A 10% overshoot implies that $\zeta = 0.591$. Substituting this value for the damping ratio into Eq. (5.23) and solving for K yields

$$K = 17.9 \tag{5.24}$$

Although we are able to design for percent overshoot in this problem, we could not have selected settling time as a design criterion because, regardless of the value of K, the real parts, -2.5, of the poles of Eq. (5.20) remain the same.

Skill-Assessment Exercise 5.2

WileyPLUS WPCS Control Solutions

PROBLEM: For a unity feedback control system with a forward-path transfer function $G(s) = \dfrac{16}{s(s+a)}$, design the value of a to yield a closed-loop step response that has 5% overshoot.

ANSWER:

$$a = 5.52$$

The complete solution is at www.wiley.com/college/nise.

TryIt 5.2

Use the following MATLAB and Control System Toolbox statements to find ζ, ω_n, $\%OS$, T_s, T_p, and T_r for the closed-loop unity feedback system described in Skill-Assessment Exercise 5.2. Start with $a = 2$ and try some other values. A step response for the closed-loop system will also be produced.

```
a=2;
numg=16;
deng=poly([0 -a]);
G=tf(numg,deng);
T=feedback(G,1);
[numt,dent]=...
 tfdata(T,'v');
wn=sqrt(dent(3))
z=dent(2)/(2*wn)
Ts=4/(z*wn)
Tp=pi/(wn*...
 sqrt(1-z^2))
pos=exp(-z*pi...
 /sqrt(1-z^2))*100
Tr=(1.76*z^3 - ...
 0.417*z^2 + 1.039*...
 z + 1)/wn
step(T)
```

5.4 Signal-Flow Graphs

Signal-flow graphs are an alternative to block diagrams. Unlike block diagrams, which consist of blocks, signals, summing junctions, and pickoff points, a signal-flow graph consists only of *branches*, which represent systems, and *nodes*, which represent signals. These elements are shown in Figure 5.17(*a*) and (*b*), respectively. A system is represented by a line with an arrow showing the direction of signal flow through the

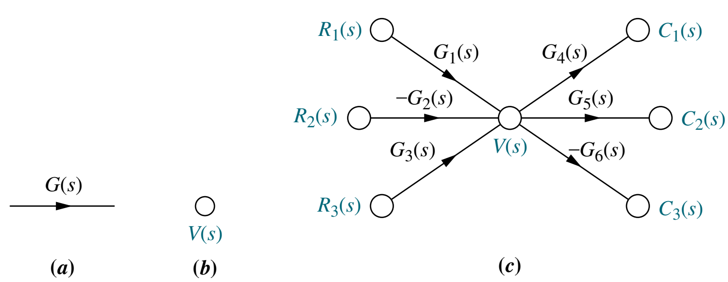

FIGURE 5.17 Signal-flow graph components: **a.** system; **b.** signal; **c.** interconnection of systems and signals

system. Adjacent to the line we write the transfer function. A signal is a node with the signal's name written adjacent to the node.

Figure 5.17(*c*) shows the interconnection of the systems and the signals. Each signal is the sum of signals flowing into it. For example, the signal $V(s) = R_1(s)G_1(s) - R_2(s)G_2(s) + R_3(s)G_3(s)$. The signal $C_2(s) = V(s)G_5(s) = R_1(s)G_1(s)G_5(s) - R_2(s)G_2(s)G_5(s) + R_3(s)G_3(s)G_5(s)$. The signal $C_3(s) = -V(s)G_6(s) = -R_1(s)G_1(s)G_6(s) + R_2(s)G_2(s)G_6(s) - R_3(s)G_3(s)G_6(s)$. Notice that in summing negative signals we associate the negative sign with the system and not with a summing junction, as in the case of block diagrams.

To show the parallel between block diagrams and signal-flow graphs, we will take some of the block diagram forms from Section 5.2 and convert them to signal-flow graphs in Example 5.5. In each case, we will first convert the signals to nodes and then interconnect the nodes with system branches. In Example 5.6, we will convert an intricate block diagram to a signal-flow graph.

