

Example 2.23

Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(a) and (b), find the transfer function, $\theta_L(s)/E_a(s)$.

SOLUTION: Begin by finding the mechanical constants, J_m and D_m , in Eq. (2.153). From Eq. (2.155), the total inertia at the armature of the motor is

$$J_m = J_a + J_L \left(\frac{N_1}{N_2} \right)^2 = 5 + 700 \left(\frac{1}{10} \right)^2 = 12 \quad (2.164)$$

and the total damping at the armature of the motor is

$$D_m = D_a + D_L \left(\frac{N_1}{N_2} \right)^2 = 2 + 800 \left(\frac{1}{10} \right)^2 = 10 \quad (2.165)$$

Now we will find the electrical constants, K_t/R_a and K_b . From the torque-speed curve of Figure 2.39(b),

$$T_{\text{stall}} = 500 \quad (2.166)$$

$$\omega_{\text{no-load}} = 50 \quad (2.167)$$

$$e_a = 100 \quad (2.168)$$

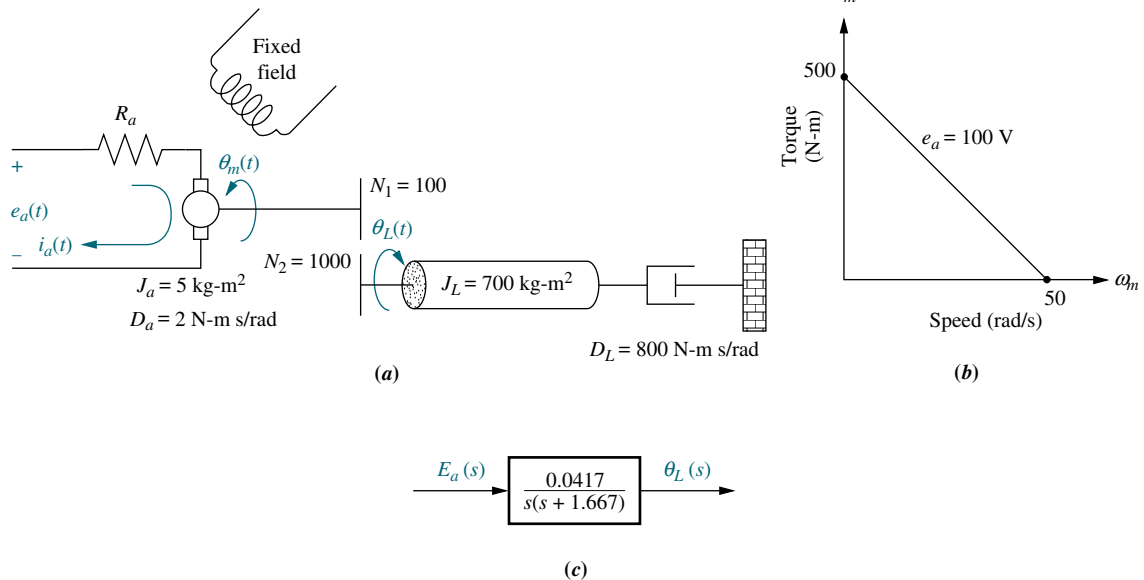


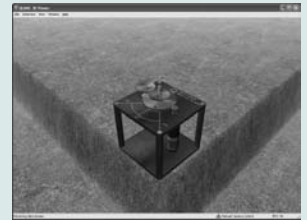
FIGURE 2.39 a. DC motor and load; b. torque-speed curve; c. block diagram

Hence the electrical constants are

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} = \frac{500}{100} = 5 \quad (2.169)$$

Virtual Experiment 2.2 Open-Loop Servo Motor

Put theory into practice exploring the dynamics of the Quanser Rotary Servo System modeled in LabVIEW. It is particularly important to know how a servo motor behaves when using them in high-precision applications such as hard disk drives.



Virtual experiments are found on WileyPLUS.

