6.1 Introduction

In Chapter 1, we saw that three requirements enter into the design of a control system: transient response, stability, and steady-state errors. Thus far we have covered transient response, which we will revisit in Chapter 8. We are now ready to discuss the next requirement, stability.

Stability is the most important system specification. If a system is unstable, transient response and steady-state errors are moot points. An unstable system cannot be designed for a specific transient response or steady-state error requirement. What, then, is stability? There are many definitions for stability, depending upon the kind of system or the point of view. In this section, we limit ourselves to linear, time-invariant systems.

In Section 1.5, we discussed that we can control the output of a system if the steady-state response consists of only the forced response. But the total response of a system is the sum of the forced and natural responses, or

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t) \tag{6.1}$$

Using these concepts, we present the following definitions of stability, instability, and marginal stability:

A linear, time-invariant system is *stable* if the natural response approaches zero as time approaches infinity.

A linear, time-invariant system is *unstable* if the natural response grows without bound as time approaches infinity.

A linear, time-invariant system is *marginally stable* if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.

Thus, the definition of stability implies that only the forced response remains as the natural response approaches zero.

These definitions rely on a description of the natural response. When one is looking at the total response, it may be difficult to separate the natural response from the forced response. However, we realize that if the input is bounded and the total response is not approaching infinity as time approaches infinity, then the natural response is obviously not approaching infinity. If the input is unbounded, we see an unbounded total response, and we cannot arrive at any conclusion about the stability of the system; we cannot tell whether the total response is unbounded because the forced response is unbounded or because the natural response is unbounded. Thus, our alternate definition of *stability*, one that regards the total response and implies the first definition based upon the natural response, is this:

A system is stable if *every* bounded input yields a bounded output.

We call this statement the bounded-input, bounded-output (BIBO) definition of stability.

Let us now produce an alternate definition for instability based on the total response rather than the natural response. We realize that if the input is bounded but the total response is unbounded, the system is unstable, since we can conclude that the natural response approaches infinity as time approaches infinity. If the input is unbounded, we will see an unbounded total response, and we cannot draw any conclusion about the stability of the system; we cannot tell whether the total response is unbounded because the forced response is unbounded or because the natural response is unbounded. Thus, our alternate definition of *instability*, one that regards the total response, is this:

A system is unstable if *any* bounded input yields an unbounded output.

These definitions help clarify our previous definition of *marginal stability*, which really means that the system is stable for some bounded inputs and unstable for others. For example, we will show that if the natural response is undamped, a bounded sinusoidal input of the same frequency yields a natural response of growing oscillations. Hence, the system appears stable for all bounded inputs except this one sinusoid. Thus, marginally stable systems by the natural response definitions are included as unstable systems under the BIBO definitions.

Let us summarize our definitions of stability for linear, time-invariant systems. Using the natural response:

1. A system is stable if the natural response approaches zero as time approaches infinity.
2. A system is unstable if the natural response approaches infinity as time approaches infinity.
3. A system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates.

Using the total response (BIBO):

1. A system is stable if *every* bounded input yields a bounded output.
2. A system is unstable if *any* bounded input yields an unbounded output.

Physically, an unstable system whose natural response grows without bound can cause damage to the system, to adjacent property, or to human life. Many times systems are designed with limited stops to prevent total runaway. From the perspective of the time response plot of a physical system, instability is displayed by transients that grow without bound and, consequently, a total response that does not approach a steady-state value or other forced response.[1]

How do we determine if a system is stable? Let us focus on the natural response definitions of stability. Recall from our study of system poles that poles in the left half-plane (lhp) yield either pure exponential decay or damped sinusoidal natural responses. These natural responses decay to zero as time approaches infinity. Thus, if the closed-loop system poles are in the left half of the plane and hence have a negative real part, the system is stable. That is, *stable systems have closed-loop transfer functions with poles only in the left half-plane.*

Poles in the right half-plane (rhp) yield either pure exponentially increasing or exponentially increasing sinusoidal natural responses. These natural responses approach infinity as time approaches infinity. Thus, if the closed-loop system poles are in the right half of the s-plane and hence have a positive real part, the system is unstable. Also, poles of multiplicity greater than 1 on the imaginary axis lead to the sum of responses of the form $At^n \cos(\omega t + \phi)$, where $n = 1, 2, \ldots$, which also approaches infinity as time approaches infinity. Thus, *unstable systems have closed-loop transfer functions with at least one pole in the right half-plane and/or poles of multiplicity greater than 1 on the imaginary axis.*

[1] Care must be taken here to distinguish between natural responses growing without bound and a forced response, such as a ramp or exponential increase, that also grows without bound. A system whose forced response approaches infinity is stable as long as the natural response approaches zero.

