9

Methods of Deduction

9.1 Formal Proof of Validity

In theory, truth tables are adequate to test the validity of any argument of the general type we have considered. In practice, however, they become unwieldy as the number of component statements increases. A more efficient method of establishing the validity of an extended argument is to deduce its conclusion from its premises by a sequence of elementary arguments, each of which is known to be valid. This technique accords fairly well with ordinary methods of argumentation.

Consider, for example, the following argument:

If Anderson was nominated, then she went to Boston.

If she went to Boston, then she campaigned there.

If she campaigned there, she met Douglas.

Anderson did not meet Douglas.

Either Anderson was nominated or someone more eligible was selected.

Therefore someone more eligible was selected.

The validity of this argument may be intuitively obvious, but let us consider the matter of proof. The discussion will be facilitated by translating the argument into symbolism as

$A \supset B$

$B \supset C$

$C \supset D$

$\sim D$

$A \vee E$

$\therefore E$

To establish the validity of this argument by means of a truth table requires a table with thirty-two rows, because five different simple statements are involved. We can prove the argument valid by deducing its conclusion instead using a sequence of just four elementary valid arguments. From the first two premises, $A \supset B$ and $B \supset C$, we validly infer that $A \supset C$ as a Hypothetical Syllogism. From $A \supset C$ and the third premise, $C \supset D$, we validly infer that $A \supset D$ as another Hypothetical Syllogism. From $A \supset D$ and the fourth premise, $\sim D$, we validly infer that $\sim A$ by *Modus Tollens*. And from $\sim A$ and the fifth premise, $A \vee E$, as a Disjunctive Syllogism we validly infer E, the conclusion of the original argument. That the conclusion can be deduced from the five premises of the original argument by four elementary valid arguments proves the original argument to be valid. Here the elementary valid argument forms Hypothetical Syllogism (H.S.), *Modus Tollens* (M.T.), and Disjunctive Syllogism (D.S.) are used as **rules of inference** in accordance with which conclusions are validly inferred or deduced from premises.

This method of deriving the conclusion of a deductive argument—using rules of inference successively to prove the validity of the argument—is as reliable as the truth-table method discussed in Chapter 8, if the rules are used with meticulous care. But it improves on the truth-table method in two ways: It is vastly more efficient, as has just been shown; and it enables us to follow the flow of the reasoning process from the premises to the conclusion and is therefore much more intuitive and more illuminating. The method is often called **natural deduction.** Using natural deduction, we can provide a *formal proof* of the validity of an argument that is valid.

A *formal proof of validity* is given by writing the premises and the statements that we deduce from them in a single column, and setting off in another column, to the right of each such statement, its "justification," or the reason we give for including it in the proof. It is convenient to list all the premises first and to write the conclusion either on a separate line, or slightly to one side and separated by a diagonal line from the premises. If all the statements in the column are numbered, the "justification" for each statement consists of the numbers of the preceding statements from which it is inferred, together with the abbreviation for the rule of inference by which it follows from them. The formal proof of the example argument is written as

1.	$A \supset B$	
2.	$B \supset C$	
3.	$C \supset D$	
4.	$\sim D$	
5.	$A \vee E$	
	$\therefore E$	
6.	$A \supset C$	1, 2, H.S.
7.	$A \supset D$	6, 3, H.S.
8.	$\sim A$	7, 4, M.T.
9.	E	5, 8, D.S.

We define a **formal proof of validity** of a given argument as a *sequence of statements, each of which is either a premise of that argument or follows from preceding statements of the sequence by an elementary valid argument, such that the last statement in the sequence is the conclusion of the argument whose validity is being proved.*

We define an **elementary valid argument** as *any argument that is a substitution instance of an elementary valid argument form.* Note that *any* substitution instance of an elementary valid argument form is an elementary valid argument. Thus the argument

$(A \bullet B) \supset [C \equiv (D \vee E)]$

$A \bullet B$

$\therefore C \equiv (D \vee E)$

is an elementary valid argument because it is a substitution instance of the elementary valid argument form *Modus Ponens* (M.P.). It results from

$p \supset q$

p

$\therefore q$

by substituting $A \bullet B$ for p and $C \equiv (D \vee E)$ for q, and it is therefore of that form even though *modus ponens* is not *the specific form* of the given argument.

Modus Ponens is a very elementary valid argument form indeed, but what *other* valid argument forms are considered to be rules of inference? We begin with a list of just nine rules of inference that can be used in constructing formal proofs of validity. With their aid, formal proofs of validity can be constructed for a wide range of more complicated arguments. The names provided are for the most part standard, and the use of their abbreviations permits formal proofs to be set down with a minimum of writing.

9.2 The Elementary Valid Argument Forms

Our object is to build a set of logical rules—rules of inference—with which we can prove the validity of deductive arguments if they are valid. We began with a few elementary valid argument forms that have already been introduced—*Modus Ponens*, for example, and Disjunctive Syllogism. These are indeed simple and common. But we need a set of rules that is more powerful. The rules of inference may be thought of as a logical toolbox, from which the tools may be taken, as needed, to prove validity. What else is needed for our toolbox? How shall we expand the list of rules of inference?

The needed rules of inference consist of two sets, each set containing rules of a different kind. The first is a set of elementary valid argument forms. The second set consists of a small group of elementary logical equivalences. In this section we discuss only the elementary valid argument forms.

To this point we have become acquainted with four elementary valid argument forms:

1. *Modus Ponens (M.P.)*

$p \supset q$
p
$\therefore p$

2. *Modus Tollens (M.T.)*

$p \supset q$
$\sim q$
$\therefore \sim p$

3. Hypothetical Syllogism (H.S.)

$p \supset q$
$q \supset r$
$\therefore p \supset r$

4. Disjunctive Syllogism (D.S.).

$p \vee q$
$\sim p$
$\therefore q$

For an effective logical toolbox we need to add five more. Let us examine these additional argument forms—each of which is valid and can be readily proved valid using a truth table.

5. Rule 5 is called **Constructive Dilemma** (C.D.) It is symbolized as

$(p \supset q) \bullet (r \supset s)$

$p \vee r$

$\therefore q \vee s$

In general, a dilemma is an argument in which one of two alternatives must be chosen. In this argument form the alternatives are two conditional propositions,

$p \supset q$ and $r \supset s$. We know from *Modus Ponens* that if we are given $p \supset q$ and p, we may infer q; and if we are given $r \supset s$ and r, we may infer s. Therefore it is clear that if we are given *both* $p \supset q$, and $r \supset s$, and *either* p or r (that is, either of the antecedents), we may infer validly either q or s (that is, one or the other of the consequents.) Constructive Dilemma is, in effect, a combination of two arguments in *Modus Ponens* form, and it is most certainly valid, as a truth table can make evident. We add Constructive Dilemma (C.D.) to our tool box.

6. Absorption (Abs.)

$p \supset q$

$\therefore p \supset (p \bullet q)$

Any proposition p always implies itself, of course. Therefore, if we know that $p \supset q$, we may validly infer that p implies both itself and q. That is all that Absorption says. Why (one may ask) do we need so elementary a rule? The need for it will become clearer as we go on; in short, we need it because it will be very convenient, even essential at times, to carry the p across the horseshoe. In effect, Absorption makes the principle of identity, one of the basic logical principles discussed in Section 8.10, always available for our use. We add Absorption (Abs.) to our logical toolbox.

The next two elementary valid argument forms are intuitively very easy to grasp if we understand the logical connectives explained earlier.

