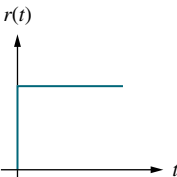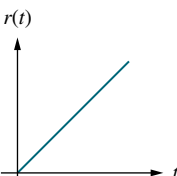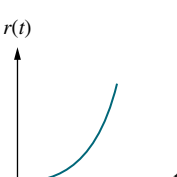# (7.1 Introduction

In Chapter 1, we saw that control systems analysis and design focus on three specifications: (1) transient response, (2) stability, and (3) steady-state errors, taking into account the robustness of the design along with economic and social considerations. Elements of transient analysis were derived in Chapter 4 for first- and second-order systems. These concepts are revisited in Chapter 8, where they are extended to higher-order systems. Stability was covered in Chapter 6, where we saw that forced responses were overpowered by natural responses that increase without bound if the system is unstable. Now we are ready to examine steady-state errors. We define the errors and derive methods of controlling them. As we progress, we find that control system design entails trade-offs between desired transient response, steady-state error, and the requirement that the system be stable.

Definition and Test Inputs

Steady-state error is the difference between the input and the output for a prescribed test input as $t \rightarrow \infty$. Test inputs used for steady-state error analysis and design are summarized in Table 7.1.

In order to explain how these test signals are used, let us assume a position control system, where the output position follows the input commanded position. Step inputs represent constant position and thus are useful in determining the ability of the control system to position itself with respect to a stationary target, such as a satellite in geostationary orbit (see Figure 7.1). An antenna position control is an example of a system that can be tested for accuracy using step inputs.

TABLE 7.1   Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\dfrac{1}{s}$
	Ramp	Constant velocity	t	$\dfrac{1}{s^2}$
	Parabola	Constant acceleration	$\dfrac{1}{2}t^2$	$\dfrac{1}{s^3}$
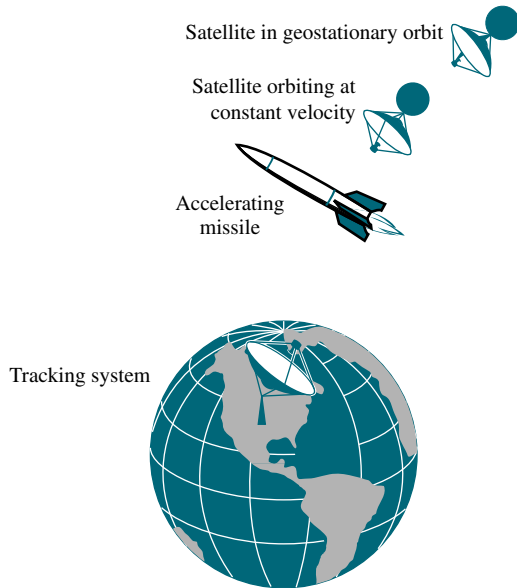
FIGURE 7.1 Test inputs for steady-state error analysis and design vary with target type

Labels in figure: Satellite in geostationary orbit; Satellite orbiting at constant velocity; Accelerating missile; Tracking system

Ramp inputs represent constant-velocity inputs to a position control system by their linearly increasing amplitude. These waveforms can be used to test a system's ability to follow a linearly increasing input or, equivalently, to track a constant-velocity target. For example, a position control system that tracks a satellite that moves across the sky at a constant angular velocity, as shown in Figure 7.1, would be tested with a ramp input to evaluate the steady-state error between the satellite's angular position and that of the control system.

Finally, parabolas, whose second derivatives are constant, represent constant-acceleration inputs to position control systems and can be used to represent accelerating targets, such as the missile in Figure 7.1, to determine the steady-state error performance.

Application to Stable Systems

Since we are concerned with the difference between the input and the output of a feedback control system after the steady state has been reached, our discussion is limited to stable systems, where the natural response approaches zero as $t \to \infty$. Unstable systems represent loss of control in the steady state and are not acceptable for use at all. The expressions we derive to calculate the steady-state error can be applied erroneously to an unstable system. Thus, the engineer must check the system for stability while performing steady-state error analysis and design. However, in order to focus on the topic, we assume that all the systems in examples and problems in this chapter are stable. For practice, you may want to test some of the systems for stability.

