gains, K . As K approaches zero (small gain),

$$T(s) \approx \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + \epsilon} \quad (8.23)$$

From Eq. (8.23) we see that the closed-loop system poles at small gains approach the combined poles of $G(s)$ and $H(s)$. We conclude that the root locus begins at the poles of $G(s)H(s)$, the open-loop transfer function.

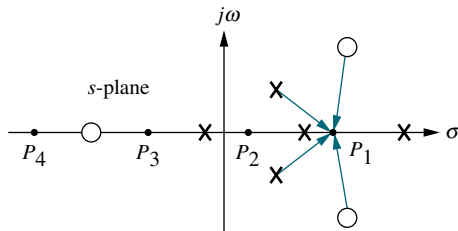


FIGURE 8.8 Poles and zeros of a general open-loop system with test points, P_i , on the real axis

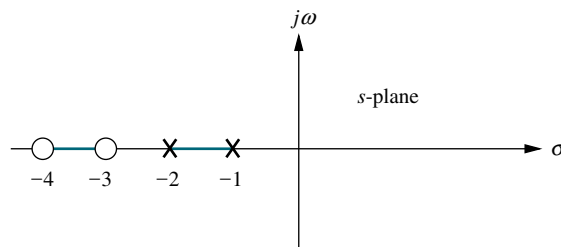


FIGURE 8.9 Real-axis segments of the root locus for the system of Figure 8.6

At high gains, where K is approaching infinity,

$$T(s) \approx \frac{KN_G(s)D_H(s)}{\epsilon + KN_G(s)N_H(s)} \quad (8.24)$$

From Eq. (8.24) we see that the closed-loop system poles at large gains approach the combined zeros of $G(s)$ and $H(s)$. Now we conclude that the root locus ends at the zeros of $G(s)H(s)$, the open-loop transfer function.

Summarizing what we have found:

The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

Remember that these poles and zeros are the open-loop poles and zeros.

In order to demonstrate this rule, look at the system in Figure 8.6(a), whose real-axis segments have been sketched in Figure 8.9. Using the rule just derived, we find that the root locus begins at the poles at -1 and -2 and ends at the zeros at -3 and -4 (see Figure 8.10). Thus, the poles start out at -1 and -2 and move through the real-axis space between the two poles. They meet somewhere between the two poles and break out into the complex plane, moving as complex conjugates. The poles return to the real axis somewhere between the zeros at -3 and -4 , where their path is completed as they move away from each other, and end up, respectively, at the two zeros of the open-loop system at -3 and -4 .

- 5. Behavior at infinity.** Consider applying Rule 4 to the following open-loop transfer function:

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)} \quad (8.25)$$

There are three finite poles, at $s = 0, -1,$ and $-2,$ and no finite zeros.

A function can also have *infinite* poles and zeros. If the function approaches infinity as s approaches infinity, then the function has a pole at infinity. If the function approaches zero as s approaches infinity, then the function has a zero at infinity. For example, the function $G(s) = s$ has a pole at infinity, since $G(s)$ approaches infinity as s approaches infinity. On the other hand, $G(s) = 1/s$ has a zero at infinity, since $G(s)$ approaches zero as s approaches infinity.

Every function of s has an equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros. In this example,

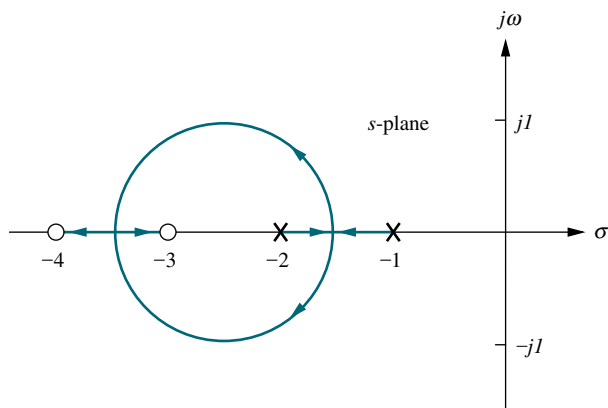


FIGURE 8.10 Complete root locus for the system of Figure 8.6

Eq. (8.25) contains three finite poles and three infinite zeros. To illustrate, let s approach infinity. The open-loop transfer function becomes