7. Simplification (Simp.)

$p \bullet q$

$\therefore p$

says only that if two propositions, p and q, are true when they are conjoined ($p \bullet q$), we may validly infer that one of them, p, is true by itself. We simplify the expression before us; we "pull" p from the conjunction and stand it on its own. Because we are given that $p \bullet q$, we know that both p and q must be true; we may therefore know with certainty that p is true.

What about q? Isn't q true for exactly the same reason? Yes, it is. Then why does the elementary argument form, Simplification, conclude only that p is true? The reason is that we want to keep our toolbox uncluttered. The rules of inference must always be applied *exactly* as they appear. We surely need a rule that will enable us to take conjunctions apart, but we do not need two such rules; one will suffice. When we may need to "pull" some q from a conjunction we will be able to put it where p is now, and then use only the one rule, Simplification, which we add to our toolbox.

8. Conjunction (Conj.)

p

q

$\therefore p \bullet q$

says only that if two propositions, *p,* and *q,* are known to be true, we can put them together into one conjunctive expression, $p \bullet q$. We may conjoin them. If they are true separately, they must also be true when they are conjoined. And in this case the order presents no problem, because we may always treat the one we seek to put on the left as *p,* and the other as *q*. That *joint* truth is what a conjunction asserts. We add Conjunction (Conj.) to our logical toolbox.

The last of the nine elementary valid argument forms is also a straightforward consequence of the meaning of the logical connectives—in this case, disjunction.

9. Addition (Add.)

p

$\therefore p \vee q$

We know that any disjunction must be true if either of its disjuncts is true. That is, $p \vee q$ is true if *p* is true, or if *q* is true, or if they are both true. That is what disjunction means. It obviously follows from this that if we know that some proposition, *p,* is true, we also know that *either* it is true or some other—any other!—proposition is true. So we can construct a disjunction, $p \vee q$, using the one proposition known to be true as *p,* and adding to it (in the logical, disjunctive sense) any proposition we care to. We call this logical addition. The additional proposition, *q,* is not conjoined to *p;* it is used with *p* to build a disjunction that we may know with certainty to be true because one of the disjuncts, *p*, is known to be true. And the disjunction we thus build will be true *no matter what that added proposition asserts*—no matter how absurd or wildly false it may be! We know that Michigan is north of Florida. Therefore we know that either Michigan is north of Florida *or* the moon is made of green cheese! Indeed, we know that either Michigan is north of Florida *or* $2 + 2 = 5$. The truth or falsity of the added proposition does not affect the truth of the disjunction we build, because that disjunction is made certainly true by the truth of the disjunct with which we began. Therefore, if we are given *p* as true, we may validly infer *for any q whatever* that $p \vee q$. This principle, Addition (Add.), we add to our logical toolbox.

Our set of nine elementary valid argument forms is now complete.

All nine of these argument forms are very plainly valid. Any one of them whose validity we may doubt can be readily proved to be valid using a truth table. Each of them is simple and intuitively clear; as a set we will find them powerful as we go on to construct formal proofs for the validity of more extended arguments.

OVERVIEW

Rules of Inferences: Elementary Valid Argument Forms

Name	Abbreviation	Form
1. *Modus Ponens*	M.P.	$p \supset q$ p $\therefore q$
2. *Modus Tollens*	M.T.	$p \supset q$ $\sim q$ $\therefore \sim p$
3. Hypothetical Syllogism	H.S.	$p \supset q$ $q \supset r$ $\therefore p \supset r$
4. Disjunctive Syllogism	D.S.	$p \vee q$ $\sim p$ $\therefore q$
5. Constructive Dilemma	C.D.	$(p \supset q) \bullet (r \supset s)$ $p \vee r$ $\therefore q \vee s$
6. Absorption	Abs.	$p \supset q$ $\therefore p \supset (p \bullet q)$
7. Simplification	Simp.	$p \bullet q$ $\therefore p$
8. Conjunction	Conj.	p q $\therefore p \bullet q$
9. Addition	Add.	p $\therefore p \vee q$

Two features of these elementary arguments must be emphasized. First, *they must be applied with exactitude.* An argument that one proves valid using *Modus Ponens* must have that exact form: $p \supset q$, p, therefore q. Each statement variable must be replaced by some statement (simple or compound) consistently and accurately. Thus, for example, if we are given $(C \vee D) \supset (J \vee K)$ and $(C \vee D)$, we may infer $(J \vee K)$ by *Modus Ponens.* But we may not infer $(K \vee J)$ by *Modus*

Ponens, even though it may be true. The elementary argument form must be fitted *precisely* to the argument with which we are working. No shortcut—no fudging of any kind—is permitted, because we seek to know with certainty that the outcome of our reasoning is valid, and that can be known only if we can demonstrate that *every link in the chain of our reasoning is absolutely solid.*

Second, these elementary valid arguments must be applied to *the entire lines* of the larger argument with which we are working. Thus, for example, if we are give $[(X \bullet Y) \supset Z] \bullet T$, we cannot validly infer X by Simplification. X is one of the conjuncts of a conjunction, but that conjunction is part of a more complex expression. X may not be true even if that more complex expression is true. We are given only that *if* X and Y are both true, then Z is true. Simplification applies only to the entire line, which must be a conjunction; its conclusion is the left side (and only the left side) of that conjunction. So, from this same line, $[(X \bullet Y) \supset Z)] \bullet T$, we may validly infer $(X \bullet Y) \supset Z$ by Simplification. But we may not infer T by Simplification, even though it may be true.

Formal proofs in deductive logic have crushing power, but they possess that power only because, when they are correct, there can be not the slightest doubt of the validity of each inference drawn. The tiniest gap destroys the power of the whole.

The nine elementary valid argument forms we have given should be committed to memory. They must be always readily in mind as we go on to construct formal proofs. Only if we comprehend these elementary argument forms fully, and can apply them immediately and accurately, may we expect to succeed in devising formal proofs of the validity of more extended arguments.

EXERCISES

Here follow a set of twenty elementary valid arguments. They are valid because each of them is exactly in the form of one of the nine elementary valid argument forms. For each of them, state the rule of inference by which its conclusion follows from its premise or premises.

EXAMPLE

1. $(A \bullet B) \supset C$

   $\therefore (A \bullet B) \supset [(A \bullet B) \bullet C]$

SOLUTION

Absorption. If $(A \bullet B)$ replaces p, and C replaces q, this argument is seen to be exactly in the form $p \supset q$, therefore $p \supset (p \bullet q)$.

