8.1 Introduction

Root locus, a graphical presentation of the closed-loop poles as a system parameter is varied, is a powerful method of analysis and design for stability and transient response (*Evans*, *1948*; *1950*). Feedback control systems are difficult to comprehend from a qualitative point of view, and hence they rely heavily upon mathematics. The root locus covered in this chapter is a graphical technique that gives us the qualitative description of a control system's performance that we are looking for and also serves as a powerful quantitative tool that yields more information than the methods already discussed.

Up to this point, gains and other system parameters were designed to yield a desired transient response for only first- and second-order systems. Even though the root locus can be used to solve the same kind of problem, its real power lies in its ability to provide solutions for systems of order higher than 2. For example, under the right conditions, a fourth-order system's parameters can be designed to yield a given percent overshoot and settling time using the concepts learned in Chapter 4.

The root locus can be used to describe qualitatively the performance of a system as various parameters are changed. For example, the effect of varying gain upon percent overshoot, settling time, and peak time can be vividly displayed. The qualitative description can then be verified with quantitative analysis.

Besides transient response, the root locus also gives a graphical representation of a system's stability. We can clearly see ranges of stability, ranges of instability, and the conditions that cause a system to break into oscillation.

Before presenting root locus, let us review two concepts that we need for the ensuing discussion: (1) the control system problem and (2) complex numbers and their representation as vectors.

The Control System Problem

We have previously encountered the control system problem in Chapter 6: Whereas the poles of the open-loop transfer function are easily found (typically, they are known by inspection and do not change with changes in system gain), the poles of the closed-loop transfer function are more difficult to find (typically, they cannot be found without factoring the closed-loop system's characteristic polynomial, the denominator of the closed-loop transfer function), and further, the closed-loop poles change with changes in system gain.

A typical closed-loop feedback control system is shown in Figure 8.1(a). The open-loop transfer function was defined in Chapter 5 as $KG(s)H(s)$. Ordinarily, we

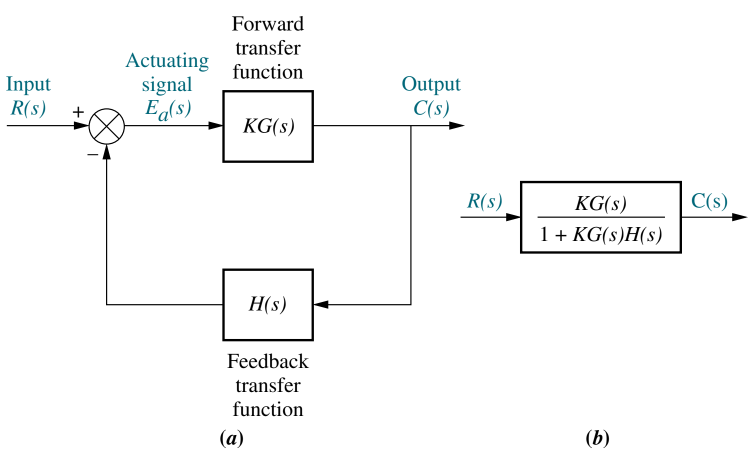


FIGURE 8.1 **a.** Closed-loop system; **b.** equivalent transfer function

can determine the poles of $KG(s)H(s)$, since these poles arise from simple cascaded first- or second-order subsystems. Further, variations in K do not affect the location of any pole of this function. On the other hand, we cannot determine the poles of $T(s) = KG(s)/[1 + KG(s)H(s)]$ unless we factor the denominator. Also, the poles of $T(s)$ change with K.