$$KG(s)H(s) \approx \frac{K}{s^3} = \frac{K}{s \cdot s \cdot s} \quad (8.26)$$

Each s in the denominator causes the open-loop function, $KG(s)H(s)$, to become zero as that s approaches infinity. Hence, Eq. (8.26) has three zeros at infinity.

Thus, for Eq. (8.25), the root locus begins at the finite poles of $KG(s)H(s)$ and ends at the infinite zeros. The question remains: Where are the infinite zeros? We must know where these zeros are in order to show the locus moving from the three finite poles to the three infinite zeros. Rule 5 helps us locate these zeros at infinity. Rule 5 also helps us locate poles at infinity for functions containing more finite zeros than finite poles.¹

We now state Rule 5, which will tell us what the root locus looks like as it approaches the zeros at infinity or as it moves from the poles at infinity. The derivation can be found in Appendix M.1 at www.wiley.com/college/nise.

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a and angle, θ_a as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.27)$$

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.28)$$

where $k = 0, \pm 1, \pm 2, \pm 3$ and the angle is given in radians with respect to the positive extension of the real axis.

Notice that the running index, k , in Eq. (8.28) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity. Let us demonstrate the concepts with an example.

Example 8.2

Sketching a Root Locus with Asymptotes

PROBLEM: Sketch the root locus for the system shown in Figure 8.11.

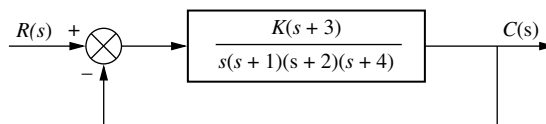


FIGURE 8.11 System for Example 8.2.

¹Physical systems, however, have more finite poles than finite zeros, since the implied differentiation yields infinite output for discontinuous input functions, such as step inputs.

SOLUTION: Let us begin by calculating the asymptotes. Using Eq. (8.27), the real-axis intercept is evaluated as

$$\sigma_a = \frac{(-1 - 2 - 4) - (-3)}{4 - 1} = -\frac{4}{3} \quad (8.29)$$

The angles of the lines that intersect at $-4/3$, given by Eq. (8.28), are

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.30a)$$

$$= \pi/3 \quad \text{for } k = 0 \quad (8.30b)$$

$$= \pi \quad \text{for } k = 1 \quad (8.30c)$$

$$= 5\pi/3 \quad \text{for } k = 2 \quad (8.30d)$$

If the value for k continued to increase, the angles would begin to repeat. The number of lines obtained equals the difference between the number of finite poles and the number of finite zeros.

Rule 4 states that the locus begins at the open-loop poles and ends at the open-loop zeros. For the example there are more open-loop poles than open-loop zeros. Thus, there must be zeros at infinity. The asymptotes tell us how we get to these zeros at infinity.

Figure 8.12 shows the complete root locus as well as the asymptotes that were just calculated. Notice that we have made use of all the rules learned so far. The real-axis segments lie to the left of an odd number of poles and/or zeros. The locus starts at the open-loop poles and ends at the open-loop zeros. For the example there is only one open-loop finite zero and three infinite zeros. Rule 5, then, tells us that the three zeros at infinity are at the ends of the asymptotes.