Evaluating Steady-State Errors

Let us examine the concept of steady-state errors. In Figure 7.2(*a*) a step input and two possible outputs are shown. Output 1 has zero steady-state error, and output 2 has a finite steady-state error, $e_2(\infty)$. A similar example is shown in Figure 7.2(*b*), where a ramp input is compared with output 1, which has zero steady-state error, and output 2, which has a finite steady-state error, $e_2(\infty)$, as measured vertically between the input and output 2 after the transients have died down. For the ramp input
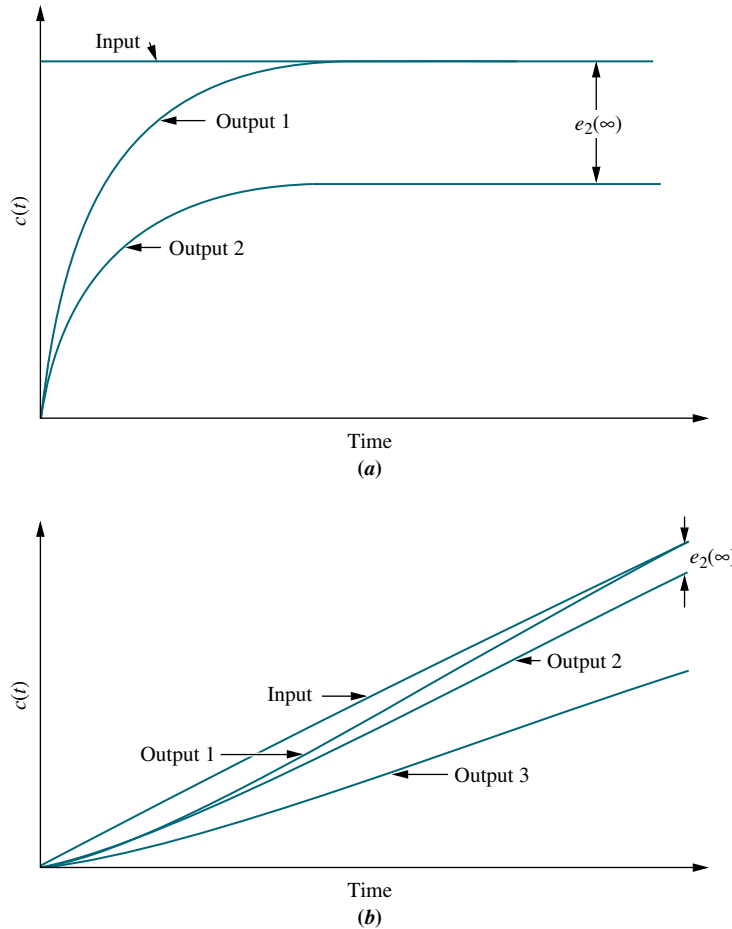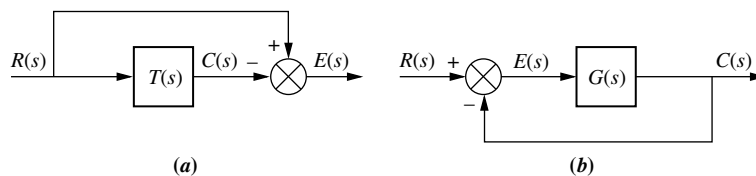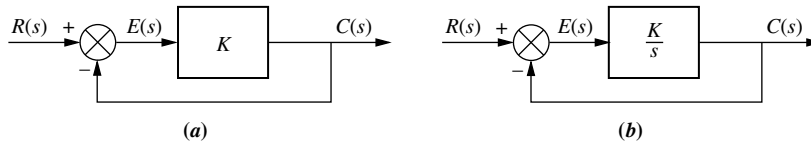
FIGURE 7.2   Steady-state error:
a. step input; **b.** ramp input

another possibility exists. If the output's slope is different from that of the input, then output 3, shown in Figure 7.2(*b*), results. Here the steady-state error is infinite as measured vertically between the input and output 3 after the transients have died down, and *t* approaches infinity.

Let us now look at the error from the perspective of the most general block diagram. Since the error is the difference between the input and the output of a system, we assume a closed-loop transfer function, $T(s)$, and form the error, $E(s)$, by taking the difference between the input and the output, as shown in Figure 7.3(*a*). Here we are interested in the steady-state, or final, value of $e(t)$. For unity feedback systems, $E(s)$ appears as shown in Figure 7.3(*b*). In this chapter, we study and derive expressions for the steady-state error for unity feedback systems first and then expand to nonunity feedback systems. Before we begin our study of steady-state errors for unity feedback systems, let us look at the sources of the errors with which we deal.