Let us demonstrate. Letting

$$G(s) = \frac{N_G(s)}{D_G(s)} \tag{8.1}$$

and

$$H(s) = \frac{N_H(s)}{D_H(s)} \tag{8.2}$$

then

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)} \tag{8.3}$$

where N and D are factored polynomials and signify numerator and denominator terms, respectively. We observe the following: Typically, we know the factors of the numerators and denominators of $G(s)$ and $H(s)$. Also, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known and in fact can change with K. For example, if $G(s) = (s + 1)/[s(s + 2)]$ and $H(s) = (s + 3)/(s + 4)$, the poles of $KG(s)H(s)$ are $0, -2$, and -4. The zeros of $KG(s)H(s)$ are -1 and -3. Now, $T(s) = K(s + 1)(s + 4)/[s^3 + (6 + K)s^2 + (8 + 4K)s + 3K]$. Thus, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known without factoring the denominator, and they are a function of K. Since the system's transient response and stability are dependent upon the poles of $T(s)$, we have no knowledge of the system's performance unless we factor the denominator for specific values of K. The root locus will be used to give us a vivid picture of the poles of $T(s)$ as K varies.

Vector Representation of Complex Numbers

Any *complex number*, $\sigma + j\omega$, described in Cartesian coordinates can be graphically represented by a vector, as shown in Figure 8.2(a). The complex number also can be described in polar form with magnitude M and angle θ, as $M\angle\theta$. If the complex number is substituted into a complex function, $F(s)$, another complex number will result. For example, if $F(s) = (s + a)$, then substituting the complex number $s = \sigma + j\omega$ yields $F(s) = (\sigma + a) + j\omega$, another complex number. This number is shown in Figure 8.2(b). Notice that $F(s)$ has a zero at $-a$. If we translate the vector a units to the left, as in Figure 8.2(c), we have an alternate representation of the complex number that originates at the zero of $F(s)$ and terminates on the point $s = \sigma + j\omega$.

We conclude that $(s + a)$ *is a complex number and can be represented by a vector drawn from the zero of the function to the point s*. For example, $(s + 7)|_{s \to 5+j2}$ is a complex number drawn from the zero of the function, -7, to the point s, which is $5 + j2$, as shown in Figure 8.2(d).

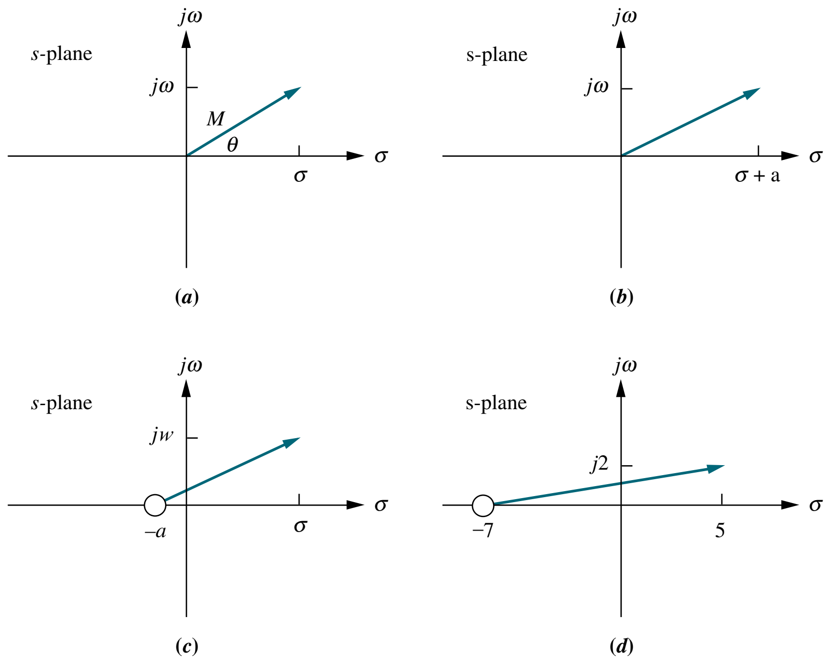


FIGURE 8.2 Vector representation of complex numbers: **a.** $s = \sigma + j\omega$; **b.** $(s + a)$; **c.** alternate representation of $(s + a)$; **d.** $(s + 7)|_{s \rightarrow 5+j2}$