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2 \quad (2.170)$$

Substituting Eqs. (2.164), (2.165), (2.169), and (2.170) into Eq. (2.153) yield

$$\frac{\theta_m(s)}{E_a(s)} = \frac{5/12}{s \left\{ s + \frac{1}{12} [10 + (5)(2)] \right\}} = \frac{0.417}{s(s + 1.667)} \quad (2.171)$$

In order to find $\theta_L(s)/E_a(s)$, we use the gear ratio, $N_1/N_2 = 1/10$, and find

$$\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s + 1.667)} \quad (2.172)$$

as shown in Figure 2.39(c).

Skill-Assessment Exercise 2.11

WileyPLUS

WPCS

Control Solutions

PROBLEM: Find the transfer function, $G(s) = \theta_L(s)/E_a(s)$, for the motor and load shown in Figure 2.40. The torque-speed curve is given by $T_m = -8\omega_m + 200$ when the input voltage is 100 volts.

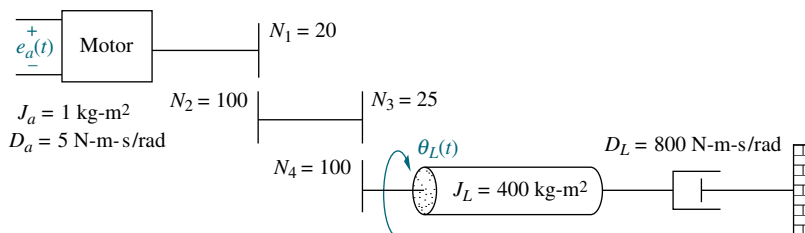


FIGURE 2.40 Electro-mechanical system for Skill-Assessment Exercise 2.11

ANSWER: $G(s) = \frac{1/20}{s[s + (15/2)]}$

The complete solution is at www.wiley.com/college/nise.

2.9 Electric Circuit Analogs

In this section, we show the commonality of systems from the various disciplines by demonstrating that the mechanical systems with which we worked can be represented by equivalent electric circuits. We have pointed out the similarity between the equations resulting from Kirchhoff's laws for electrical systems and the equations of motion of mechanical systems. We now show this commonality even more convincingly by producing electric circuit equivalents for mechanical systems. The variables of the electric circuits behave exactly as the analogous

variables of the mechanical systems. In fact, converting mechanical systems to electrical networks before writing the describing equations is a problem-solving approach that you may want to pursue.

An electric circuit that is analogous to a system from another discipline is called an electric circuit *analog*. Analogs can be obtained by comparing the describing equations, such as the equations of motion of a mechanical system, with either electrical mesh or nodal equations. When compared with mesh equations, the resulting electrical circuit is called a *series analog*. When compared with nodal equations, the resulting electrical circuit is called a *parallel analog*.

Series Analog

Consider the translational mechanical system shown in Figure 2.41(a), whose equation of motion is

$$(Ms^2 + f_v s + K)X(s) = F(s) \quad (2.173)$$

Kirchhoff's mesh equation for the simple series *RLC* network shown in Figure 2.41(b) is

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E(s) \quad (2.174)$$

As we previously pointed out, Eq. (2.173) is not directly analogous to Eq. (2.174) because displacement and current are not analogous. We can create a direct analogy by operating on Eq. (2.173) to convert displacement to velocity by dividing and multiplying the left-hand side by s , yielding

$$\frac{Ms^2 + f_v s + K}{s} sX(s) = \left(Ms + f_v + \frac{K}{s}\right)V(s) = F(s) \quad (2.175)$$

Comparing Eqs. 2.174 and 2.175, we recognize the sum of impedances and draw the circuit shown in Figure 2.41(c). The conversions are summarized in Figure 2.41(d).