FIGURE 7.3   Closed-loop control
system error: **a.** general
representation; **b.** representation
for unity feedback systems

FIGURE 7.4   System with **a.** finite steady-state error for a step input; **b.** zero steady-state error for step input

Sources of Steady-State Error

Many steady-state errors in control systems arise from nonlinear sources, such as backlash in gears or a motor that will not move unless the input voltage exceeds a threshold. Nonlinear behavior as a source of steady-state errors, although a viable topic for study is beyond the scope of a text on linear control systems. The steady-state errors we study here are errors that arise from the configuration of the system itself and the type of applied input.

For example, look at the system of Figure 7.4(a), where $R(s)$ is the input, $C(s)$ is the output, and $E(s) = R(s) - C(s)$ is the error. Consider a step input. In the steady state, if $c(t)$ equals $r(t)$, $e(t)$ will be zero. But with a pure gain, K, the error, $e(t)$, cannot be zero if $c(t)$ is to be finite and nonzero. Thus, by virtue of the configuration of the system (a pure gain of K in the forward path), an error must exist. If we call $c_\text{steady-state}$ the steady-state value of the output and $e_\text{steady-state}$ the steady-state value of the error, then $c_\text{steady-state} = K e_\text{steady-state}$, or

$$e_\text{steady-state} = \frac{1}{K} c_\text{steady-state} \tag{7.1}$$

Thus, the larger the value of K, the smaller the value of $e_\text{steady-state}$ would have to be to yield a similar value of $c_\text{steady-state}$. The conclusion we can draw is that with a pure gain in the forward path, there will always be a steady-state error for a step input. This error diminishes as the value of K increases.

If the forward-path gain is replaced by an integrator, as shown in Figure 7.4(b), there will be zero error in the steady state for a step input. The reasoning is as follows: As $c(t)$ increases, $e(t)$ will decrease, since $e(t) = r(t) - c(t)$. This decrease will continue until there is zero error, but there will still be a value for $c(t)$ since an integrator can have a constant output without any input. For example, a motor can be represented simply as an integrator. A voltage applied to the motor will cause rotation. When the applied voltage is removed, the motor will stop and remain at its present output position. Since it does not return to its initial position, we have an angular displacement output without an input to the motor. Therefore, a system similar to Figure 7.4(b), which uses a motor in the forward path, can have zero steady-state error for a step input.

We have examined two cases qualitatively to show how a system can be expected to exhibit various steady-state error characteristics, depending upon the system configuration. We now formalize the concepts and derive the relationships between the steady-state errors and the system configuration generating these errors.

7.2   Steady-State Error for Unity Feedback Systems

Steady-state error can be calculated from a system's closed-loop transfer function, $T(s)$, or the open-loop transfer function, $G(s)$, for unity feedback systems. We begin by deriving the system's steady-state error in terms of the closed-loop transfer

function, $T(s)$, in order to introduce the subject and the definitions. Next we obtain insight into the factors affecting steady-state error by using the open-loop transfer function, $G(s)$, in unity feedback systems for our calculations. Later in the chapter we generalize this discussion to nonunity feedback systems.

Steady-State Error in Terms of $T(s)$

Consider Figure 7.3(a). To find $E(s)$, the error between the input, $R(s)$, and the output, $C(s)$, we write

$$E(s) = R(s) - C(s) \tag{7.2}$$

But

$$C(s) = R(s)T(s) \tag{7.3}$$

Substituting Eq. (7.3) into Eq. (7.2), simplifying, and solving for $E(s)$ yields

$$E(s) = R(s)[1 - T(s)] \tag{7.4}$$

Although Eq. (7.4) allows us to solve for $e(t)$ at any time, t, we are interested in the final value of the error, $e(\infty)$. Applying the final value theorem,[1] which allows us to use the final value of $e(t)$ without taking the inverse Laplace transform of $E(s)$, and then letting t approach infinity, we obtain

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \tag{7.5}[2]$$

Substituting Eq. (7.4) into Eq. (7.5) yields

$$e(\infty) = \lim_{s \to \infty} sR(s)[1 - T(s)] \tag{7.6}$$

Let us look at an example.