Example 5.5

Converting Common Block Diagrams to Signal-Flow Graphs

PROBLEM: Convert the cascaded, parallel, and feedback forms of the block diagrams shown in Figures 5.3(*a*), 5.5(*a*), and 5.6(*b*), respectively, into signal-flow graphs.

SOLUTION: In each case, we start by drawing the signal nodes for that system. Next we interconnect the signal nodes with system branches. The signal nodes for the cascaded, parallel, and feedback forms are shown in Figure 5.18(*a*), (*c*), and (*e*), respectively. The interconnection of the nodes with branches that represent the subsystems is shown in Figure 5.18(*b*), (*d*), and (*f*) for the cascaded, parallel, and feedback forms, respectively.

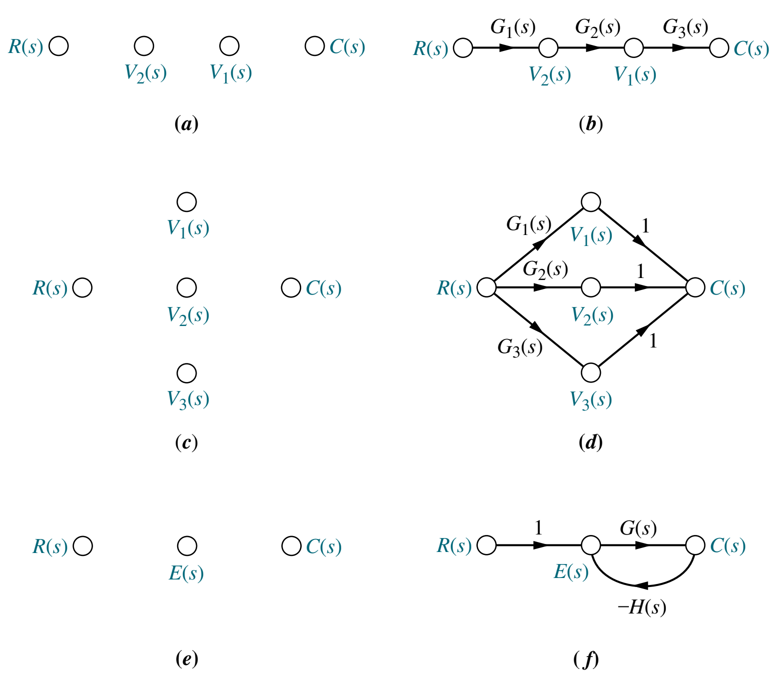

FIGURE 5.18 Building signal-flow graphs: **a.** cascaded system nodes (from Figure 5.3(*a*)); **b.** cascaded system signal-flow graph; **c.** parallel system nodes (from Figure 5.5(*a*)); **d.** parallel system signal-flow graph; **e.** feedback system nodes (from Figure 5.6(*b*)); **f.** feedback system signal-flow graph

Example 5.6

Converting a Block Diagram to a Signal-Flow Graph

PROBLEM: Convert the block diagram of Figure 5.11 to a signal-flow graph.

SOLUTION: Begin by drawing the signal nodes, as shown in Figure 5.19(*a*). Next, interconnect the nodes, showing the direction of signal flow and identifying each transfer function. The result is shown in Figure 5.19(*b*). Notice that the negative signs at the summing junctions of the block diagram are represented by the negative transfer functions of the signal-flow graph. Finally, if desired, simplify the signal-flow graph to the one shown in Figure 5.19(*c*) by eliminating signals that have a single flow in and a single flow out, such as $V_2(s)$, $V_6(s)$, $V_7(s)$, and $V_8(s)$.