Now let us apply the concepts to a complicated function. Assume a function

$$F(s) = \frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = \frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}} \tag{8.4}$$

where the symbol $\prod$ means "product," m = number of zeros, and n = number of poles. Each factor in the numerator and each factor in the denominator is a complex number that can be represented as a vector. The function defines the complex arithmetic to be performed in order to evaluate $F(s)$ at any point, s. Since each complex factor can be thought of as a vector, the magnitude, M, of $F(s)$ at any point, s, is

$$M = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}} = \frac{\prod_{i=1}^{m}|(s + z_i)|}{\prod_{j=1}^{n}|(s + p_j)|} \tag{8.5}$$

where a zero length, $|(s + z_i)|$, is the magnitude of the vector drawn from the zero of $F(s)$ at $-z_i$ to the point s, and a pole length, $|(s + p_j)|$, is the magnitude of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s. The angle, θ, of $F(s)$ at any point, s, is

$$\begin{aligned} \theta &= \sum \text{zero angles} - \sum \text{pole angles} \\ &= \sum_{i=1}^{m} \angle(s + z_i) - \sum_{j=1}^{n} \angle(s + p_j) \end{aligned} \tag{8.6}$$

where a zero angle is the angle, measured from the positive extension of the real axis, of a vector drawn from the zero of $F(s)$ at $-z_i$ to the point s, and a pole angle is the angle, measured from the positive extension of the real axis, of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s.

As a demonstration of the above concept, consider the following example.

Example 8.1

Evaluation of a Complex Function via Vectors

PROBLEM: Given

$$F(s) = \frac{(s+1)}{s(s+2)} \tag{8.7}$$

find $F(s)$ at the point $s = -3 + j4$.

SOLUTION: The problem is graphically depicted in Figure 8.3, where each vector, $(s+\alpha)$, of the function is shown terminating on the selected point $s = -3 + j4$. The vector originating at the zero at -1 is

$$\sqrt{20}\angle 116.6^\circ \tag{8.8}$$

The vector originating at the pole at the origin is

$$5\angle 126.9^\circ \tag{8.9}$$

The vector originating at the pole at -2 is

$$\sqrt{17}\angle 104.0^\circ \tag{8.10}$$

Substituting Eqs. (8.8) through (8.10) into Eqs. (8.5) and (8.6) yields

$$M\angle\theta = \frac{\sqrt{20}}{5\sqrt{17}}\angle 116.6^\circ - 126.9^\circ - 104.0^\circ = 0.217\angle - 114.3^\circ \tag{8.11}$$

as the result for evaluating $F(s)$ at the point $-3 + j4$.

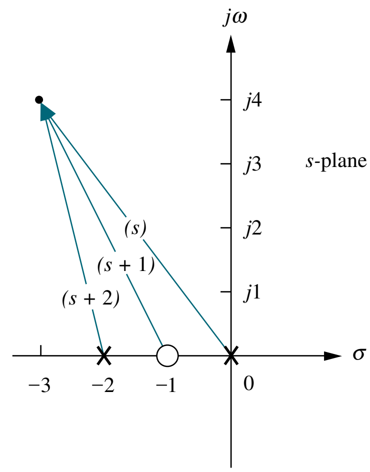


FIGURE 8.3 Vector representation of Eq. (8.7)

Skill-Assessment Exercise 8.1

PROBLEM: Given

$$F(s) = \frac{(s+2)(s+4)}{s(s+3)(s+6)}$$

find $F(s)$ at the point $s = -7 + j9$ the following ways:

a. Directly substituting the point into $F(s)$

b. Calculating the result using vectors

ANSWER:

$$-0.0339 - j0.0899 = 0.096\angle - 110.7^\circ$$

The complete solution is at www.wiley.com/college/nise.