---

[1] The final value theorem is derived from the Laplace transform of the derivative. Thus,

$$\mathscr{L}[\dot{f}(t)] = \int_{0-}^{\infty} \dot{f}(t)e^{st}dt = sF(s) - f(0-)$$

As $s \to 0$,

$$\int_{0-}^{\infty} \dot{f}(t)dt = f(\infty) - f(0-) = \lim_{s \to 0} sF(s) - f(0-)$$

or

$$f(\infty) = \lim_{s \to 0} sF(s)$$

For finite steady-state errors, the final value theorem is valid only if $F(s)$ has poles only in the left half-plane and, at most, one pole at the origin. However, correct results that yield steady-state errors that are infinite can be obtained if $F(s)$ has more than one pole at the origin (see *D'Azzo and Houpis, 1988*). If $F(s)$ has poles in the right half-plane or poles on the imaginary axis other than at the origin, the final value theorem is invalid.

[2] Valid only if (1) $E(s)$ has poles only in the left half-plane and at the origin, and (2) the closed-loop transfer function, $T(s)$, is stable. Notice that by using Eq. (7.5), numerical results can be obtained for unstable systems. These results, however, are meaningless.

Example 7.1

Steady-State Error in Terms of *T(s)*

PROBLEM: Find the steady-state error for the system of Figure 7.3(*a*) if $T(s) = 5/(s^2 + 7s + 10)$ and the input is a unit step.

SOLUTION: From the problem statement, $R(s) = 1/s$ and $T(s) = 5/(s^2 + 7s + 10)$. Substituting into Eq. (7.4) yields

$$E(s) = \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)} \tag{7.7}$$

Since $T(s)$ is stable and, subsequently, $E(s)$ does not have right–half-plane poles or $j\omega$ poles other than at the origin, we can apply the final value theorem. Substituting Eq. (7.7) into Eq. (7.5) gives $e(\infty) = 1/2$.

Steady-State Error in Terms of *G(s)*

Many times we have the system configured as a unity feedback system with a forward transfer function, $G(s)$. Although we can find the closed-loop transfer function, $T(s)$, and then proceed as in the previous subsection, we find more insight for analysis and design by expressing the steady-state error in terms of $G(s)$ rather than $T(s)$.

Consider the feedback control system shown in Figure 7.3(*b*). Since the feedback, $H(s)$, equals 1, the system has unity feedback. The implication is that $E(s)$ is actually the error between the input, $R(s)$, and the output, $C(s)$. Thus, if we solve for $E(s)$, we will have an expression for the error. We will then apply the final value theorem, Item 11 in Table 2.2, to evaluate the steady-state error.

Writing $E(s)$ from Figure 7.3(*b*), we obtain

$$E(s) = R(s) - C(s) \tag{7.8}$$

But

$$C(s) = E(s)G(s) \tag{7.9}$$

Finally, substituting Eq. (7.9) into Eq. (7.8) and solving for $E(s)$ yields

$$E(s) = \frac{R(s)}{1 + G(s)} \tag{7.10}$$

We now apply the final value theorem, Eq. (7.5). At this point in a numerical calculation, we must check to see whether the closed-loop system is stable, using, for example, the Routh-Hurwitz criterion. For now, though, assume that the closed-loop system is stable and substitute Eq. (7.10) into Eq. (7.5), obtaining

$$e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} \tag{7.11}$$

Equation (7.11) allows us to calculate the steady-state error, $e(\infty)$, given the input, $R(s)$, and the system, $G(s)$. We now substitute several inputs for $R(s)$ and then

draw conclusions about the relationships that exist between the open-loop system, $G(s)$, and the nature of the steady-state error, $e(\infty)$.

The three test signals we use to establish specifications for a control system's steady-state error characteristics are shown in Table 7.1. Let us take each input and evaluate its effect on the steady-state error by using Eq. (7.11).

Step Input. Using Eq. (7.11) with $R(s) = 1/s$, we find

$$e(\infty) = e_{\text{step}}(\infty) = \lim_{s \to 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)} \tag{7.12}$$

The term

$$\lim_{s \to 0} G(s)$$

is the dc gain of the forward transfer function, since s, the frequency variable, is approaching zero. In order to have zero steady-state error,

$$\lim_{s \to 0} G(s) = \infty \tag{7.13}$$

Hence, to satisfy Eq. (7.13), $G(s)$ must take on the following form:

$$G(s) \equiv \frac{(s + z_1)(s + z_2) \cdots}{s^n (s + p_1)(s + p_2) \cdots} \tag{7.14}$$

and for the limit to be infinite, the denominator must be equal to zero as s goes to zero. Thus, $n \geq 1$; that is, at least one pole must be at the origin. Since division by s in the frequency domain is integration in the time domain (see Table 2.2, Item 10), we are also saying that at least one pure integration must be present in the forward path. The steady-state response for this case of zero steady-state error is similar to that shown in Figure 7.2(a), output 1.