When we have more than one degree of freedom, the impedances associated with a motion appear as series electrical elements in a mesh, but

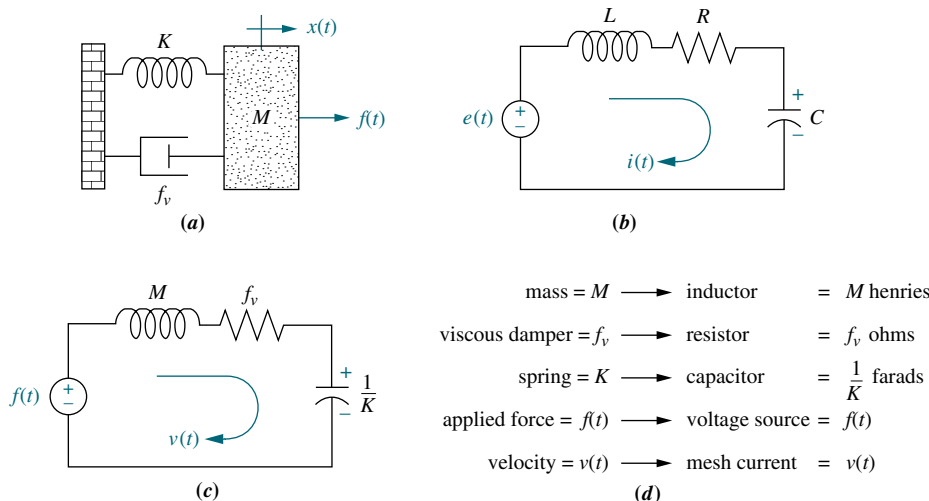


FIGURE 2.41 Development of series analog: **a.** mechanical system; **b.** desired electrical representation; **c.** series analog; **d.** parameters for series analog

the impedances between adjacent motions are drawn as series electrical impedances between the two corresponding meshes. We demonstrate with an example.

Example 2.24

Converting a Mechanical System to a Series Analog

PROBLEM: Draw a series analog for the mechanical system of Figure 2.17(a).

SOLUTION: Equations (2.118) are analogous to electrical mesh equations after conversion to velocity. Thus,

$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s} \right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s} \right) V_2(s) = F(s) \quad (2.176a)$$

$$- \left(f_{v_3} + \frac{K_2}{s} \right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s} \right] V_2(s) = 0 \quad (2.176b)$$

Coefficients represent sums of electrical impedance. Mechanical impedances associated with M_1 form the first mesh, where impedances between the two masses are common to the two loops. Impedances associated with M_2 form the second mesh. The result is shown in Figure 2.42, where $v_1(t)$ and $v_2(t)$ are the velocities of M_1 and M_2 , respectively.

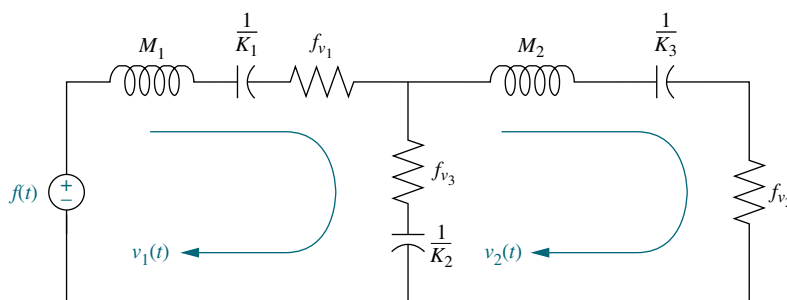


FIGURE 2.42 Series analog of mechanical system of Figure 2.17(a)

Parallel Analog

A system can also be converted to an equivalent parallel analog. Consider the translational mechanical system shown in Figure 2.43(a), whose equation of motion is given by Eq. (2.175). Kirchhoff's nodal equation for the simple parallel RLC network shown in Figure 2.43(b) is

$$\left(Cs + \frac{1}{R} + \frac{1}{Ls} \right) E(s) = I(s) \quad (2.177)$$

Comparing Eqs. (2.175) and (2.177), we identify the sum of admittances and draw the circuit shown in Figure 2.43(c). The conversions are summarized in Figure 2.43(d).