*1. $(A \bullet B) \supset C$
   $\therefore (A \bullet B) \supset [(A \bullet B) \bullet C]$

2. $(D \vee E) \bullet (F \vee G)$
   $\therefore D \vee E$

3. $H \supset I$
$\therefore (H \supset I) \vee (H \supset {\sim}I)$

4. ${\sim}(J \bullet K) \bullet (L \supset {\sim}M)$
$\therefore {\sim}(J \bullet K)$

***5.** $[N \supset (O \bullet P)] \bullet [Q \supset (O \bullet R)]$
$N \vee Q$
$\therefore (O \bullet P) \vee (O \bullet R)$

6. $(X \vee Y) \supset {\sim}(Z \bullet {\sim}A)$
${\sim}{\sim}(Z \bullet {\sim}A)$
$\therefore {\sim}(X \vee Y)$

7. $(S \equiv T) \vee [(U \bullet V) \vee (U \bullet W)]$
${\sim}(S \equiv T)$
$\therefore (U \bullet V) \vee (U \bullet W)$

8. ${\sim}(B \bullet C) \supset (D \vee E)$
${\sim}(B \bullet C)$
$\therefore D \vee E$

9. $(F \equiv G) \supset {\sim}(G \bullet {\sim}F)$
${\sim}(G \bullet {\sim}F) \supset (G \supset F)$
$\therefore (F \equiv G) \supset (G \supset F)$

***10.** ${\sim}(H \bullet {\sim}I) \supset (H \supset I)$
$(I \equiv H) \supset {\sim}(H \bullet {\sim}I)$
$\therefore (I \equiv H) \supset (H \supset I)$

11. $(A \supset B) \supset (C \vee D)$
$A \supset B$
$\therefore C \vee D$

12. $[E \supset (F \equiv {\sim}G)] \vee (C \vee D)$
${\sim}[E \supset (F \equiv {\sim}G)]$
$\therefore C \vee D$

13. $(C \vee D) \supset [(J \vee K) \supset (J \bullet K)]$
${\sim}[(J \vee K) \supset (J \bullet K)]$
$\therefore {\sim}(C \vee D)$

14. ${\sim}[L \supset (M \supset N)] \supset {\sim}(C \vee D)$
${\sim}[L \supset (M \supset N)]$
$\therefore {\sim}(C \vee D)$

***15.** $(J \supset K) \bullet (K \supset L)$
$L \supset M$
$\therefore [(J \supset K) \bullet (K \supset L)] \bullet (L \supset M)$

16. $N \supset (O \vee P)$
$Q \supset (O \vee R)$
$\therefore [Q \supset (O \vee R)] \bullet [N \supset (O \vee P)]$

17. $(S \supset T) \supset (U \supset V)$
$\therefore (S \supset T) \supset [(S \supset T) \bullet (U \supset V)]$

18. $(W \bullet {\sim}X) \equiv (Y \supset Z)$
$\therefore [(W \bullet {\sim}X) \equiv (Y \supset Z)] \vee (X \equiv {\sim}Z)$

19. $[(H \bullet {\sim}I) \supset C] \bullet [(I \bullet {\sim}H) \supset D]$
$(H \bullet {\sim}I) \vee (I \bullet {\sim}H)$
$\therefore C \vee D$

***20.** $[(O \supset P) \supset Q] \supset {\sim}(C \vee D)$
$(C \vee D) \supset [(O \supset P) \supset Q]$
$\therefore (C \vee D) \supset {\sim}(C \vee D)$

9.3 Formal Proofs of Validity Exhibited

We have defined a formal proof of validity for a given argument as a sequence of statements, each of which is either a premise of that argument or follows from preceding statements of the sequence by an elementary valid argument, such that the last statement in the sequence is the conclusion of the argument whose validity is being proved. Our task will be to build such sequences, to prove the validity of arguments with which we are confronted.

Doing this can be a challenge. Before attempting to construct such sequences, it will be helpful to become familiar with the look and character of formal proofs. In this section we examine a number of complete formal proofs, to see how they work and to get a "feel" for constructing them.

Our first step is not to devise such proofs, but to understand and appreciate them. A sequence of statements is put before us in each case. Every statement in that sequence will be either a premise, or will follow from preceding statements in the sequence using one of the elementary valid argument forms—just as in the illustration that was presented in Section 9.1. When we confront such a proof, but the rule of inference that justifies each step in the proof is not given, we know (having been told that these are completed proofs) that every line in the proof that is not itself a premise can be deduced from the preceding lines. To understand those deductions, the nine elementary valid argument forms must be kept in mind.

Let us look at some proofs that exhibit this admirable solidity. Our first example is Exercise 1 in the set of exercises on pages 382–383.

EXAMPLE 1

1. $A \bullet B$

2. $(A \vee C) \supset D$

$\therefore A \bullet D$

3. A

4. $A \vee C$

5. D

6. $A \bullet D$

The first two lines of this proof are seen to be premises, because they appear before the "therefore" symbol (∴); what appears immediately to the right of that symbol is the conclusion of this argument, $A \bullet D$. The very last line of the sequence is (as it must be if the formal proof is correct) that same conclusion, $A \bullet D$. What about the steps between the premises and the conclusion? Line 3, A, we can deduce from line 1, $A \bullet B$, by Simplification. So we put, to the right of line 3, the line number from which it comes and the rule by which it is inferred from that line, "1, Simp." Line 4 is $A \vee C$. How can that be inferred from the lines above it? We can*not* infer it from line 2 by Simplification. But we can infer it from line 3, A, by Addition. Addition tells us that if p is true, then $p \vee q$ is true, whatever q may be. Using that logical pattern precisely, we may infer from A that $A \vee C$ is true. To the right of line 4 we therefore put "3, Add." Line 5 is D. D appears in line 2 as the consequent of a conditional statement $(A \vee C) \supset D$. We proved on line 4 that $A \vee C$ is true; now, using *Modus Ponens*, we combine this with the conditional on line 2 to prove D. To the right of line 5 we therefore write "2, 4, M.P." A has been proved true (on line 3) and D has been proved true (on line 5). We may therefore validly conjoin them, which is what line 6 asserts: $A \bullet D$. To the right of line 6 we therefore write "3, 5, Conj." This line, $A \bullet D$, is the conclusion of the argument, and it is therefore the last statement in the sequence of statements that constitutes this proof. The proof, which had been presented to us complete, has thus been "fleshed out" by specifying the justification of each step within it.

In this example, and the exercises that follow, every line of each proof can be justified by using one of the elementary valid argument forms in our logical toolbox. No other inferences of any kind are permitted, however plausible they may seem. When we had occasion to refer to an argument form that has two premises (e.g., M.P. or D.S.), we indicated first, in the justification, the numbers of the lines used, *in the order in which they appear in the elementary valid form.* Thus, line 5 in Example 1 is justified by 2, 4, M.P.

To become proficient in the construction of formal proofs, we must become fully familiar with the shape and rhythm of the nine elementary argument forms—the first nine of the rules of inference that we will be using extensively.

EXERCISES

Each of the following exercises presents a flawless formal proof of validity for the indicated argument. For each, state the justification for each numbered line that is not a premise.

1.
1. $A \bullet B$
2. $(A \vee C) \supset D$
   $\therefore A \bullet D$
3. A
4. $A \vee C$
5. D
6. $A \bullet D$

2.
1. $(E \vee F) \bullet (G \vee H)$
2. $(E \supset G) \bullet (F \supset H)$
3. $\sim G$
   $\therefore H$
4. $E \vee F$
5. $G \vee H$
6. H

3.
1. $I \supset J$
2. $J \supset K$
3. $L \supset M$
4. $I \vee L$
   $\therefore K \vee M$
5. $I \supset K$
6. $(I \supset K) \bullet (L \supset M)$
7. $K \vee M$

4.
1. $N \supset O$
2. $(N \bullet O) \supset P$
3. $\sim(N \bullet P)$
   $\therefore \sim N$
4. $N \supset (N \bullet O)$
5. $N \supset P$
6. $N \supset (N \bullet P)$
7. $\sim N$

***5.**
1. $Q \supset R$
2. $\sim S \supset (T \supset U)$
3. $S \vee (Q \vee T)$
4. $\sim S$

   $\therefore R \vee U$

5. $T \supset U$
6. $(Q \supset R) \bullet (T \supset U)$
7. $Q \vee T$
8. $R \vee U$