If there are no integrations, then $n = 0$. Using Eq. (7.14), we have

$$\lim_{s \to 0} G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \tag{7.15}$$

which is finite and yields a finite error from Eq. (7.12). Figure 7.2(a), output 2, is an example of this case of finite steady-state error.

In summary, for a step input to a unity feedback system, the steady-state error will be zero if there is at least one pure integration in the forward path. If there are no integrations, then there will be a nonzero finite error. This result is comparable to our qualitative discussion in Section 7.1, where we found that a pure gain yields a constant steady-state error for a step input, but an integrator yields zero error for the same type of input. We now repeat the development for a ramp input.

Ramp Input. Using Eq. (7.11) with, $R(s) = 1/s^2$, we obtain

$$e(\infty) = e_{\text{ramp}}(\infty) = \lim_{s \to 0} \frac{s(1/s^2)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim_{s \to 0} sG(s)} \tag{7.16}$$

To have zero steady-state error for a ramp input, we must have

$$\lim_{s \to 0} sG(s) = \infty \tag{7.17}$$

To satisfy Eq. (7.17), $G(s)$ must take the same form as Eq. (7.14), except that $n \geq 2$. In other words, there must be at least two integrations in the forward path. An example of zero steady-state error for a ramp input is shown in Figure 7.2(b), output 1.

If only one integration exists in the forward path, then, assuming Eq. (7.14),

$$\lim_{s \to 0} sG(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \tag{7.18}$$

which is finite rather than infinite. Using Eq. (7.16), we find that this configuration leads to a constant error, as shown in Figure 7.2(b), output 2.

If there are no integrations in the forward path, then

$$\lim_{s \to 0} sG(s) = 0 \tag{7.19}$$

and the steady-state error would be infinite and lead to diverging ramps, as shown in Figure 7.2(b), output 3. Finally, we repeat the development for a parabolic input.

Parabolic Input. Using Eq. (7.11) with $R(s) = 1/s^3$, we obtain

$$e(\infty) = e_{\text{parabola}}(\infty) = \lim_{s \to 0} \frac{s(1/s^3)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s \to 0} s^2 G(s)} \tag{7.20}$$

In order to have zero steady-state error for a parabolic input, we must have

$$\lim_{s \to 0} s^2 G(s) = \infty \tag{7.21}$$

To satisfy Eq. (7.21), $G(s)$ must take on the same form as Eq. (7.14), except that $n \geq 3$. In other words, there must be at least three integrations in the forward path.

If there are only two integrations in the forward path, then

$$\lim_{s \to 0} s^2 G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \tag{7.22}$$

is finite rather than infinite. Using Eq. (7.20), we find that this configuration leads to a constant error.

If there is only one or less integration in the forward path, then

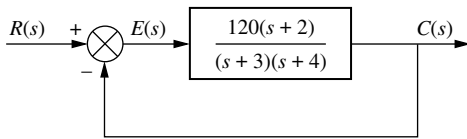$$\lim_{s \to 0} s^2 G(s) = 0 \tag{7.23}$$

and the steady-state error is infinite. Two examples demonstrate these concepts.

Example 7.2

Steady-State Errors for Systems with No Integrations

PROBLEM: Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, and $5t^2 u(t)$ to the system shown in Figure 7.5. The function $u(t)$ is the unit step.