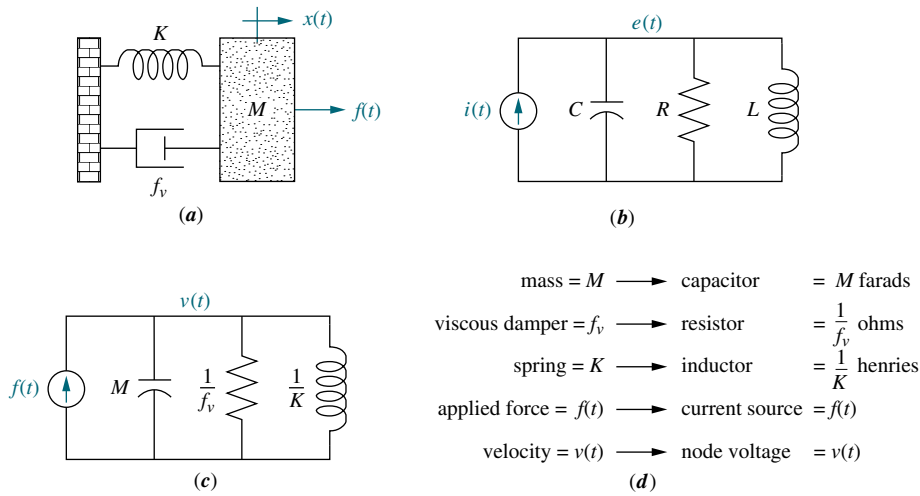


FIGURE 2.43 Development of parallel analog: **a.** mechanical system; **b.** desired electrical representation; **c.** parallel analog; **d.** parameters for parallel analog

When we have more than one degree of freedom, the components associated with a motion appear as parallel electrical elements connected to a node, but the components of adjacent motions are drawn as parallel electrical elements between two corresponding nodes. We demonstrate with an example.

Example 2.25

Converting a Mechanical System to a Parallel Analog

PROBLEM: Draw a parallel analog for the mechanical system of Figure 2.17(a).

SOLUTION: Equation (2.176) is also analogous to electrical node equations. Coefficients represent sums of electrical admittances. Admittances associated with M_1 form the elements connected to the first node, where mechanical admittances between the two masses are common to the two nodes. Mechanical admittances associated with M_2 form the elements connected to the second node. The result is shown in Figure 2.44, where $v_1(t)$ and $v_2(t)$ are the velocities of M_1 and M_2 , respectively.

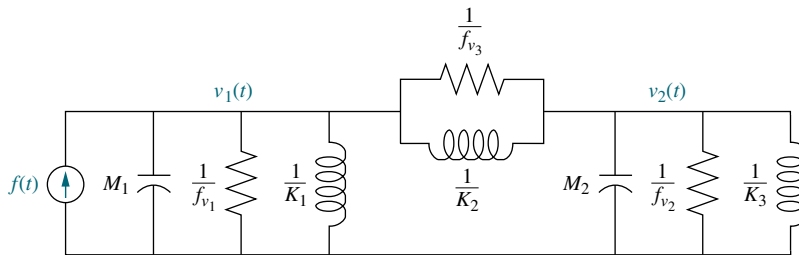


FIGURE 2.44 Parallel analog of mechanical system of Figure 2.17(a)

Skill-Assessment Exercise 2.12

PROBLEM: Draw a series and parallel analog for the rotational mechanical system of Figure 2.22.

ANSWER: The complete solution is at www.wiley.com/college/nise.

- (2) describe quantitatively the transient response of the open-loop system;
 - (3) derive the expression for the open-loop angular velocity output for a step voltage input; (4) obtain the open-loop state-space representation; (5) plot the open-loop velocity step response using a computer simulation.
- Given the block diagram for the Unmanned Free-Swimming Submersible (UFSS) vehicle's pitch control system shown on the back endpapers, you will be able to predict, find, and plot the response of the vehicle dynamics to a step input command. Further, you will be able to evaluate the effect of system zeros and higher-order poles on the response. You also will be able to evaluate the roll response of a ship at sea.

4.1 Introduction

In Chapter 2, we saw how transfer functions can represent linear, time-invariant systems. In Chapter 3, systems were represented directly in the time domain via the state and output equations. After the engineer obtains a mathematical representation of a subsystem, the subsystem is analyzed for its transient and steady-state responses to see if these characteristics yield the desired behavior. This chapter is devoted to the analysis of system transient response.