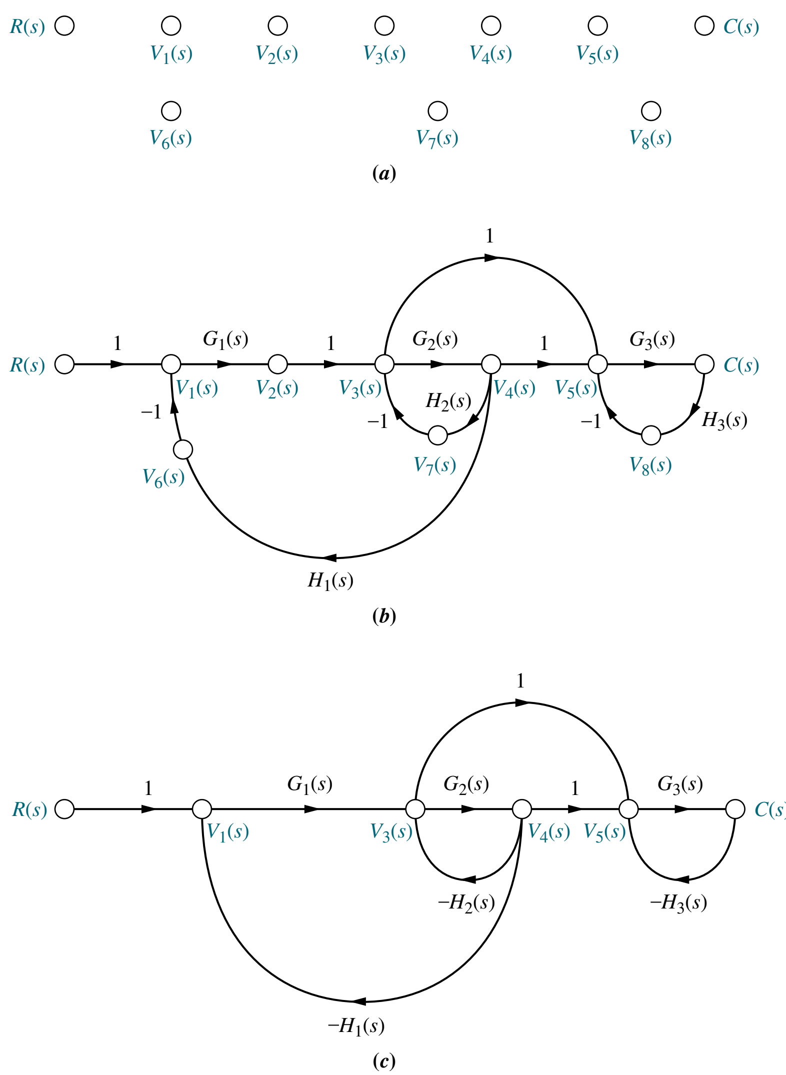

FIGURE 5.19 Signal-flow graph development: **a.** signal nodes; **b.** signal-flow graph; **c.** simplified signal-flow graph

Skill-Assessment Exercise 5.3

PROBLEM: Convert the block diagram of Figure 5.13 to a signal-flow graph.

ANSWER: The complete solution is at www.wiley.com/college/nise.

5.5 Mason's Rule

Earlier in this chapter, we discussed how to reduce block diagrams to single transfer functions. Now we are ready to discuss a technique for reducing signal-flow graphs to single transfer functions that relate the output of a system to its input.

The block diagram reduction technique we studied in Section 5.2 requires successive application of fundamental relationships in order to arrive at the system transfer function. On the other hand, Mason's rule for reducing a signal-flow graph to a single transfer function requires the application of one formula. The formula was derived by S. J. Mason when he related the signal-flow graph to the simultaneous equations that can be written from the graph (*Mason*, *1953*).

In general, it can be complicated to implement the formula without making mistakes. Specifically, the existence of what we will later call nontouching loops increases the complexity of the formula. However, many systems do not have non-touching loops. For these systems, you may find Mason's rule easier to use than block diagram reduction.