Finally, a system that has imaginary axis poles of multiplicity 1 yields pure sinusoidal oscillations as a natural response. These responses neither increase nor decrease in amplitude. Thus, *marginally stable systems have closed-loop transfer functions with only imaginary axis poles of multiplicity 1 and poles in the left half-plane.*

As an example, the unit step response of the stable system of Figure 6.1(a) is compared to that of the unstable system of Figure 6.1(b). The responses, also shown in Figure 6.1, show that while the oscillations for the stable system diminish, those for the unstable system increase without bound. Also notice that the stable system's response in this case approaches a steady-state value of unity.

It is not always a simple matter to determine if a feedback control system is stable. Unfortunately, a typical problem that arises is shown in Figure 6.2. Although we know the poles of the forward transfer function in Figure 6.2(a), we do not know the location of the poles of the equivalent closed-loop system of Figure 6.2(b) without factoring or otherwise solving for the roots.

However, under certain conditions, we can draw some conclusions about the stability of the system. First, if the closed-loop transfer function has only

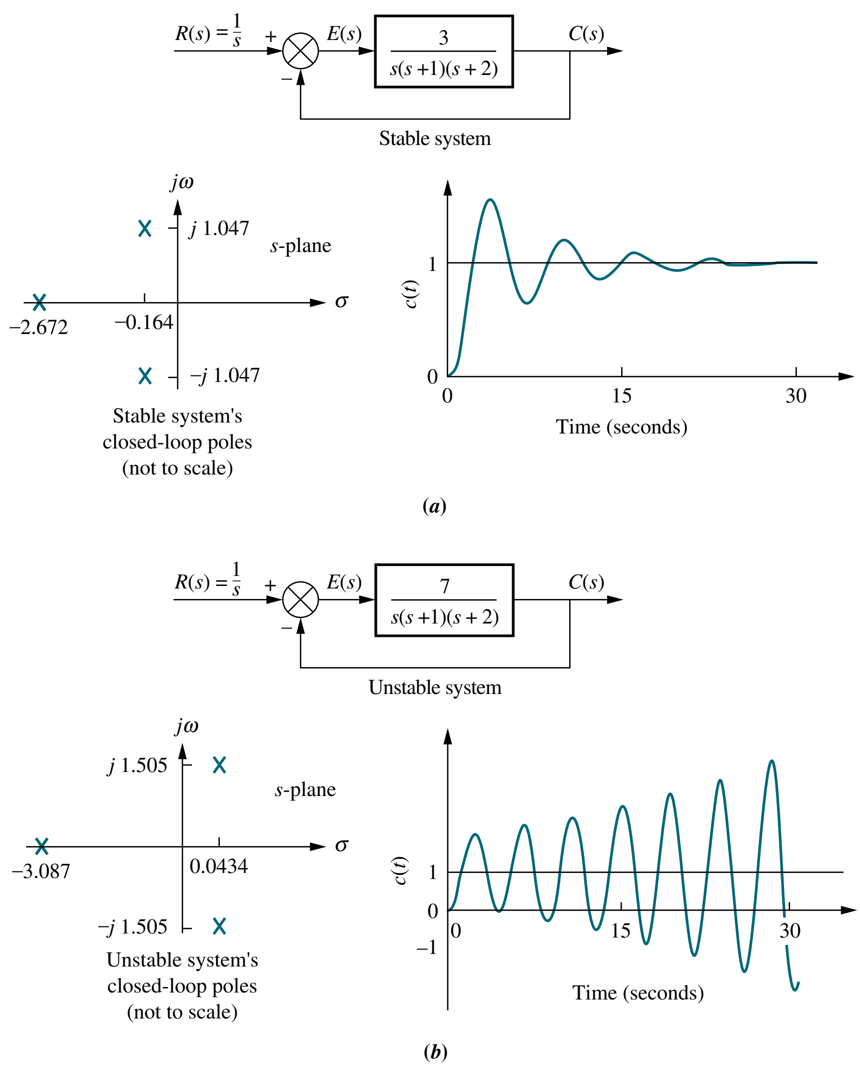

FIGURE 6.1 Closed-loop poles and response: **a.** stable system; **b.** unstable system