7.
1. $(A \vee B) \supset C$
2. $(C \vee B) \supset [A \supset (D \equiv E)]$
3. $A \bullet D$

   $\therefore D \equiv E$

4. A
5. $A \vee B$
6. C
7. $C \vee B$
8. $A \supset (D \equiv E)$
9. $D \equiv E$

9.
1. $I \supset J$
2. $I \vee (\sim\sim K \bullet \sim\sim J)$
3. $L \supset \sim K$
4. $\sim(I \bullet J)$

   $\therefore \sim L \vee \sim J$

5. $\supset (I \bullet J)$
6. $\sim I$
7. $\sim\sim K \bullet \sim\sim J$
8. $\sim\sim K$
9. $\sim L$
10. $\sim L \vee \sim J$

6.
1. $W \supset X$
2. $(W \supset Y) \supset (Z \vee X)$
3. $(W \bullet X) \supset Y$
4. $\sim Z$

   $\therefore X$

5. $W \supset (W \bullet X)$
6. $W \supset Y$
7. $Z \vee X$
8. X

8.
1. $F \supset \sim G$
2. $\sim F \supset (H \supset \sim G)$
3. $(\sim I \vee \sim H) \supset \sim\sim G$
4. $\sim I$

   $\therefore \sim H$

5. $\sim I \vee \sim H$
6. $\sim\sim G$
7. $\sim F$
8. $H \supset \sim G$
9. $\sim H$

***10.**
1. $(L \supset M) \supset (N \equiv O)$
2. $(P \supset \sim Q) \supset (M \equiv \sim Q)$
3. $\{[(P \supset \sim Q) \vee (R \equiv S)] \bullet (N \vee O)\} \supset [(R \equiv S) \supset (L \supset M)]$
4. $(P \supset \sim Q) \vee (R \equiv S)$
5. $N \vee O$

   $\therefore (M \equiv \sim Q) \vee (N \equiv O)$

6. $[(P \supset \sim Q) \vee (R \equiv S)] \bullet (N \vee O)$
7. $(R \equiv S) \supset (L \supset M)$
8. $(R \equiv S) \supset (N \equiv O)$
9. $[(P \supset \sim Q) \supset (M \equiv \sim Q)] \bullet [(R \equiv S) \supset (N \equiv O)]$
10. $(M \equiv \sim Q) \vee (N \equiv O)$

9.4 Constructing Formal Proofs of Validity

We turn now to one of the central tasks of deductive logic: proving formally that valid arguments really are valid. In the preceding sections we examined formal proofs that needed only to be supplemented by the justifications of the steps taken. From this point, however, we confront arguments whose formal proofs must be constructed. This is an easy task for many arguments, a more challenging task for some. But whether the proof needed is short and simple, or long and complex, the rules of inference are in every case our instruments. Success requires *mastery* of these rules. Having the list of rules before one will probably not be sufficient. One must be able to call on the rules "from within" as the proofs are being devised. The ability to do this will grow rapidly with practice, and yields many satisfactions.

We begin by constructing proofs for simple arguments. The only rules needed (or available for our use) are the nine elementary valid argument forms with which we have been working. This limitation we will later overcome, but even with only these nine rules in our logical toolbox, very many arguments can be formally proved valid. We begin with arguments that require, in addition to the premises, no more than two additional statements.

We will look first at two examples, the first two in the set of exercises on pages 385–386.

First example: Consider the argument:

1. A

B

$\therefore (A \vee C) \bullet B$

The conclusion of this argument $(A \vee C) \bullet B$ is a conjunction; we see immediately that the second conjunct, B, is readily at hand as a premise in line 2. All that is now needed is the statement of the disjunction, $(A \vee C)$, which may then be conjoined with B to complete the proof. $(A \vee C)$ is easily obtained from the premise A, in line 1; we simply add C using the rule Addition, which tells us that to any given p we may add (disjunctively) any q whatever. In this example we have been told that A is true, so we may infer by this rule that $A \vee C$ must be true. The third line of this proof is "3. A $\vee$ C, 1, Add." In line 4 we can conjoin this disjunction (line 3) with the premise B (line 2): "4. $(A \vee C) \cdot B$, 3, 2, Conj." This final line of the sequence is the conclusion of the argument being proved. The formal proof is complete.

Here is a second example of an argument whose formal proof requires only two additional lines in the sequence:

2. $D \supset E$

$D \bullet F$

$\therefore E$

The conclusion of this argument, E, is the consequent of the conditional statement $D \supset E$, which is given as the first premise. We know that we will be able to infer the truth of E by *Modus Ponens* if we can establish the truth of *D*. We can establish the truth of *D*, of course, by Simplification from the second premise, *D* • *F*. So the complete formal proof consists of the following four lines:

1. $D \supset E$
2. $D \bullet F$ $/\therefore E$
3. D 2, Simp.
4. E 1, 3, M.P.

In each of these examples, and in all the exercises immediately following, a formal proof for each argument may be constructed by adding just two additional statements. This will be an easy task *if the nine elementary valid argument forms are clearly in mind.* Bear in mind that the final line in the sequence of each proof is always the conclusion of the argument being proved.

EXERCISES

1. A
B
$\therefore (A \vee C) \bullet B$

2. $D \supset E$
$D \bullet F$
$\therefore E$

3. G
H
$(G \bullet H) \vee I$

4. $J \supset K$
J
$\therefore K \vee L$

***5.** $M \vee N$
$\sim M \bullet \sim O$
$\therefore N$

6. $P \bullet Q$
R
$\therefore P \bullet R$

7. $S \supset T$
$\sim T \bullet \sim U$
$\therefore \sim S$

8. $V \vee W$
$\sim V$
$\therefore W \vee X$

9. $Y \supset Z$
Y
$\therefore Y \bullet Z$

***10.** $A \supset B$
$(A \bullet B) \supset C$
$\therefore A \supset C$

11. $D \supset E$
$(E \supset F) \bullet (F \supset D)$
$\therefore D \supset F$

12. $(G \supset H) \bullet (I \supset J)$
G
$\therefore H \vee J$

13. $\sim(K \bullet L)$
$K \supset L$
$\therefore \sim K$

14. $(M \supset N) \bullet (M \supset O)$
$N \supset O$
$\therefore M \supset O$

***15.** $(P \supset Q) \bullet (R \supset S)$
$(P \vee R) \bullet (Q \vee R)$
$\therefore Q \vee S$

16. $(T \supset U) \bullet (T \supset V)$
T
$\therefore U \vee V$

17. $(W \lor X) \supset Y$
W
$\therefore Y$

18. $(Z \bullet A) \supset (B \bullet C)$
$Z \supset A$
$\therefore Z \supset (B \bullet C)$

19. $D \supset E$
$[D \supset (D \bullet E)] \supset (F \supset \sim G)$
$\therefore F \supset \sim G$

***20.** $(\sim H \lor I) \lor J$
$\sim(\sim H \lor I)$
$\therefore J \lor \sim H$

21. $(K \supset L) \supset M$
$\sim M \bullet \sim(L \supset K)$
$\therefore \sim(K \supset L)$

22. $(N \supset O) \supset (P \supset Q)$
$[P \supset (N \supset O)] \bullet [N \supset (P \supset Q)]$
$\therefore P \supset (P \supset Q)$

23. $R \supset S$
$S \supset (S \bullet R)$
$\therefore [R \supset (R \bullet S)] \bullet [S \supset (S \bullet R)]$