It may appear more logical to continue with Chapter 5, which covers the modeling of closed-loop systems, rather than to break the modeling sequence with the analysis presented here in Chapter 4. However, the student should not continue too far into system representation without knowing the application for the effort expended. Thus, this chapter demonstrates applications of the system representation by evaluating the transient response from the system model. Logically, this approach is not far from reality, since the engineer may indeed want to evaluate the response of a subsystem prior to inserting it into the closed-loop system.

After describing a valuable analysis and design tool, poles and zeros, we begin analyzing our models to find the step response of first- and second-order systems. The order refers to the order of the equivalent differential equation representing the system—the order of the denominator of the transfer function after cancellation of common factors in the numerator or the number of simultaneous first-order equations required for the state-space representation.

4.2 Poles, Zeros, and System Response

The output response of a system is the sum of two responses: the *forced response* and the *natural response*.¹ Although many techniques, such as solving a differential equation or taking the inverse Laplace transform, enable us to evaluate this output response, these techniques are laborious and time-consuming. Productivity is aided by analysis and design techniques that yield results in a minimum of time. If the technique is so rapid that we feel we derive the desired result by inspection, we sometimes use the attribute *qualitative* to describe the method. The use of poles and

¹The forced response is also called the *steady-state response* or *particular solution*. The natural response is also called the *homogeneous solution*.

zeros and their relationship to the time response of a system is such a technique. Learning this relationship gives us a qualitative “handle” on problems. The concept of poles and zeros, fundamental to the analysis and design of control systems, simplifies the evaluation of a system’s response. The reader is encouraged to master the concepts of poles and zeros and their application to problems throughout this book. Let us begin with two definitions.

Poles of a Transfer Function

The *poles* of a transfer function are (1) the values of the Laplace transform variable, s , that cause the transfer function to become infinite or (2) **any roots of the denominator of the transfer function that are common to roots of the numerator.**

Strictly speaking, the poles of a transfer function satisfy part (1) of the definition. For example, the roots of the characteristic polynomial in the denominator are values of s that make the transfer function infinite, so they are thus poles. However, if a factor of the denominator can be canceled by the same factor in the numerator, the root of this factor no longer causes the transfer function to become infinite. In control systems, we often refer to the root of the canceled factor in the denominator as a pole even though the transfer function will not be infinite at this value. Hence, we include part (2) of the definition.

Zeros of a Transfer Function

The *zeros* of a transfer function are (1) the values of the Laplace transform variable, s , that cause the transfer function to become zero, or (2) **any roots of the numerator of the transfer function that are common to roots of the denominator.**

Strictly speaking, the zeros of a transfer function satisfy part (1) of this definition. For example, the roots of the numerator are values of s that make the transfer function zero and are thus zeros. However, if a factor of the numerator can be canceled by the same factor in the denominator, the root of this factor no longer causes the transfer function to become zero. In control systems, we often refer to the root of the canceled factor in the numerator as a zero even though the transfer function will not be zero at this value. Hence, we include part (2) of the definition.

Poles and Zeros of a First-Order System: An Example

Given the transfer function $G(s)$ in Figure 4.1(a), a pole exists at $s = -5$, and a zero exists at -2 . These values are plotted on the complex s -plane in Figure 4.1(b), using an \times for the pole and a \circ for the zero. To show the properties of the poles and zeros, let us find the unit step response of the system. Multiplying the transfer function of Figure 4.1(a) by a step function yields