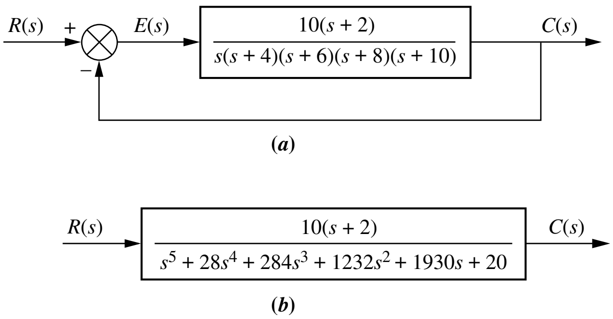

FIGURE 6.2 Common cause of problems in finding closed-loop poles: **a.** original system; **b.** equivalent system

left–half-plane poles, then the factors of the denominator of the closed-loop system transfer function consist of products of terms such as $(s + a_i)$, where a_i is real and positive, or complex with a positive real part. The product of such terms is a polynomial with all positive coefficients.[2] No term of the polynomial can be missing, since that would imply cancellation between positive and negative coefficients or imaginary axis roots in the factors, which is not the case. Thus, a sufficient condition for a system to be unstable is that all signs of the coefficients of the denominator of the closed-loop transfer function are not the same. If powers of s are missing, the system is either unstable or, at best, marginally stable. Unfortunately, if all coefficients of the denominator are positive and not missing, we do not have definitive information about the system's pole locations.

If the method described in the previous paragraph is not sufficient, then a computer can be used to determine the stability by calculating the root locations of the denominator of the closed-loop transfer function. Today some hand-held calculators can evaluate the roots of a polynomial. There is, however, another method to test for stability without having to solve for the roots of the denominator. We discuss this method in the next section.

6.2 Routh-Hurwitz Criterion

In this section, we learn a method that yields stability information without the need to solve for the closed-loop system poles. Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the $j\omega$-axis. (Notice that we say *how many*, not *where*.) We can find the number of poles in each section of the s-plane, but we cannot find their coordinates. The method is called the *Routh-Hurwitz criterion* for stability (*Routh*, *1905*).

The method requires two steps: (1) Generate a data table called a *Routh table* and (2) interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the $j\omega$-axis. You might wonder why we study the Routh-Hurwitz criterion when modern calculators and computers can tell us the exact location of system poles. The power of the method lies in design rather than analysis. For example, if you have an unknown parameter in the denominator of a transfer function, it is difficult to determine via a calculator the range of this parameter to yield stability. You would probably rely on trial and error to answer the

[2] The coefficients can also be made all negative by multiplying the polynomial by -1. This operation does not change the root location.

stability question. We shall see later that the Routh-Hurwitz criterion can yield a closed-form expression for the range of the unknown parameter.

In this section, we make and interpret a basic Routh table. In the next section, we consider two special cases that can arise when generating this data table.

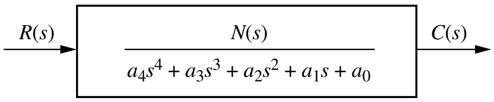

FIGURE 6.3 Equivalent closed-loop transfer function

Generating a Basic Routh Table

Look at the equivalent closed-loop transfer function shown in Figure 6.3. Since we are interested in the system poles, we focus our attention on the denominator. We first create the Routh table shown in Table 6.1. Begin by labeling the rows with powers of s from the highest power of the denominator of the closed-loop transfer function to s^0. Next start with the coefficient of the highest power of s in the denominator and list, horizontally in the first row, every other coefficient. In the second row, list horizontally, starting with the next highest power of s, every coefficient that was skipped in the first row.