24. $[T \supset (U \lor V)] \bullet [U \supset (T \lor V)]$
$(T \lor U) \bullet (U \lor V)$
$\therefore (U \lor V) \lor (T \lor V)$

***25.** $(W \bullet X) \supset (Y \bullet Z)$
$\sim[(W \bullet X) \bullet (Y \bullet Z)]$
$\therefore \sim(W \bullet X)$

26. $A \supset B$
$A \lor C$
$C \supset D$
$\therefore B \lor D$

27. $(E \bullet F) \lor (G \supset H)$
$I \supset G$
$\sim(E \bullet F)$
$\therefore I \supset H$

28. $J \lor \sim K$
$K \lor (L \supset J)$
$\sim J$
$\therefore L \supset J$

29. $(M \supset N) \bullet (O \supset P)$
$N \supset P$
$(N \supset P) \supset (M \lor O)$
$\therefore N \lor P$

***30.** $Q \supset (R \lor S)$
$(T \bullet U) \supset R$
$(R \lor S) \supset (T \bullet U)$
$\therefore Q \supset R$

9.5 Constructing More Extended Formal Proofs

Arguments whose formal proof requires only two additional statements are quite simple. We now advance to construct formal proofs of the validity of more complex arguments. However, the process will be the same: The target for the final statement of the sequence will always be the conclusion of the argument, and the rules of inference will always be our only logical tools.

Let us look closely at an example—the first exercise of Set A on page 387, an argument whose proof requires three additional statements:

1. $A \lor (B \supset A)$
$\sim A \bullet C$
$\therefore \sim B$

In devising the proof of this argument (as in most cases), we need some plan of action, some strategy with which we can progress, using our rules, toward the conclusion sought. Here that conclusion is $\sim B$. We ask ourselves: Where in the premises does B appear? Only as the antecedent of the hypothetical ($B \supset A$), which is a component of the first premise. How might $\sim B$ be derived? Using *Modus Tollens*, we can infer it from $B \supset A$ if we can establish that hypothetical separately and also establish $\sim A$. Both of those needed steps can be readily accomplished. $\sim A$ is inferred from line 2 by Simplification:

3. $\sim A$ 2, Simp.

We can then apply $\sim A$ to line 1, using Disjunctive Syllogism to infer ($B \supset A$):

4. $(B \supset A)$ 1, 3, D.S.

The proof may then be completed using *Modus Tollens* on lines 4 and 3:

5. $\sim B$ 4, 3, M.T.

The strategy used in this argument is readily devised. In the case of some proofs, devising the needed strategy will not be so simple, but it is almost always helpful to ask: What statement(s) will enable one to infer the conclusion? And what statement(s) will enable one to infer *that*? And so on, moving backward from the conclusion toward the premises given.

EXERCISES

A. For each of the following arguments, it is possible to provide a formal proof of validity by adding just three statements to the premises. Writing these out, carefully and accurately, will strengthen your command of the rules of inference, a needed preparation for the construction of proofs that are more extended and more complex.

1. $A \vee (B \supset A)$
$\sim A \bullet C$
$\therefore \sim B$

2. $(D \vee E) \supset (F \bullet G)$
D
$\therefore F$

3. $(H \supset I) \bullet (H \supset J)$
$H \bullet (I \vee J)$
$\therefore I \vee J$

4. $(K \bullet L) \supset M$
$K \supset L$
$\therefore K \supset [(K \bullet L) \bullet M]$

***5.** $N \supset [(N \bullet O) \supset P]$
$N \bullet O$
$\therefore P$

6. $Q \supset R$
$R \supset S$
$\sim S$
$\therefore \sim Q \bullet \sim R$

7. $T \supset U$
$V \vee \sim U$
$\sim V \bullet \sim W$
$\therefore \sim T$

8. $\sim X \supset Y$
$Z \supset X$
$\sim X$
$\therefore Y \bullet \sim Z$

9. $(A \vee B) \supset {\sim}C$
$C \vee D$
A
$\therefore D$

***10.** $E \vee {\sim}F$
$F \vee (E \vee G)$
${\sim}E$
$\therefore G$

11. $(H \supset I) \bullet (J \supset K)$
$K \vee H$
${\sim}K$
$\therefore I$

12. $L \vee (M \supset N)$
${\sim}L \supset (N \supset O)$
${\sim}L$
$\therefore M \supset O$

13. $(P \supset Q) \bullet (Q \supset P)$
$R \supset S$
$P \vee R$
$\therefore Q \vee S$

14. $(T \supset U) \bullet (V \supset W)$
$(U \supset X) \bullet (W \supset Y)$
T
$\therefore X \vee Y$

***15.** $(Z \bullet A) \supset B$
$B \supset A$
$(B \bullet A) \supset (A \bullet B)$
$\therefore (Z \bullet A) \supset (A \bullet B)$

Formal proofs most often require more than two or three lines to be added to the premises. Some are very lengthy. Whatever their length, however, the same process and the same strategic techniques are called for in devising the needed proofs. In this section we rely entirely on the nine elementary valid argument forms that serve as our rules of inference.

As we begin to construct longer and more complicated proofs, let us look closely at an example of such proofs—the first exercise of Set B on page 389. It is not difficult, but it is more extended than those we have worked with so far.

1. $A \supset B$
$A \vee (C \bullet D)$
${\sim}B \bullet {\sim}E$
$\therefore C$

The strategy needed for the proof of this argument is not hard to see: To obtain C we must break apart the premise in line 2; to do that we will need ${\sim}A$; to establish ${\sim}A$ we will need to apply *Modus Tollens* to line 1 using ${\sim}B$. Therefore we continue the sequence with the fourth line of the proof by applying Simplification to line 3:

1. $A \supset B$
2. $A \vee (C \bullet D)$
3. ${\sim}B \bullet {\sim}E$ $/\therefore C$
4. ${\sim}B$ 3, Simp.

Using line 4 we can obtain ~A from line 1:

5. ~A 1, 4, M.T.

With ~A established we can break line 2 apart, as we had planned, using D.S.:

6. C • D 2, 5, D.S.

The conclusion may be pulled readily from the sixth line by Simplification.

7. C 6, Simp.

Seven lines (including the premises) are required for this formal proof. Some proofs require very many more lines than this, but the object and the method remain always the same.

It sometimes happens, as one is devising a formal proof, that a statement is correctly inferred and added to the numbered sequence but turns out not to be needed; a solid proof may be given without using that statement. In such a case it is usually best to rewrite the proof, eliminating the unneeded statement. However, if the unneeded statement is retained, and the proof remains accurately constructed using other statements correctly inferred, the inclusion of the unneeded statement (although perhaps inelegant) does not render the proof incorrect. Logicians tend to prefer shorter proofs, proofs that move to the conclusion as directly as the rules of inference permit. But if, as one is constructing a more complicated proof, it becomes apparent that some much earlier statement(s) has been needlessly inferred, it may be more efficient to allow such statement(s) to remain in place, using (as one goes forward) the more extended numbering that that inclusion makes necessary. *Logical solidity* is the critical objective. A solid formal proof, one in which *each step is correctly derived and the conclusion is correctly linked to the premises by an unbroken chain of arguments using the rules of inference correctly,* remains a proof—even if it is not as crisp and elegant as some other proof that could be devised.

EXERCISES

B. For each of the following arguments, a formal proof of validity can be constructed without great difficulty, although some of the proofs may require a sequence of eight or nine lines (including premises) for their completion.