TryIt 8.1

Use the following MATLAB statements to solve the problem given in Skill-Assessment Exercise 8.1.

```
s=-7+9j;
G=(s+2)*(s+4)/...
 (s*(s+3)*(s+6));
Theta=(180/pi)*...
 angle(G)
M=abs(G)
```

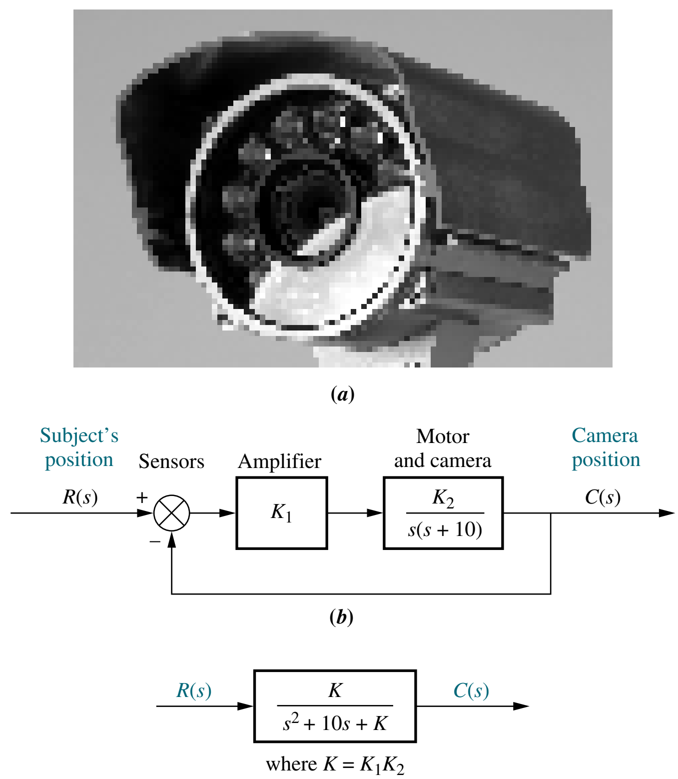
We are now ready to begin our discussion of the root locus.

8.2 Defining the Root Locus

A security camera system similar to that shown in Figure 8.4(a) can automatically follow a subject. The tracking system monitors pixel changes and positions the camera to center the changes.

The root locus technique can be used to analyze and design the effect of loop gain upon the system's transient response and stability. Assume the block diagram representation of a tracking system as shown in Figure 8.4(b), where the closed-loop poles of the system change location as the gain, K, is varied. Table 8.1, which was formed by applying the quadratic formula to the denominator of the transfer function in Figure 8.4(c), shows the variation of pole location for different values of gain, K. The data of Table 8.1 is graphically displayed in Figure 8.5(a), which shows each pole and its gain.

As the gain, K, increases in Table 8.1 and Figure 8.5(a), the closed-loop pole, which is at -10 for $K = 0$, moves toward the right, and the closed-loop pole, which is at 0 for $K = 0$, moves toward the left. They meet at -5, break away from the real axis, and move into the complex plane. One closed-loop pole moves upward while the other moves downward. We cannot tell which pole moves up or which moves down. In Figure 8.5(b), the individual closed-loop pole locations are removed and their paths are represented with solid lines. It is this *representation of the paths of the*



(c)

FIGURE 8.4 **a.** Security cameras with auto tracking can be used to follow moving objects automatically; **b.** block diagram; **c.** closed-loop transfer function