The remaining entries are filled in as follows. Each entry is a negative determinant of entries in the previous two rows divided by the entry in the first column directly above the calculated row. The left-hand column of the determinant is always the first column of the previous two rows, and the right-hand column is the elements of the column above and to the right. The table is complete when all of the rows are completed down to s^0. Table 6.2 is the completed Routh table. Let us look at an example.

TABLE 6.1 Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Example 6.1

Creating a Routh Table

PROBLEM: Make the Routh table for the system shown in Figure 6.4(a).

SOLUTION: The first step is to find the equivalent closed-loop system because we want to test the denominator of this function, not the given forward transfer

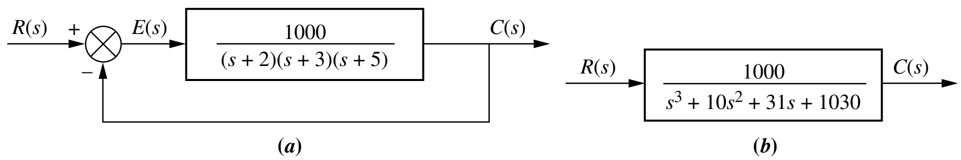

FIGURE 6.4 **a.** Feedback system for Example 6.1; **b.** equivalent closed-loop system

TABLE 6.3 Completed Routh table for Example 6.1

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$

function, for pole location. Using the feedback formula, we obtain the equivalent system of Figure 6.4(b). The Routh-Hurwitz criterion will be applied to this denominator. First label the rows with powers of s from s^3 down to s^0 in a vertical column, as shown in Table 6.3. Next form the first row of the table, using the coefficients of the denominator of the closed-loop transfer function. Start with the coefficient of the highest power and skip every other power of s. Now form the second row with the coefficients of the denominator skipped in the previous step. Subsequent rows are formed with determinants, as shown in Table 6.2.

For convenience, any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below. This can be proved by examining the expressions for the entries and verifying that any multiplicative constant from a previous row cancels out. In the second row of Table 6.3, for example, the row was multiplied by 1/10. We see later that care must be taken not to multiply the row by a negative constant.

Interpreting the Basic Routh Table

Now that we know how to generate the Routh table, let us see how to interpret it. The basic Routh table applies to systems with poles in the left and right half-planes. Systems with imaginary poles and the kind of Routh table that results will be discussed in the next section. Simply stated, the Routh-Hurwitz criterion declares that *the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column*.

If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table. For example, Table 6.3 has two sign changes in the first column. The first sign change occurs from 1 in the s^2 row to -72 in the s^1 row. The second occurs from -72 in the s^1 row to 103 in the s^0 row. Thus, the system of Figure 6.4 is unstable since two poles exist in the right half-plane.

Skill-Assessment Exercise 6.1

PROBLEM: Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane.

$$P(s) = 3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$$

ANSWER: Four in the right half-plane (rhp), three in the left half-plane (lhp).

The complete solution is at www.wiley.com/college/nise.

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Control Solutions

Now that we have described how to generate and interpret a basic Routh table, let us look at two special cases that can arise.

6.3 Routh-Hurwitz Criterion: Special Cases

Two special cases can occur: (1) The Routh table sometimes will have a zero *only in the first column* of a row, or (2) the Routh table sometimes will have an *entire row* that consists of zeros. Let us examine the first case.

Zero Only in the First Column

If the first element of a row is zero, division by zero would be required to form the next row. To avoid this phenomenon, an epsilon, ϵ, is assigned to replace the zero in the first column. The value ϵ is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined. Let us look at an example.

Example 6.2

Stability via Epsilon Method

TryIt 6.1

Use the following MATLAB statement to find the poles of the closed-loop transfer function in Eq. (6.2).

```
roots([1 2 3 6 5 3])
```

PROBLEM: Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \tag{6.2}$$

SOLUTION: The solution is shown in Table 6.4. We form the Routh table by using the denominator of Eq. (6.2). Begin by assembling the Routh table down to the row where a zero appears *only* in the first column (the s^3 row). Next replace the zero by a small number, ϵ, and complete the table. To begin the interpretation, we must first assume a sign, positive or negative, for the quantity ϵ. Table 6.5 shows the first column of Table 6.4 along with the resulting signs for choices of ϵ positive and ϵ negative.