1. $A \supset B$
$A \vee (C \bullet D)$
~B • ~E
∴ C

2. $(F \supset G) \bullet (H \supset I)$
$J \supset K$
$(F \vee J) \bullet (H \vee L)$
∴ $G \vee K$

3. (~M • ~N) $\supset (O \supset N)$
$N \supset M$
~M
∴ ~O

4. $(K \vee L) \supset (M \vee N)$
$(M \vee N) \supset (O \bullet P)$
K
∴ O

***5.** $(Q \supset R) \bullet (S \supset T)$
$(U \supset V) \bullet (W \supset X)$
$Q \vee U$
$\therefore R \vee V$

6. $W \supset X$
$(W \bullet X) \supset Y$
$(W \bullet Y) \supset Z$
$\therefore W \supset Z$

7. $A \supset B$
$C \supset D$
$A \vee C$
$\therefore (A \bullet B) \vee (C \bullet D)$

8. $(E \vee F) \supset (G \bullet H)$
$(G \vee H) \supset I$
E
$\therefore I$

9. $J \supset K$
$K \vee L$
$(L \bullet {\sim}J) \supset (M \bullet {\sim}J)$
${\sim}K$
$\therefore M$

***10.** $(N \vee O) \supset P$
$(P \vee Q) \supset R$
$Q \vee N$
${\sim}Q$
$\therefore R$

In the study of logic, our aim is to evaluate arguments in a natural language, such as English. When an argument in everyday discourse confronts us, we can prove it to be valid (if it really is valid) by first translating the statements (from English, or from any other natural language) into our symbolic language, and then constructing a formal proof of that symbolic translation. The symbolic version of the argument may reveal that the argument is, in fact, more simple (or possibly more complex) than one had supposed on first hearing or reading it. Consider the following example (the first in the set of exercises that immediately follow):

1. If either Gertrude or Herbert wins, then both Jens and Kenneth lose. Gertrude wins. Therefore Jens loses. (*G*—Gertrude wins; *H*—Herbert wins; *J*—Jens loses; *K*—Kenneth loses.)

Abbreviations for each statement are provided in this context because, without them, those involved in the discussion of these arguments would be likely to employ various abbreviations, making communication difficult. Using the abbreviations suggested greatly facilitates discussion.

Translated from the English into symbolic notation, this first argument appears as

1. $(G \vee H) \supset (J \bullet K)$
2. G $\quad /\therefore J$

The formal proof of this argument is short and straightforward:

3. $G \vee H$ 2, Add.
4. $J \bullet K$ 1, 3, M. P.
5. J 4, Simp.

EXERCISES

C. Each of the following arguments in English may be similarly translated, and for each, a formal proof of validity (using only the nine elementary valid argument forms as rules of inference) may be constructed. These proofs vary in length, some requiring a sequence of thirteen statements (including the premises) to complete the formal proofs. The suggested abbreviations should be used for the sake of clarity. Bear in mind that, as one proceeds to produce a formal proof of an argument presented in a natural language, it is of the utmost importance that *the translation into symbolic notation of the statements appearing discursively in the argument be perfectly accurate;* if it is not, one will be working with an argument that is *different* from the original one, and in that case any proof devised will be useless, being not applicable to the original argument.

1. If either Gertrude or Herbert wins, then both Jens and Kenneth lose. Gertrude wins. Therefore Jens loses. (*G*—Gertrude wins; *H*—Herbert wins; *J*—Jens loses; *K*—Kenneth loses.)

2. If Adriana joins, then the club's social prestige will rise; and if Boris joins, then the club's financial position will be more secure. Either Adriana or Boris will join. If the club's social prestige rises, then Boris will join; and if the club's financial position becomes more secure, then Wilson will join. Therefore either Boris or Wilson will join. (*A*—Adriana joins; *S*—The club's social prestige rises; *B*—Boris joins; *F*—The club's financial position is more secure; *W*—Wilson joins.)

3. If Brown received the message, then she took the plane; and if she took the plane, then she will not be late for the meeting. If the message was incorrectly addressed, then Brown will be late for the meeting. Either Brown received the message or the message was incorrectly addressed. Therefore either Brown took the plane or she will be late for the meeting. (*R*—Brown received the message; *P*—Brown took the plane; *L*—Brown will be late for the meeting; *T*—The message was incorrectly addressed.)

4. If Nihar buys the lot, then an office building will be constructed; whereas if Payton buys the lot, then it will be quickly sold again. If Rivers buys the lot, then a store will be constructed; and if a store is constructed, then Thompson will offer to lease it. Either Nihar or Rivers will buy the lot. Therefore either an office building or a store will be constructed. (*N*—Nihar buys the lot; *O*—An office building will be constructed; *P*—Payton buys the lot; *Q*—The lot will be quickly sold again; *R*—Rivers buys the lot; *S*—A store will be constructed; *T*—Thompson will offer to lease it.)

***5.** If rain continues, then the river rises. If rain continues and the river rises, then the bridge will wash out. If the continuation of rain would cause the bridge to wash out, then a single road is not sufficient for the town. Either a single road is sufficient for the town or the traffic engineers have made a mistake. Therefore the traffic engineers have made a mistake. (*C*—Rain continues; *R*—The river rises; *B*—The bridge washes out; *S*—A single road is sufficient for the town; *M*—The traffic engineers have made a mistake.)

6. If Jonas goes to the meeting, then a complete report will be made; but if Jonas does not go to the meeting, then a special election will be required. If a complete report is made, then an investigation will be launched. If Jonas's going to the meeting implies that a complete report will be made, and the making of a complete report implies that an investigation will be launched, then either Jonas goes to the meeting and an investigation is launched or Jonas does not go to the meeting and no investigation is launched. If Jonas goes to the meeting and an investigation is launched, then some members will have to stand trial. But if Jonas does not go to the meeting and no investigation is launched, then the organization will disintegrate very rapidly. Therefore either some members will have to stand trial or the organization will disintegrate very rapidly. (*J*—Jonas goes to the meeting; *R*—A complete report is made; *E*—A special election is required; *I*—An investigation is launched; *T*—Some members have to stand trial; *D*—The organization disintegrates very rapidly.)

7. If Ann is present, then Bill is present. If Ann and Bill are both present, then either Charles or Doris will be elected. If either Charles or Doris is elected, then Elmer does not really dominate the club. If Ann's presence implies that Elmer does not really dominate the club, then Florence will be the new president. So Florence will be the new president. (*A*—Ann is present; *B*—Bill is present; *C*—Charles will be elected; *D*—Doris will be elected; *E*—Elmer really dominates the club; *F*—Florence will be the new president.)

8. If Mr. Jones is the manager's next-door neighbor, then Mr. Jones's annual earnings are exactly divisible by 3. If Mr. Jones's annual earnings are exactly divisible by 3, then $40,000 is exactly divisible by 3. But $40,000 is not exactly divisible by 3. If Mr. Robinson is the manager's next-door neighbor, then Mr. Robinson lives halfway between Detroit and Chicago. If Mr. Robinson lives in Detroit, then he does not live halfway between Detroit and Chicago. Mr. Robinson lives in Detroit. If Mr. Jones is not the manager's next-door neighbor, then either Mr. Robinson or Mr. Smith is the manager's next-door neighbor. Therefore Mr. Smith is the manager's next-door neighbor. (*J*—Mr. Jones is the manager's next-door neighbor; *E*—Mr. Jones's annual earnings are exactly divisible by 3; *T*—$40,000 is exactly divisible by 3; *R*—Mr. Robinson is the manager's next-door neighbor; *H*—Mr. Robinson lives halfway between Detroit and Chicago; *D*—Mr. Robinson lives in Detroit; *S*—Mr. Smith is the manager's next-door neighbor.)