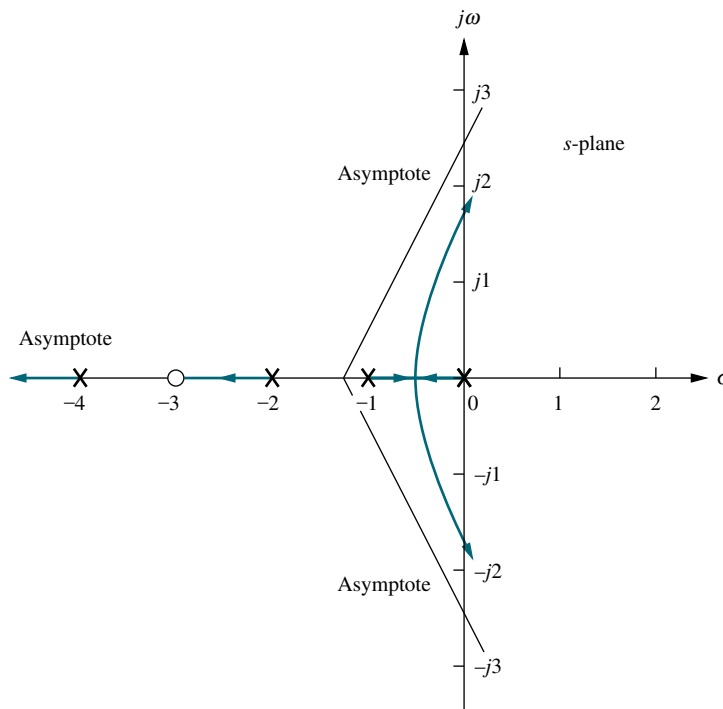


FIGURE 8.12 Root locus and asymptotes for the system of Figure 8.11

Skill-Assessment Exercise 8.3

PROBLEM: Sketch the root locus and its asymptotes for a unity feedback system that has the forward transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

ANSWER: The complete solution is at www.wiley.com/college/nise.

8.5 Refining the Sketch

The rules covered in the previous section permit us to sketch a root locus rapidly. If we want more detail, we must be able to accurately find important points on the root locus along with their associated gain. Points on the real axis where the root locus enters or leaves the complex plane—real-axis breakaway and break-in points—and the $j\omega$ -axis crossings are candidates. We can also derive a better picture of the root locus by finding the angles of departure and arrival from complex poles and zeros, respectively.

In this section, we discuss the calculations required to obtain specific points on the root locus. Some of these calculations can be made using the basic root locus relationship that the sum of the zero angles minus the sum of the pole angles equals an odd multiple of 180° , and the gain at a point on the root locus is found as the ratio of (1) the product of pole lengths drawn to that point to (2) the product of zero lengths drawn to that point. We have yet to address how to implement this task. In the past, an inexpensive tool called a SpiruleTM added the angles together rapidly and then quickly multiplied and divided the lengths to obtain the gain. Today we can rely on hand-held or programmable calculators as well as personal computers.

Students pursuing MATLAB will learn how to apply it to the root locus at the end of Section 8.6. Other alternatives are discussed in Appendix H.2 at www.wiley.com/college/nise. The discussion can be adapted to programmable hand-held calculators. All readers are encouraged to select a computational aid at this point. Root locus calculations can be labor intensive if hand calculations are used.

We now discuss how to refine our root locus sketch by calculating real-axis breakaway and break-in points, $j\omega$ -axis crossings, angles of departure from complex poles, and angles of arrival to complex zeros. We conclude by showing how to find accurately any point on the root locus and calculate the gain.

Real-Axis Breakaway and Break-In Points

Numerous root loci appear to break away from the real axis as the system poles move from the real axis to the complex plane. At other times the loci appear to return to the real axis as a pair of complex poles becomes real. We illustrate this in Figure 8.13. This locus is sketched using the first four rules: (1) number of branches, (2) symmetry, (3) real-axis segments, and (4) starting and ending points. The figure shows a root locus leaving the real axis between -1 and -2 and returning to the real axis between $+3$ and $+5$. The point where the locus leaves the real axis, $-\sigma_1$, is called the *breakaway point*, and the point where the locus returns to the real axis, σ_2 , is called the *break-in point*.