TABLE 6.4 Completed Routh table for Example 6.2

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon-7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	0	0
s^0	3	0	0

TABLE 6.5 Determining signs in first column of a Routh table with zero as first element in a row

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon-7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	+	+
s^0	3	+	+

If ϵ is chosen positive, Table 6.5 will show a sign change from the s^3 row to the s^2 row, and there will be another sign change from the s^2 row to the s^1 row. Hence, the system is unstable and has two poles in the right half-plane.

Alternatively, we could choose ϵ negative. Table 6.5 would then show a sign change from the s^4 row to the s^3 row. Another sign change would occur from the s^3 row to the s^2 row. Our result would be exactly the same as that for a positive choice for ϵ. Thus, the system is unstable, with two poles in the right half-plane.

Symbolic Math

SM

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch6sp1 in Appendix F at www.wiley.com/college/nise. You will learn how to use the Symbolic Math Toolbox to calculate the values of cells in a Routh table even if the table contains symbolic objects, such as ϵ. You will see that the Symbolic Math Toolbox and MATLAB yield an alternate way to generate the Routh table for Example 6.2.

Another method that can be used when a zero appears only in the first column of a row is derived from the fact that a polynomial that has the reciprocal roots of the original polynomial has its roots distributed the same—right half-plane, left half-plane, or imaginary axis—because taking the reciprocal of the root value does not move it to another region. Thus, if we can find the polynomial that has the reciprocal roots of the original, it is possible that the Routh table for the new polynomial will not have a zero in the first column. This method is usually computationally easier than the epsilon method just described.

We now show that the polynomial we are looking for, the one with the reciprocal roots, is simply the original polynomial with its coefficients written in reverse order (*Phillips, 1991*). Assume the equation

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{6.3}$$

If s is replaced by $1/d$, then d will have roots which are the reciprocal of s. Making this substitution in Eq. (6.3),

$$\left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \cdots + a_1\left(\frac{1}{d}\right) + a_0 = 0 \tag{6.4}$$

Factoring out $(1/d)^n$,

$$\left(\frac{1}{d}\right)^n \left[1 + a_{n-1}\left(\frac{1}{d}\right)^{-1} + \cdots + a_1\left(\frac{1}{d}\right)^{(1-n)} + a_0\left(\frac{1}{d}\right)^{-n}\right]$$

$$= \left(\frac{1}{d}\right)^n [1 + a_{n-1}d + \cdots + a_1 d^{(n-1)} + a_0 d^n] = 0 \tag{6.5}$$

Thus, the polynomial with reciprocal roots is a polynomial with the coefficients written in reverse order. Let us redo the previous example to show the computational advantage of this method.

Example 6.3

Stability via Reverse Coefficients

PROBLEM: Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \tag{6.6}$$

SOLUTION: First write a polynomial that has the reciprocal roots of the denominator of Eq. (6.6). From our discussion, this polynomial is formed by writing the denominator of Eq. (6.6) in reverse order. Hence,

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1 \tag{6.7}$$

We form the Routh table as shown in Table 6.6 using Eq. (6.7). Since there are two sign changes, the system is unstable and has two right-half-plane poles. This is the same as the result obtained in Example 6.2. Notice that Table 6.6 does not have a zero in the first column.

TABLE 6.6 Routh table for Example 6.3

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

Entire Row is Zero

We now look at the second special case. Sometimes while making a Routh table, we find that an entire row consists of zeros because there is an even polynomial that is a factor of the original polynomial. This case must be handled differently from the case of a zero in only the first column of a row. Let us look at an example that demonstrates how to construct and interpret the Routh table when an entire row of zeros is present.

Example 6.4

Stability via Routh Table with Row of Zeros

PROBLEM: Determine the number of right-half-plane poles in the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56} \tag{6.8}$$

SOLUTION: Start by forming the Routh table for the denominator of Eq. (6.8) (see Table 6.7). At the second row we multiply through by 1/7 for convenience. We stop at the third row, since the entire row consists of zeros, and use the following

TABLE 6.7 Routh table for Example 6.4

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

procedure. First we return to the row immediately above the row of zeros and form an auxiliary polynomial, using the entries in that row as coefficients. The polynomial will start with the power of s in the label column and continue by skipping every other power of s. Thus, the polynomial formed for this example is

$$P(s) = s^4 + 6s^2 + 8 \tag{6.9}$$

Next we differentiate the polynomial with respect to s and obtain

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0 \tag{6.10}$$

Finally, we use the coefficients of Eq. (6.10) to replace the row of zeros. Again, for convenience, the third row is multiplied by 1/4 after replacing the zeros.