9. If Mr. Smith is the manager's next-door neighbor, then Mr. Smith lives halfway between Detroit and Chicago. If Mr. Smith lives halfway between Detroit and Chicago, then he does not live in Chicago. Mr. Smith is the manager's next-door neighbor. If Mr. Robinson lives in Detroit, then he does not live in Chicago. Mr. Robinson lives in Detroit. Mr. Smith lives in Chicago or else either Mr. Robinson or Mr. Jones lives in Chicago. If Mr. Jones lives in Chicago, then the manager is Jones. Therefore the manager is Jones. (*S*—Mr. Smith is the manager's next-door neighbor; *W*—Mr. Smith lives halfway between Detroit and Chicago; *L*—Mr. Smith lives in Chicago; *D*—Mr. Robinson lives in Detroit; *I*—Mr. Robinson lives in Chicago; *C*—Mr. Jones lives in Chicago; *B*—The manager is Jones.)

***10.** If Smith once beat the editor at billiards, then Smith is not the editor. Smith once beat the editor at billiards. If the manager is Jones, then Jones is not the editor. The manager is Jones. If Smith is not the editor and Jones is not the editor, then Robinson is the editor. If the manager is Jones and Robinson is the editor, then Smith is the publisher. Therefore Smith is the publisher. (*O*—Smith once beat the editor at billiards; *M*—Smith is the editor; *B*—The manager is Jones; *N*—Jones is the editor; *F*—Robinson is the editor; *G*—Smith is the publisher.)

9.6 Expanding the Rules of Inference: Replacement Rules

The nine elementary valid argument forms with which we have been working are powerful tools of inference, but they are not powerful enough. There are very many valid truth-functional arguments whose validity cannot be proved using only the nine rules thus far developed. We need to expand the set of rules, to increase the power of our logical toolbox.

To illustrate the problem, consider the following simple argument, which is plainly valid:

> If you travel directly from Chicago to Los Angeles, you must cross the Mississippi River. If you travel only along the Atlantic seaboard, you will not cross the Mississippi River. Therefore if you travel directly from Chicago to Los Angeles, you will not travel only along the Atlantic seaboard.

Translated into symbolic notation, this argument appears as

$D \supset C$

$A \supset \sim C$

$/\therefore D \supset \sim A$

This conclusion certainly does follow from the given premises. But, try as we may, there is no way to prove that it is valid using only the elementary valid argument forms. Our logical toolbox is not fully adequate.

What is missing? Chiefly, what is missing is the ability to replace one statement by another that is logically equivalent to it. We need to be able to put, in place of any given statement, any other statement whose meaning is exactly the same as that of the statement being replaced. And we need rules that identify legitimate replacements precisely.

Such rules are available to us. Recall that the only compound statements that concern us here (as we noted in Section 8.2) are *truth-functional* compound statements, and in a truth-functional compound statement, if we replace any component by another statement having the same truth value, the truth value of the compound statement remains unchanged. Therefore we may accept as an additional principle of inference what may be called the general **rule of replacement**—a rule that permits us to infer from any statement the result of replacing any component of that statement by any other statement that is *logically equivalent* to the component replaced.

The correctness of such replacements is intuitively obvious. To illustrate, the principle of *Double Negation* (D.N.) asserts that p is logically equivalent to $\sim\sim p$. Using the rule of replacement we may say, correctly, that from the statement $A \supset \sim\sim B$, any one of the following statements may be validly inferred:

$A \supset B$,

$\sim\sim A \supset \sim\sim B$,

$\sim\sim(A \supset \sim\sim B)$, and even

$A \supset \sim\sim\sim\sim B$.

When we put any one of these in place of $A \supset \sim\sim B$, we do no more than exchange one statement for another that is its logical equivalent.

This rule of replacement is a powerful enrichment of our rules of inference. In its general form, however, its application is problematic because its content is not definite; we are not always sure what statements are indeed logically equivalent to some other statements, and thus (if we have the rule only in its general form) we may be unsure whether that rule applies in a given case. To overcome this problem in a way that makes the rule of replacement applicable with indubitable accuracy, we make the rule *definite* by listing *ten* *specific logical equivalences* to which the rule of replacement may certainly be applied. Each of these equivalences—they are all logically true biconditionals—will serve as a separate rule of inference. We list the ten logical equivalences here, as ten rules, and we number them consecutively to follow the first nine rules of inference already set forth in the preceding sections of this chapter.

OVERVIEW

The Rules of Replacement: Logically Equivalent Expressions

Any of the following logically equivalent expressions may replace each other wherever they occur.

Name	Abbreviation	Form
10. De Morgan's theorems	De M.	$\sim(p \bullet q) \overset{T}{\equiv} (\sim p \vee \sim q)$ $\sim(p \vee q) \overset{T}{\equiv} (\sim p \bullet \sim q)$
11. Commutation	Com.	$(p \vee q) \overset{T}{\equiv} (q \vee p)$ $(p \bullet q) \overset{T}{\equiv} (q \bullet p)$
12. Association	Assoc.	$[p \vee (q \vee r)] \overset{T}{\equiv} [(p \vee q) \vee r]$ $[p \bullet (q \bullet r)] \overset{T}{\equiv} [(p \bullet q) \bullet r]$
13. Distribution	Dist.	$[p \bullet (q \vee r)] \overset{T}{\equiv} [(p \bullet q) \vee (p \bullet r)]$ $[p \vee (q \bullet r)] \overset{T}{\equiv} [(p \vee q) \bullet (p \vee r)]$
14. Double Negation	D.N.	$p \overset{T}{\equiv} \sim\sim p$
15. Transposition	Trans.	$(p \supset q) \overset{T}{\equiv} (\sim q \supset \sim p)$
16. Material Implication	Impl.	$(p \supset q) \overset{T}{\equiv} (\sim p \vee q)$
17. Material Equivalence	Equiv.	$(p \equiv q) \overset{T}{\equiv} [(p \supset q) \bullet (q \supset p)]$ $(p \equiv q) \overset{T}{\equiv} [(p \bullet q) \vee (\sim p \bullet \sim q)]$
18. Exportation	Exp.	$[(p \bullet q) \supset r] \overset{T}{\equiv} [p \supset (q \supset r)]$
19. Tautology	Taut.	$p \overset{T}{\equiv} (p \vee p)$ $p \overset{T}{\equiv} (p \bullet p)$

Let us now examine each of these ten logical equivalences. We will use them frequently and will rely on them in constructing formal proofs of validity, and therefore we must grasp their force as deeply, and control them as fully, as we do the nine elementary valid argument forms. We take these ten in order, giving for each the name, the abbreviation commonly used for it, and its exact logical form(s).

10. De Morgan's Theorems **De M.**

$$\sim(p \bullet q) \overset{T}{\equiv} (\sim p \vee \sim q)$$
$$\sim(p \vee q) \overset{T}{\equiv} (\sim p \bullet \sim q)$$

This logical equivalence was explained in detail in Section 8.9. De Morgan's theorems have two variants. One variant asserts that when we deny that two propositions are *both* true, that is logically equivalent to asserting that either one of them is false, or the other one is false, or they are both false. (The negation of a conjunction is logically equivalent to the disjunction of the negation of the conjuncts.) The second variant of De Morgan's theorems asserts that when we deny that *either* of two propositions is true, that is logically equivalent to asserting that both of them are false. (The negation of a disjunction is logically equivalent to the conjunction of the negations of the disjuncts.)