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5} \quad (4.1)$$

where

$$A = \left. \frac{(s+2)}{(s+5)} \right|_{s \rightarrow 0} = \frac{2}{5}$$

$$B = \left. \frac{(s+2)}{s} \right|_{s \rightarrow -5} = \frac{3}{5}$$

Thus,

$$c(t) = \frac{2}{5} + \frac{3}{5} e^{-5t} \quad (4.2)$$

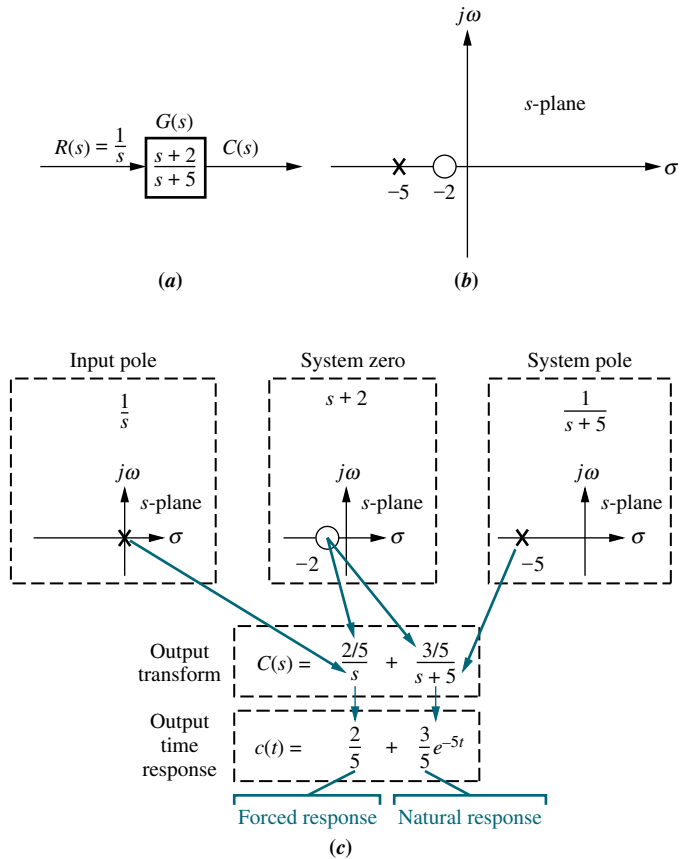


FIGURE 4.1 **a.** System showing input and output; **b.** pole-zero plot of the system; **c.** evolution of a system response. Follow blue arrows to see the evolution of the response component generated by the pole or zero.

From the development summarized in Figure 4.1(c), we draw the following conclusions:

1. A pole of the input function generates the form of the *forced response* (that is, the pole at the origin generated a step function at the output).
2. A pole of the transfer function generates the form of the *natural response* (that is, the pole at -5 generated e^{-5t}).
3. A pole on the real axis generates an *exponential* response of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis. Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero (again, the pole at -5 generated e^{-5t} ; see Figure 4.2 for the general case).
4. The zeros and poles generate the *amplitudes* for both the forced and natural responses (this can be seen from the calculation of A and B in Eq. (4.1)).

Let us now look at an example that demonstrates the technique of using poles to obtain the form of the system response. We will learn to write the form of the response by inspection. Each pole of the system transfer function that is on the real axis generates an exponential response that is a component of the natural response. The input pole generates the forced response.

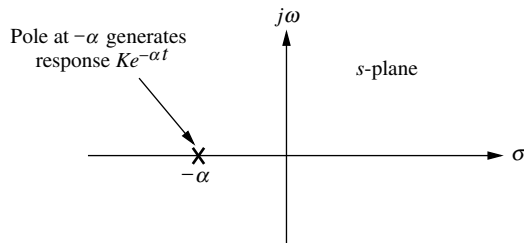


FIGURE 4.2 Effect of a real-axis pole upon transient response.

Example 4.1

Evaluating Response Using Poles

PROBLEM: Given the system of Figure 4.3, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

SOLUTION: By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

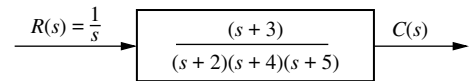


FIGURE 4.3 System for Example 4.1

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}} \quad (4.3)$$

Taking the inverse Laplace transform, we get

$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}} \quad (4.4)$$

Skill-Assessment Exercise 4.1

PROBLEM: A system has a transfer function, $G(s) = \frac{10(s+4)(s+6)}{(s+1)(s+7)(s+8)(s+10)}$.

Write, by inspection, the output, $c(t)$, in general terms if the input is a unit step.

ANSWER: $c(t) \equiv A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$

In this section, we learned that poles determine the nature of the time response: Poles of the input function determine the form of the forced response, and poles of the transfer function determine the form of the natural response. Zeros and poles of the input or transfer function contribute to the amplitudes of the component parts of the total response. Finally, poles on the real axis generate exponential responses.