The remainder of the table is formed in a straightforward manner by following the standard form shown in Table 6.2. Table 6.7 shows that all entries in the first column are positive. Hence, there are no right–half-plane poles.

Let us look further into the case that yields an entire row of zeros. An entire row of zeros will appear in the Routh table when a purely even or purely odd polynomial is a factor of the original polynomial. For example, $s^4 + 5s^2 + 7$ is an even polynomial; it has only even powers of s. Even polynomials only have roots that are symmetrical about the origin.[3] This symmetry can occur under three conditions of root position: (1) The roots are symmetrical and real, (2) the roots are symmetrical and imaginary, or (3) the roots are quadrantal. Figure 6.5 shows examples of these cases. Each case or combination of these cases will generate an even polynomial.

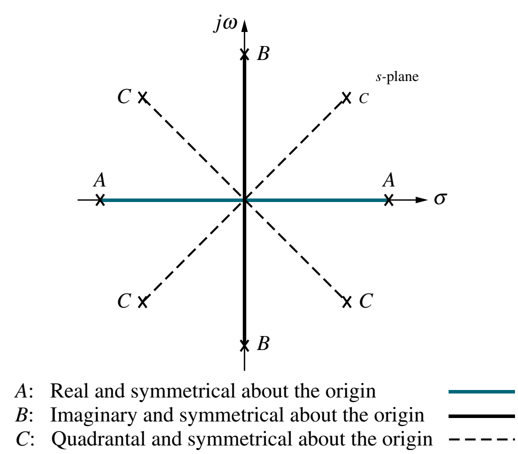


FIGURE 6.5 Root positions to generate even polynomials: A, B, C, or any combination

It is this even polynomial that causes the row of zeros to appear. Thus, the row of zeros tells us of the existence of an even polynomial whose roots are symmetric about the origin. Some of these roots could be on the $j\omega$-axis. On the other hand, since $j\omega$ roots are symmetric about the origin, if we do not have a row of zeros, we cannot possibly have $j\omega$ roots.

Another characteristic of the Routh table for the case in question is that the row previous to the row of zeros contains the even polynomial that is a factor of the original polynomial. Finally, everything from the row containing the even polynomial down to the end of the Routh table is a test of only the even polynomial. Let us put these facts together in an example.

[3] The polynomial $s^5 + 5s^3 + 7s$ is an example of an odd polynomial; it has only odd powers of s. Odd polynomials are the product of an even polynomial and an odd power of s. Thus, the constant term of an odd polynomial is always missing.

Example 6.5

Pole Distribution via Routh Table with Row of Zeros

PROBLEM: For the transfer function

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20} \quad (6.11)$$

tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis.

SOLUTION: Use the denominator of Eq. (6.11) and form the Routh table in Table 6.8. For convenience the s^6 row is multiplied by 1/10, and the s^5 row is multiplied by 1/20. At the s^3 row we obtain a row of zeros. Moving back one row to s^4, we extract the even polynomial, $P(s)$, as

$$P(s) = s^4 + 3s^2 + 2 \quad (6.12)$$

TABLE 6.8 Routh table for Example 6.5

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	~~−10~~ −1	~~−20~~ −2	~~10~~ 1	~~20~~ 2	0
s^5	~~20~~ 1	~~60~~ 3	~~40~~ 2	0	0
s^4	1	3	2	0	0
s^3	~~0~~ ~~4~~ 2	~~0~~ ~~6~~ 3	~~0~~ ~~0~~ 0	0	0
s^2	~~$\frac{3}{2}$~~ 3	~~2~~ 4	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

This polynomial will divide evenly into the denominator of Eq. (6.11) and thus is a factor. Taking the derivative with respect to s to obtain the coefficients that replace the row of zeros in the s^3 row, we find

$$\frac{dP(s)}{ds} = 4s^3 + 6s + 0 \quad (6.13)$$

Replace the row of zeros with 4, 6, and 0 and multiply the row by 1/2 for convenience. Finally, continue the table to the s^0 row, using the standard procedure.

