7.10. Hermite Differential Equation

The differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2py = 0 \quad \text{...(7.163)}$$

where p is a constant, is called Hermite differential equation. There is no singularity in the finite plane of the differential equation and hence it can be solved by the method of series integration. The series solution of Hermite equation may be expressed as

$$y = \sum_{m=0}^{\infty} a_m x^{m+n} \quad \text{...(7.164)}$$

On differentiation and substitution in (7.163) gives

$$\sum_m a_m (m+n)(m+n-1)x^{m+n-2} - \sum_m 2a_m (m+n-p)x^{m+n} = 0 \quad \text{...(7.165)}$$

The indicial equation is obtained by equating the co-efficient of lower power of x equal to zero. The indicial equation is

$$a_0 n(n-1) = 0 \quad \text{...(7.166)}$$

As $a_0 \neq 0$ being the co-efficient of first term, therefore we must have either

$$n = 0 \text{ or } n = 1 \quad \text{...(7.167)}$$

Equating the co-efficient x^{n-1} equal to zero we have

$$a_1 n(n+1) = 0 \quad \text{...(7.168)}$$

Since $n + 1 \neq 0$ for any value of n given by (7.167) hence eq. (7.168) implies that either

$$n = 0 \text{ or } a_1 = 0 \text{ or both are zero.}$$

The recurrence relation between the co-efficient is obtained by equating co-efficient of $x^{m+n} = 0$. Thus we have

$$a_{m+2}(m+2+n)(m+n+1) = 2(m+n-p)a_m$$

or

$$a_{m+2} = \frac{2(m+n-p)}{(m+n+2)(m+n+1)} a_m \quad \text{...(7.169)}$$

Since we have two possible values of n, now there arise two cases.

Case (i). When $n = 0$ we have from (7.169)

$$a_{m+2} = \frac{2m-2p}{(m+2)(m+1)} a_m \quad \text{...(7.70)}$$

This gives

$$a_2 = -\frac{2p}{2!} a_0$$

and
$$a_4 = \frac{4-2p}{4 \cdot 3} \cdot a_2 = \frac{4-2p}{4 \cdot 3} \cdot \frac{(-2p)}{2!} a_0 = -\frac{4-2p}{4!} 2p \cdot a_0$$

$$= \frac{(-2)^2 p(p-2)}{4!} \cdot a_0$$

Thus
$$a_{2m} = (-2)^m \frac{p(p-2)......(p-2m+2)}{(2m)!} a_0$$

and similarly odd co-efficients are given by

$$a_{2m+1} = \frac{(-2)^m (p-1)(p-3)......(p-2m+1)}{(2m+1)!} a_1$$

$$\therefore \quad y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

$$= a_0 \left[1 - \frac{2p}{2!} x^2 + \frac{2^2 p(p-2)}{4!} x^4 - ... + \frac{(-2)^m p(p-2)......(p-2m+2)}{(2m)!} x^{2m} + ... \right]$$

$$+ a_1 \left[x - \frac{2(p-1)}{3!} x^3 + ... + \frac{(-2)^m (p-1)(p-3)......(p-2m......+1)}{(2m+1)!} x^{2m+1} + ... \right] \quad \text{...(7.171)}$$

Case (*ii*). When $n = 1$, it will yield the solution which is 2nd part of equation (7.171) *i.e.*

$$a_1 \left[x - \frac{2(p-1)}{3!} x^3 + ... \right.$$

$$+\frac{(-2)^m (p-1)(p-3)\dots(p-2m+1)}{(2m+1)!}x^{2m+1}+\dots\Big]$$

Hence we conclude that in (7.171) $a_1 = 0$ and the two separate solutions of Hermite equation are

$$y_1 = a_0 \sum_{m=0}^{\infty} \frac{(-2)^m p(p-2)\dots(p-2m+2)x^{2m}}{(2m)!} \quad \dots(7.172)$$

and

$$y_2 = a_1 \sum_{m=1}^{\infty} \frac{(-2)^m (p-1)(p-3)\dots(p-2m+1)}{(2m+1)!}x^{2m+1} \quad \dots(7.173)$$

and the general solution of Hermite equation is

$$y = Ay_1 + By_2 \quad \dots(7.174)$$

where A and B are arbitrary constants.

Hermite Polynomials

When p is an even integer and $a_0 = (-1)^{P/2}\frac{p!}{(p/2)!}$ obviously the series (7.172) terminates at the $\left(\frac{p}{2}+1\right)$th term. This value of y is known as the Hermite polynomial of order p, for p even *i.e.*

$$H_p(x) = (2x)^p - \frac{p(p-1)}{1!}(2x)^{p-2} + \frac{p(p-1)(p-2)(p-3)}{2!}(2x)^{p-4}\dots$$

$$\mp(-1)^p \frac{p(p-1)\dots\dots(p-2r+1)(2x)^{p-2r}}{r!}\dots\dots+(-1)^{p/2}\frac{p!}{(p/2)!} \quad \dots(7.175)$$

When p is an odd integer and $a_1 = (-1)^{p-1/2}\frac{(p+1)!}{(p+1)/2!}$, the series (7.173) terminates at $\left(\frac{p-1}{2}\right)$th term. This value of y is known as the Hermite polynomial of order p for p odd *i.e.*

$$H_p(x) = \Big\{(2x)^p - \frac{p(p-1)}{1!}(2x)^{p-2} + \dots$$

$$\ldots+(-1)^r \frac{p(p-1)\ldots\ldots(p-2r+1)}{r!}(2x)^{p-2r}$$

$$+\ldots\ldots+(-1)^{p-1/2}\frac{(p+1)!}{(p+1)/2\,!}\,x\Big\} \qquad \ldots(7.176)$$

Thus we have Hermite polynomials of degree p, p being a positive integer

$$H_p(x)=\sum_{r=0}^{m}(-1)^r\frac{p!}{r!(p-2r)!}(2x)^{p-2r}$$

where $$m=\begin{cases} p/2 \text{ if } p \text{ is even} \\ \frac{1}{2}(p-1) \text{ if } p \text{ odd}\end{cases}$$

Generating Function of Hermite Polynomials

The generating function for Hermite polynomials is

$$G(x,t)=e^{-t^2+2tx}=\sum_{p=0}^{\infty}\frac{H_p(x)}{p!}t^p \qquad \ldots(7.178)$$

We have

$$G(x,t)=e^{2tx}\cdot e^{-t^2}=\sum_{r=1}^{\infty}\frac{(2tx)^r}{r!}\sum_{s}\frac{(-t^2)^s}{s!}$$

$$=\sum_{r,S}(-1)^s\frac{(2x)^r(t)^{r+2S}}{r!\,s!}$$

The co-efficient of t^p (for fixed value of s) on R.H.S. is obtained by putting $r+2s=p$ i.e. $r=p-2s$ and is given by

$$(-1)^S\frac{(2x)^{p-2S}}{(p-2s)!\,s!}.$$

The total co-efficient t^p is obtained by summing overall allowed values of s and since $r=p-2s$

Thus if p is even, s goes from 0 to p/2 and if p is odd s goes from 0 to $\frac{p-1}{2}$.