4.3 First-Order Systems

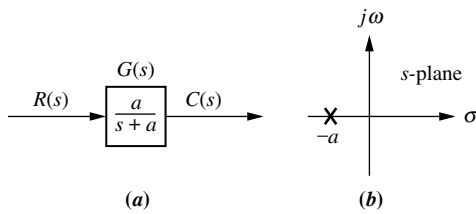


FIGURE 4.4 a. First-order system; b. pole plot

We now discuss first-order systems without zeros to define a performance specification for such a system. A first-order system without zeros can be described by the transfer function shown in Figure 4.4(a). If the input is a unit step, where $R(s) = 1/s$, the Laplace transform of the step response is $C(s)$, where

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)} \quad (4.5)$$

Taking the inverse transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at} \quad (4.6)$$

where the input pole at the origin generated the forced response $c_f(t) = 1$, and the system pole at $-a$, as shown in Figure 4.4(b), generated the natural response $c_n(t) = -e^{-at}$. Equation (4.6) is plotted in Figure 4.5.

Let us examine the significance of parameter a , the only parameter needed to describe the transient response. When $t = 1/a$,

$$e^{-at}|_{t=1/a} = e^{-1} = 0.37 \quad (4.7)$$

or

$$c(t)|_{t=1/a} = 1 - e^{-at}|_{t=1/a} = 1 - 0.37 = 0.63 \quad (4.8)$$

We now use Eqs. (4.6), (4.7), and (4.8) to define three transient response performance specifications.

Time Constant

We call $1/a$ the **time constant of the response**. From Eq. (4.7), the time constant can be described as the time for e^{-at} to decay to 37% of its initial value. Alternately, from Eq. (4.8) **the time constant is the time it takes for the step response to rise to 63% of its final value** (see Figure 4.5).

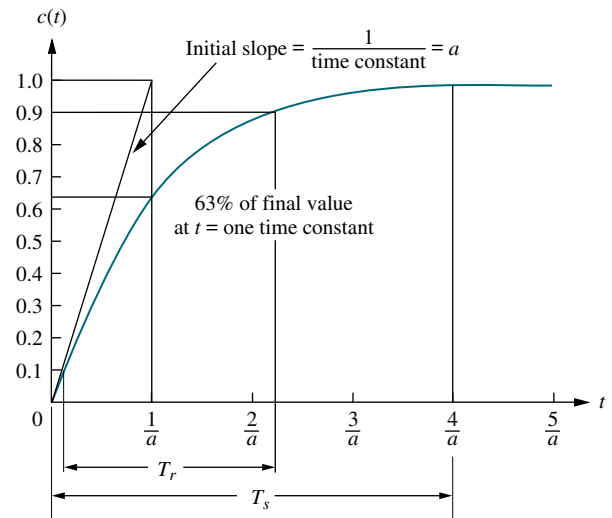
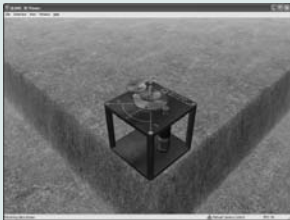


FIGURE 4.5 First-order system response to a unit step

Virtual Experiment 4.1 First-Order Open-Loop Systems

Put theory into practice and find a first-order transfer function representing the Quanser Rotary Servo. Then validate the model by simulating it in LabVIEW. Such a servo motor is used in mechatronic gadgets such as cameras.



Virtual experiments are found on WileyPLUS.

The reciprocal of the time constant has the units (1/seconds), or frequency. Thus, we can call the parameter a the *exponential frequency*. Since the derivative of e^{-at} is $-a$ when $t = 0$, a is the initial rate of change of the exponential at $t = 0$. Thus, the time constant can be considered a transient response specification for a first-order system, since it is related to the speed at which the system responds to a step input.

The time constant can also be evaluated from the pole plot (see Figure 4.4(b)). Since the pole of the transfer function is at $-a$, we can say the pole is located at the *reciprocal* of the time constant, and the farther the pole from the imaginary axis, the faster the transient response.

Let us look at other transient response specifications, such as rise time, T_r , and settling time, T_s , as shown in Figure 4.5.