How do we now interpret this Routh table? Since all entries from the even polynomial at the s^4 row down to the s^0 row are a test of the even polynomial, we begin to draw some conclusions about the roots of the even polynomial. No sign changes exist from the s^4 row down to the s^0 row. Thus, the even polynomial does not have right–half-plane poles. Since there are no right–half-plane poles, no left–half-plane poles are present because of the requirement for symmetry. Hence, the even polynomial, Eq. (6.12), must have all four of its poles on the $j\omega$-axis.[4] These results are summarized in the first column of Table 6.9.

[4] A necessary condition for stability is that the $j\omega$ roots have unit multiplicity. The even polynomial must be checked for multiple $j\omega$ roots. For this case, the existence of multiple $j\omega$ roots would lead to a perfect, fourth-order square polynomial. Since Eq. (6.12) is not a perfect square, the four $j\omega$ roots are distinct.

TABLE 6.9 Summary of pole locations for Example 6.5

	Polynomial		
Location	**Even (fourth-order)**	**Other (fourth-order)**	**Total (eighth-order)**
Right half-plane	0	2	2
Left half-plane	0	2	2
$j\omega$	4	0	4

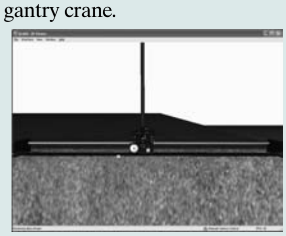
The remaining roots of the total polynomial are evaluated from the s^8 row down to the s^4 row. We notice two sign changes: one from the s^7 row to the s^6 row and the other from the s^6 row to the s^5 row. Thus, the other polynomial must have two roots in the right half-plane. These results are included in Table 6.9 under "Other". The final tally is the sum of roots from each component, the even polynomial and the other polynomial, as shown under "Total" in Table 6.9. Thus, the system has two poles in the right half-plane, two poles in the left half-plane, and four poles on the $j\omega$-axis; it is unstable because of the right–half-plane poles.

We now summarize what we have learned about polynomials that generate entire rows of zeros in the Routh table. These polynomials have a purely even factor with roots that are symmetrical about the origin. The even polynomial appears in the Routh table in the row directly above the row of zeros. Every entry in the table from the even polynomial's row to the end of the chart applies only to the even polynomial. Therefore, the number of sign changes from the even polynomial to the end of the table equals the number of right-half-plane roots of the even polynomial. Because of the symmetry of roots about the origin, the even polynomial must have the same number of left–half-plane roots as it does right–half-plane roots. Having accounted for the roots in the right and left half-planes, we know the remaining roots must be on the $j\omega$-axis.

Every row in the Routh table from the beginning of the chart to the row containing the even polynomial applies only to the other factor of the original polynomial. For this factor, the number of sign changes, from the beginning of the table down to the even polynomial, equals the number of right–half-plane roots. The remaining roots are left–half-plane roots. There can be no $j\omega$ roots contained in the other polynomial.

Virtual Experiment 6.1
Stability

Put theory into practice and evaluate the stability of the Quanser Linear Inverted Pendulum in LabVIEW. When in the upward balanced position, this system addresses the challenge of stabilizing a rocket during take-off. In the downward position it emulates the construction gantry crane.



Virtual experiments are found on WileyPLUS.

Skill-Assessment Exercise 6.2

PROBLEM: Use the Routh-Hurwitz criterion to find how many poles of the following closed-loop system, $T(s)$, are in the rhp, in the lhp, and on the $j\omega$-axis:

$$T(s) = \frac{s^3 + 7s^2 - 21s + 10}{s^6 + s^5 - 6s^4 + 0s^3 - s^2 - s + 6}$$

ANSWER: Two rhp, two lhp, and two $j\omega$

The complete solution is at www.wiley.com/college/nise.

Let us demonstrate the usefulness of the Routh-Hurwitz criterion with a few additional examples.