Mason's formula has several components that must be evaluated. First, we must be sure that the definitions of the components are well understood. Then we must exert care in evaluating the components. To that end, we discuss some basic definitions applicable to signal-flow graphs; then we state Mason's rule and do an example.

Definitions

Loop gain. The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once. For examples of loop gains, see Figure 5.20. There are four loop gains:

1. $G_2(s)H_1(s)$ (5.25a)
2. $G_4(s)H_2(s)$ (5.25b)
3. $G_4(s)G_5(s)H_3(s)$ (5.25c)
4. $G_4(s)G_6(s)H_3(s)$ (5.25d)

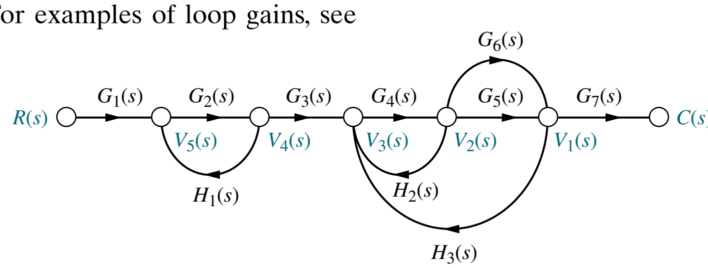

FIGURE 5.20 Signal-flow graph for demonstrating Mason's rule

Forward-path gain. The product of gains found by traversing a path from the input node to the output node of the signal-flow graph in the direction of signal flow. Examples of forward-path gains are also shown in Figure 5.20. There are two forward-path gains:

1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$ (5.26a)
2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$ (5.26b)

Nontouching loops. Loops that do not have any nodes in common. In Figure 5.20, loop $G_2(s)H_1(s)$ does not touch loops $G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, and $G_4(s)G_6(s)H_3(s)$.

Nontouching-loop gain. The product of loop gains from nontouching loops taken two, three, four, or more at a time. In Figure 5.20 the product of loop gain $G_2(s)H_1(s)$ and loop gain $G_4(s)H_2(s)$ is a nontouching-loop gain taken two at a time. In summary, all three of the nontouching-loop gains taken two at a time are

1. $[G_2(s)H_1(s)][G_4(s)H_2(s)]$ (5.27a)

2. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$ (5.27b)

3. $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$ (5.27c)

The product of loop gains $[G_4(s)G_5(s)H_3(s)][G_4(s)G_6(s)H_3(s)]$ is not a nontouching-loop gain since these two loops have nodes in common. In our example there are no nontouching-loop gains taken three at a time since three nontouching loops do not exist in the example.

We are now ready to state Mason's rule.

Mason's Rule

The transfer function, $C(s)/R(s)$, of a system represented by a signal-flow graph is

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta} \tag{5.28}$$

where

k = number of forward paths
T_k = the kth forward-path gain
$\Delta = 1 - \Sigma$ loop gains $+ \Sigma$ nontouching-loop gains taken two at a time $- \Sigma$ nontouching-loop gains taken three at a time $+ \Sigma$ nontouching-loop gains taken four at a time $- \ldots$
$\Delta_k = \Delta - \Sigma$ loop gain terms in Δ that touch the kth forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the kth forward path.

Notice the alternating signs for the components of Δ. The following example will help clarify Mason's rule.

Example 5.7

Transfer Function via Mason's Rule

PROBLEM: Find the transfer function, $C(s)/R(s)$, for the signal-flow graph in Figure 5.21.