FIGURE 7.5 Feedback control system for
Example 7.2

SOLUTION: First we verify that the closed-loop system is indeed
stable. For this example we leave out the details. Next, for the input
$5u(t)$, whose Laplace transform is $5/s$, the steady-state error will be
five times as large as that given by Eq. (7.12), or

$$e(\infty) = e_{\text{step}}(\infty) = \frac{5}{1 + \lim\limits_{s \to 0} G(s)} = \frac{5}{1 + 20} = \frac{5}{21} \tag{7.24}$$

which implies a response similar to output 2 of Figure 7.2(a).
   For the input $5tu(t)$, whose Laplace transform is $5/s^2$, the steady-state error
will be five times as large as that given by Eq. (7.16), or

$$e(\infty) = e_{\text{ramp}}(\infty) = \frac{5}{\lim\limits_{s \to 0} sG(s)} = \frac{5}{0} = \infty \tag{7.25}$$
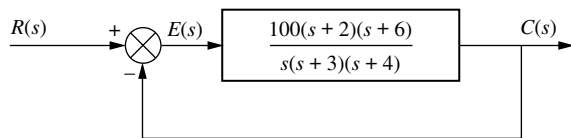
which implies a response similar to output 3 of Figure 7.2(b).
   For the input $5t^2 u(t)$, whose Laplace transform is $10/s^3$, the steady-state error
will be 10 times as large as that given by Eq. (7.20), or

$$e(\infty) = e_{\text{parabola}}(\infty) = \frac{10}{\lim\limits_{s \to 0} s^2 G(s)} = \frac{10}{0} = \infty \tag{7.26}$$

Example 7.3

Steady-State Errors for Systems with One Integration

FIGURE 7.6   Feedback control system for Example 7.3

PROBLEM: Find the steady-state errors for inputs of $5u(t)$,
$5tu(t)$, and $5t^2 u(t)$ to the system shown in Figure 7.6. The
function $u(t)$ is the unit step.

SOLUTION: First verify that the closed-loop system is in-
deed stable. For this example we leave out the details. Next
note that since there is an integration in the forward path, the
steady-state errors for some of the input waveforms will be
less than those found in Example 7.2. For the input $5u(t)$, whose Laplace transform is
$5/s$, the steady-state error will be five times as large as that given by Eq. (7.12), or

$$e(\infty) = e_{\text{step}}(\infty) = \frac{5}{1 + \lim\limits_{s \to 0} G(s)} = \frac{5}{\infty} = 0 \tag{7.27}$$

which implies a response similar to output 1 of Figure 7.2(a). Notice that the
integration in the forward path yields zero error for a step input, rather than the
finite error found in Example 7.2.
   For the input $5tu(t)$, whose Laplace transform is $5/s^2$, the steady-state error
will be five times as large as that given by Eq. (7.16), or

$$e(\infty) = e_{\text{ramp}}(\infty) = \frac{5}{\lim\limits_{s \to 0} sG(s)} = \frac{5}{100} = \frac{1}{20} \tag{7.28}$$

which implies a response similar to output 2 of Figure 7.2(*b*). Notice that the integration in the forward path yields a finite error for a ramp input, rather than the infinite error found in Example 7.2.

For the input, $5t^2u(t)$, whose Laplace transform is $10/s^3$, the steady-state error will be 10 times as large as that given by Eq. (7.20), or

$$e(\infty) = e_{\text{parabola}}(\infty) = \frac{10}{\lim\limits_{s \to 0} s^2 G(s)} = \frac{10}{0} = \infty \qquad (7.29)$$

Notice that the integration in the forward path does not yield any improvement in steady-state error over that found in Example 7.2 for a parabolic input.

Skill-Assessment Exercise 7.1

PROBLEM: A unity feedback system has the following forward transfer function:

$$G(s) = \frac{10(s+20)(s+30)}{s(s+25)(s+35)}$$

a. Find the steady-state error for the following inputs: $15u(t)$, $15tu(t)$, and $15t^2u(t)$.

b. Repeat for

$$G(s) = \frac{10(s+20)(s+30)}{s^2(s+25)(s+35)(s+50)}$$

ANSWERS:

a. The closed-loop system is stable. For $15u(t)$, $e_{\text{step}}(\infty) = 0$; for $15tu(t)$, $e_{\text{ramp}}(\infty) = 2.1875$; for $15(t^2)u(t)$, $e_{\text{parabola}}(\infty) = \infty$.

b. The closed-loop system is unstable. Calculations cannot be made.

The complete solution is at www.wiley.com/college/nise.

7.3   Static Error Constants and System Type

We continue our focus on unity negative feedback systems and define parameters that we can use as steady-state error performance specifications, just as we defined damping ratio, natural frequency, settling time, percent overshoot, and so on as performance specifications for the transient response. These steady-state error performance specifications are called *static error constants*. Let us see how they are defined, how to calculate them, and, in the next section, how to use them for design.