TABLE 8.1 Pole location as function of gain for the system of Figure 8.4

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

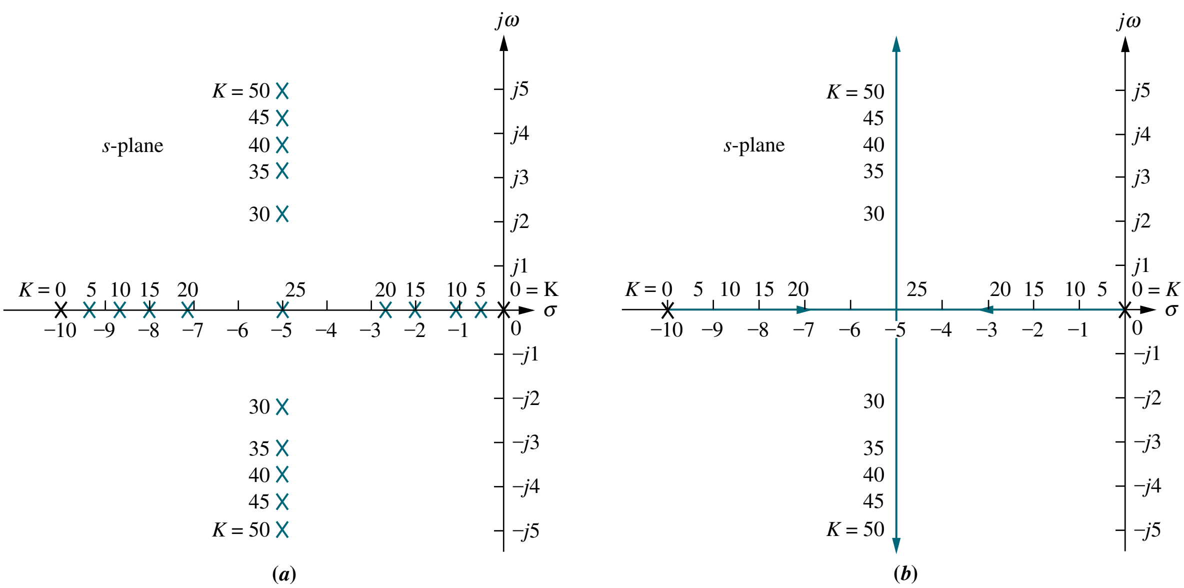


FIGURE 8.5 **a.** Pole plot from Table 8.1; **b.** root locus

closed-loop poles as the gain is varied that we call a *root locus*. For most of our work, the discussion will be limited to positive gain, or $K \geq 0$.

The root locus shows the changes in the transient response as the gain, K, varies. First of all, the poles are real for gains less than 25. Thus, the system is overdamped. At a gain of 25, the poles are real and multiple and hence critically damped. For gains above 25, the system is underdamped. Even though these preceding conclusions were available through the analytical techniques covered in Chapter 4, the following conclusions are graphically demonstrated by the root locus.

Directing our attention to the underdamped portion of the root locus, we see that regardless of the value of gain, the real parts of the complex poles are always the same.

Since the settling time is inversely proportional to the real part of the complex poles for this second-order system, the conclusion is that regardless of the value of gain, the settling time for the system remains the same under all conditions of underdamped responses.

Also, as we increase the gain, the damping ratio diminishes, and the percent overshoot increases. The damped frequency of oscillation, which is equal to the imaginary part of the pole, also increases with an increase in gain, resulting in a reduction of the peak time. Finally, since the root locus never crosses over into the right half-plane, the system is always stable, regardless of the value of gain, and can never break into a sinusoidal oscillation.

These conclusions for such a simple system may appear to be trivial. What we are about to see is that the analysis is applicable to systems of order higher than 2. For these systems, it is difficult to tie transient response characteristics to the pole location. The root locus will allow us to make that association and will become an important technique in the analysis and design of higher-order systems.

8.3 Properties of the Root Locus

In Section 8.2, we arrived at the root locus by factoring the second-order polynomial in the denominator of the transfer function. Consider what would happen if that polynomial were of fifth or tenth order. Without a computer, factoring the polynomial would be quite a problem for numerous values of gain.

We are about to examine the properties of the root locus. From these properties we will be able to make a rapid *sketch* of the root locus for higher-order systems without having to factor the denominator of the closed-loop transfer function.

The properties of the root locus can be derived from the general control system of Figure 8.1(a). The closed-loop transfer function for the system is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \tag{8.12}$$

From Eq. (8.12), a pole, s, exists when the characteristic polynomial in the denominator becomes zero, or

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ \quad k = 0, \pm1, \pm2, \pm3, \ldots \tag{8.13}$$

where -1 is represented in polar form as $1\angle(2k+1)180^\circ$. Alternately, a value of s is a closed-loop pole if

$$|KG(s)H(s)| = 1 \tag{8.14}$$

and

$$\angle KG(s)H(s) = (2k+1)180^\circ \tag{8.15}$$

Equation (8.13) implies that if a value of s is substituted into the function $KG(s)H(s)$, a complex number results. If the angle of the complex number is an odd multiple of 180°, that value of s is a system pole for some particular value of K. What