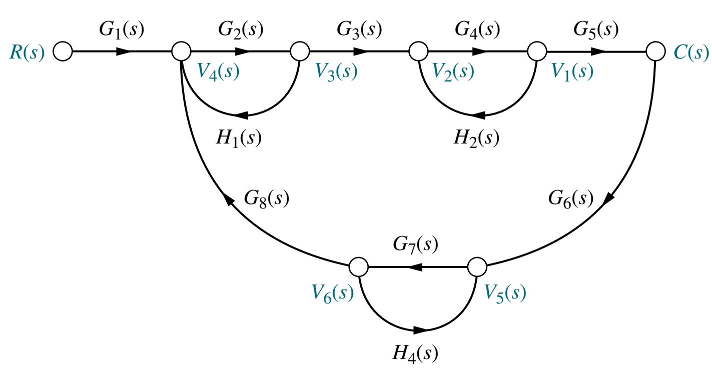

FIGURE 5.21 Signal-flow graph for Example 5.7

SOLUTION: First, identify the *forward-path gains*. In this example there is only one:

$$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s) \tag{5.29}$$

Second, identify the *loop gains*. There are four, as follows:

1. $G_2(s)H_1(s)$ (5.30a)

2. $G_4(s)H_2(s)$ (5.30b)

3. $G_7(s)H_4(s)$ (5.30c)

4. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$ (5.30d)

Third, identify the *nontouching loops taken two at a time*. From Eqs. (5.30) and Figure 5.21, we can see that loop 1 does not touch loop 2, loop 1 does not touch loop 3, and loop 2 does not touch loop 3. Notice that loops 1, 2, and 3 all touch loop 4. Thus, the combinations of nontouching loops taken two at a time are as follows:

$$\text{Loop 1 and loop 2}: \; G_2(s)H_1(s)G_4(s)H_2(s) \tag{5.31a}$$

$$\text{Loop 1 and loop 3}: \; G_2(s)H_1(s)G_7(s)H_4(s) \tag{5.31b}$$

$$\text{Loop 2 and loop 3}: \; G_4(s)H_2(s)G_7(s)H_4(s) \tag{5.31c}$$

Finally, the *nontouching loops taken three at a time* are as follows:

$$\text{Loops 1, 2, and 3}: \; G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s) \tag{5.32}$$

Now, from Eq. (5.28) and its definitions, we form Δ and Δ_k. Hence,

$$\begin{aligned}\Delta = 1 \; &-[G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) \\ &\qquad + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] \\ &+[G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) \\ &\qquad + G_4(s)H_2(s)G_7(s)H_4(s)] \\ &-[G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]\end{aligned} \tag{5.33}$$

We form Δ_k by eliminating from Δ the loop gains that touch the kth forward path:

$$\Delta_1 = 1 - G_7(s)H_4(s) \tag{5.34}$$

Expressions (5.29), (5.33), and (5.34) are now substituted into Eq. (5.28), yielding the transfer function:

$$G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1 - G_7(s)H_4(s)]}{\Delta} \tag{5.35}$$

Since there is only one forward path, $G(s)$ consists of only one term, rather than a sum of terms, each coming from a forward path.

Skill-Assessment Exercise 5.4

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PROBLEM: Use Mason's rule to find the transfer function of the signal-flow diagram shown in Figure 5.19(*c*). Notice that this is the same system used in Example 5.2 to find the transfer function via block diagram reduction.

ANSWER:

$$T(s) = \frac{G_1(s)G_3(s)[1 + G_2(s)]}{[1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)][1 + G_3(s)H_3(s)]}$$

The complete solution is at www.wiley.com/college/nise.

5.6 Signal-Flow Graphs of State Equations

State Space
SS

In this section, we draw signal-flow graphs from state equations. At first this process will help us visualize state variables. Later we will draw signal-flow graphs and then write alternate representations of a system in state space.

Consider the following state and output equations:

$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r \tag{5.36a}$$

$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r \tag{5.36b}$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r \tag{5.36c}$$

$$y = -4x_1 + 6x_2 + 9x_3 \tag{5.36d}$$

First, identify three nodes to be the three state variables, x_1, x_2, and x_3; also identify three nodes, placed to the left of each respective state variable, to be the derivatives of the state variables, as in Figure 5.22(*a*). Also identify a node as the input, r, and another node as the output, y.