Rise Time, T_r

Rise time is defined as **the time for the waveform to go from 0.1 to 0.9 of its final value**. Rise time is found by solving Eq. (4.6) for the difference in time at $c(t) = 0.9$ and $c(t) = 0.1$. Hence,

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a} \quad (4.9)$$

Settling Time, T_s

Settling time is defined as **the time for the response to reach, and stay within, 2% of its final value**.² Letting $c(t) = 0.98$ in Eq. (4.6) and solving for time, t , we find the settling time to be

$$T_s = \frac{4}{a} \quad (4.10)$$

First-Order Transfer Functions via Testing

Often it is not possible or practical to obtain a system's transfer function analytically. Perhaps the system is closed, and the component parts are not easily identifiable. Since the transfer function is a representation of the system from input to output, the system's step response can lead to a representation even though the inner construction is not known. With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

Consider a simple first-order system, $G(s) = K/(s + a)$, whose step response is

$$C(s) = \frac{K}{s(s + a)} = \frac{K/a}{s} - \frac{K/a}{(s + a)} \quad (4.11)$$

If we can identify K and a from laboratory testing, we can obtain the transfer function of the system.

For example, assume the unit step response given in Figure 4.6. We determine that it has the first-order characteristics we have seen thus far, such as no overshoot and nonzero initial slope. From the response, we measure the time constant, that is, the time for the amplitude to reach 63% of its final value. Since the final value is

² Strictly speaking, this is the definition of the 2% *settling time*. Other percentages, for example 5%, also can be used. We will use *settling time* throughout the book to mean 2% settling time.

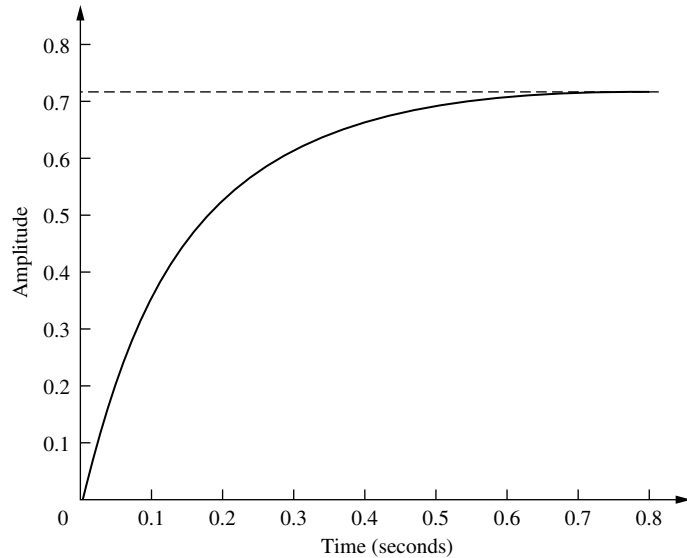


FIGURE 4.6 Laboratory results of a system step response test

about 0.72, the time constant is evaluated where the curve reaches $0.63 \times 0.72 = 0.45$, or about 0.13 second. Hence, $a = 1/0.13 = 7.7$.

To find K , we realize from Eq. (4.11) that the forced response reaches a steady-state value of $K/a = 0.72$. Substituting the value of a , we find $K = 5.54$. Thus, the transfer function for the system is $G(s) = 5.54/(s + 7.7)$. It is interesting to note that the response of Figure 4.6 was generated using the transfer function $G(s) = 5/(s + 7)$.

Skill-Assessment Exercise 4.2

PROBLEM: A system has a transfer function, $G(s) = \frac{50}{s + 50}$. Find the time constant, T_c , settling time, T_s , and rise time, T_r .

ANSWER: $T_c = 0.02$ s, $T_s = 0.08$ s, and $T_r = 0.044$ s.

The complete solution is located at www.wiley.com/college/nise.

4.4 Second-Order Systems: Introduction

Let us now extend the concepts of poles and zeros and transient response to second-order systems. Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described. Whereas varying a first-order system's parameter simply changes the speed of the response, changes in the parameters of a second-order system can change the *form* of the response. For example, a second-order system can display characteristics much