Next interconnect the state variables and their derivatives with the defining integration, $1/s$, as shown in Figure 5.22(*b*). Then using Eqs. (5.36), feed to each node the indicated signals. For example, from Eq. (5.36*a*), $\dot{x}_1$ receives $2x_1 - 5x_2 + 3x_3 + 2r$, as shown in Figure 5.22(*c*). Similarly, $\dot{x}_2$ receives $-6x_1 - 2x_2 + 2x_3 + 5r$, as shown in Figure 5.22(*d*), and $\dot{x}_3$ receives $x_1 - 3x_2 - 4x_3 + 7r$, as shown in Figure 5.22(*e*). Finally, using Eq. (5.36*d*), the output, y, receives $-4x_1 + 6x_2 + 9x_3$, as shown in Figure 5.19(*f*), the final phase-variable representation, where the state variables are the outputs of the integrators.

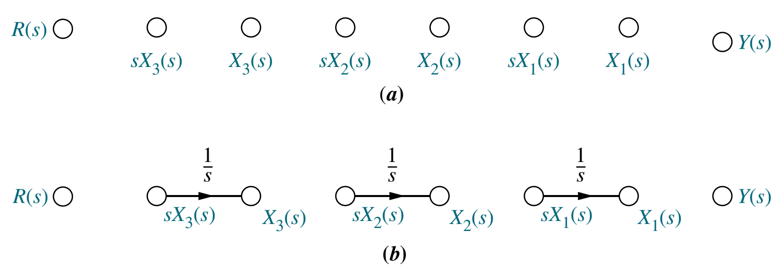

FIGURE 5.22 Stages of development of a signal-flow graph for the system of Eqs. (5.36): **a.** Place nodes; **b.** interconnect state variables and derivatives; (*figure continues*)

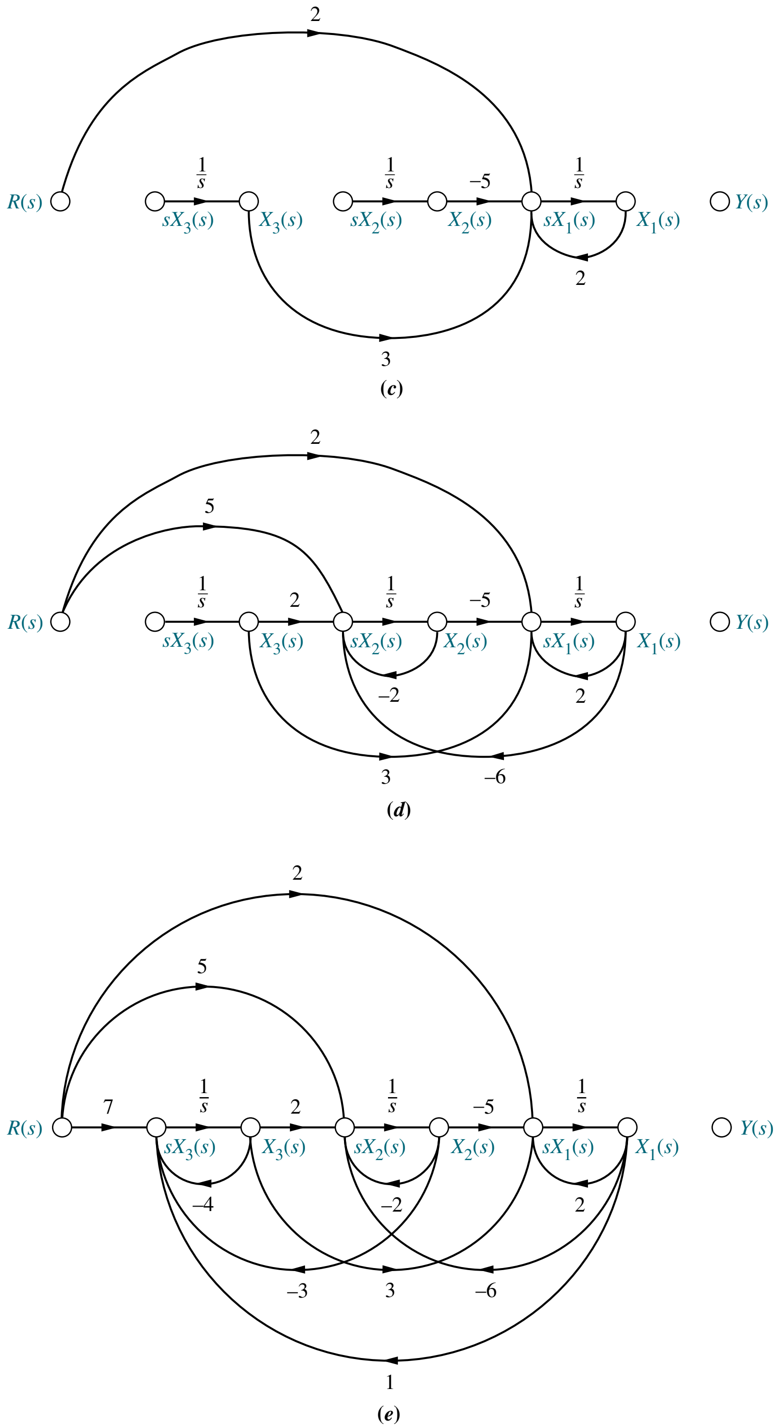

FIGURE 5.22 (*Continued*) **c.** form dx_1/dt; **d.** form dx_2/dt; **e.** form dx_3/dt; (*figure continues*)

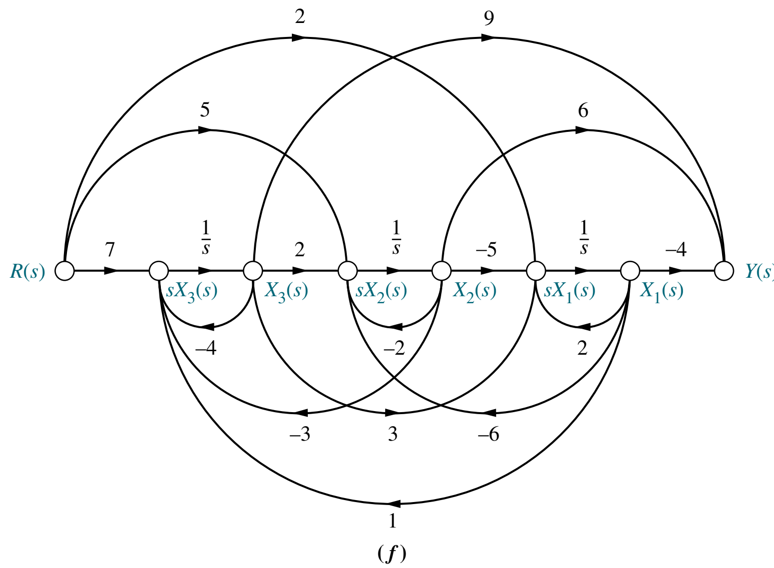

FIGURE 5.22 (*Continued*) **f.** form output (*figure end*)

Skill-Assessment Exercise 5.5

PROBLEM: Draw a signal-flow graph for the following state and output equations:

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [0 \quad 1 \quad 0]\mathbf{x}$$

ANSWER: The complete solution is at www.wiley.com/college/nise.

In the next section, the signal-flow model will help us visualize the process of determining alternative representations in state space of the same system. We will see that even though a system can be the same with respect to its input and output terminals, the state-space representations can be many and varied.

5.7 Alternative Representations in State Space

State Space
SS

In Chapter 3, systems were represented in state space in phase-variable form. However, system modeling in state space can take on many representations other than the phase-variable form. Although each of these models yields the same output for a given input, an engineer may prefer a particular one for several reasons. For example, one set of state variables, with its unique representation, can model actual physical variables of a system, such as amplifier and filter outputs.