These two biconditionals are tautologies, of course. That is, the expression of the material equivalence of the two sides of each is *always* true, and thus can have no false substitution instance. All ten of the logical equivalences now being recognized as rules of inference are tautological biconditionals in exactly this sense.

11. Commutation	**Com.**	$(p \lor q) \overset{T}{\equiv} (q \lor p)$
		$(p \bullet q) \overset{T}{\equiv} (q \bullet p)$

These two equivalences simply assert that the *order* of statement of the elements of a conjunction, or of a disjunction, does not matter. We are always permitted to turn them around, to *commute* them, because, whichever order happens to appear, the meanings remain exactly the same.

Recall that Rule 7, Simplification, permitted us to pull p from the conjunction $p \bullet q$, but not q. Now, with Commutation, we can always replace $p \bullet q$ with $q \bullet p$—so that, with Simplification and Commutation both at hand, we can readily establish the truth of each of the conjuncts in any conjunction we know to be true.

12. Association	**Assoc.**	$[p \lor (q \lor r)] \overset{T}{\equiv} [(p \lor q) \lor r]$
		$[p \bullet (q \bullet r)] \overset{T}{\equiv} [(p \bullet q) \bullet r]$

These two equivalences do no more than allow us to group statements differently. If we know three different statements to be true, to assert that p is true along with q and r clumped, is logically equivalent to asserting that p and q clumped is true along with r. Equivalence also holds if the three are grouped as disjuncts: p or the disjunction of $q \lor r$, is a grouping logically equivalent to the disjunction $p \lor q$, or r.

13. Distribution	**Dist.**	$[p \bullet (q \lor r)] \overset{T}{\equiv} [(p \bullet q) \lor (p \bullet r)]$
		$[p \lor (q \bullet r)] \overset{T}{\equiv} [(p \lor q) \bullet (p \lor r)]$

Of all the rules permitting replacement, this one may be the least obvious—but it too is a tautology, of course. Its also has two variants. The first variant asserts merely that the *conjunction* of one statement with the disjunction of two other statements is logically equivalent to either the disjunction of the first with the second or the disjunction of the first with the third. The second variant asserts merely that the *disjunction* of one statement with the conjunction of two others is logically equivalent to the conjunction of the disjunction of the first and the second and the disjunction of the first and the third. The rule is named *Distribution* because it *distributes* the first element of the three, exhibiting its logical connections with each of the other two statements separately.

14. Double Negation **D.N.** $p \overset{T}{\equiv} \sim\sim p$

Intuitively clear to everyone, this rule simply asserts that any statement is logically equivalent to the negation of the negation of that statement.

15. Transposition **Trans.** $(p \supset q) \overset{T}{\equiv} (\sim q \supset \sim p)$

This logical equivalence permits us to turn any conditional statement around. We know that if any conditional statement is true, then if its consequent is false its antecedent must also be false. Therefore any conditional statement is logically equivalent to the conditional statement asserting that the negation of its consequent implies the negation of its antecedent.

16. Material Implication **Impl.** $(p \supset q) \overset{T}{\equiv} (\sim p \vee q)$

This logical equivalence does no more than formulate the definition of material implication explained in Section 8.9 as a replacement that can serve as a rule of inference. There we saw that $p \supset q$ simply means that either the antecedent, p, is false or the consequent, q, is true.

As we go on to construct formal proofs, this definition of material implication will become very important, because it is often easier to manipulate or combine two statements if they have the same basic form—that is, if they are both in disjunctive form, or if they are both in implicative form. If one is in disjunctive form and the other is in implicative form, we can, using this rule, transform one of them into the form of the other. This will be very convenient.

17. Material Equivalence **Equiv.** $(p \equiv q) \overset{T}{\equiv} [(p \bullet q) \vee (\sim p \bullet \sim q)]$

$(p \equiv q) \overset{T}{\equiv} [(p \supset q) \bullet (q \supset p)]$

The two variants of this rule simply assert the two essential meanings of material equivalence, explained in detail in Section 8.8. There we explained that two statements are materially equivalent if they both have the same truth value; therefore (first variant) the assertion of their material equivalence (with the tribar, $\equiv$) is logically equivalent to asserting that they are both true, or that they are both false. We also explained at that point that if two statements are both true, they must materially imply one another, and likewise if they are both false, they must materially imply one another; therefore (second variant) the statement that they are materially equivalent is *logically* equivalent to the statement that they imply one another.

18. Exportation **Exp.** $[(p \bullet q) \supset r] \overset{\mathrm{T}}{\equiv} [p \supset (q \supset r)]$

This replacement rule states a logical biconditional that is intuitively clear upon reflection: If one asserts that two propositions conjoined are known to imply a third, that is logically equivalent to asserting that if one of those two propositions is known to be true, then the truth of the other must imply the truth of the third. Like all the others, this logical equivalence may be readily confirmed using a truth table.

19. Tautology **Taut.** $p \overset{\mathrm{T}}{\equiv} (p \vee p)$
$p \overset{\mathrm{T}}{\equiv} (p \bullet p)$

The two variants of this last rule are patently obvious but very useful. They say simply that any statement is logically equivalent to the disjunction of itself with itself, and that any statement is logically equivalent to the conjunction of itself with itself. It sometimes happens that, as the outcome of a series of inferences, we learn that either the proposition we seek to establish is true *or* that it is true. From this disjunction we may readily infer (using this rule) that the proposition in question is true. The same applies to the conjunction of a statement with itself.

It should be noted that the word "tautology" is used in three different senses. It can mean (1) a *statement form* all of whose substitution instances are true; in this sense the statement form $(p \supset q) \supset [p \supset (p \supset q)]$ is a tautology. It can mean (2) a *statement*—for example, $(A \supset B) \supset [A \supset (A \supset B)]$ whose specific form is a tautology in sense (1). And it can mean (3) *the particular logical equivalence* we have just introduced, number 19 in our list of rules of inference.

As we look back on these ten rules, we should be clear about what it is they make possible. They are not rules of "substitution" as that term is correctly used; we *substitute* statements for statement variables, as when we say that $A \supset B$ is a substitution instance of the expression $p \supset q$. In such operations we may substitute *any* statement for any statement variable so long as it is substituted for *every other* occurrence of that statement variable. But when these listed rules of replacement are applied, we exchange, or *replace*, a component of one statement *only by a statement that we know (by one of these ten rules) to be logically equivalent to that component*. For example, by transposition we may replace $A \supset B$ by $\sim B \supset \sim A$. And these rules permit us to replace one occurrence of that component without having to replace any other occurrence of it.

EXERCISES

The following set of arguments involves, in each case, one step only, in which one of the ten logical equivalences set forth in this section has been employed. Here are two examples, the first two in the exercise set immediately following.

EXAMPLE 1

$(A \supset B) \bullet (C \supset D)$

$\therefore (A \supset B) \bullet (\sim D \supset \sim C)$

SOLUTION

The conclusion of this simple argument is exactly like its premise, except for the fact that the second conjunct in the premise, $(C \supset D)$, has been replaced by the logically equivalent expression $(\sim D \supset \sim C)$. That replacement is plainly justified by the rule we call Transposition (Trans.):

$[(p \supset q) \overset{T}{\equiv} (\sim q \supset \sim p)$

EXAMPLE 2

$(E \supset F) \bullet (G \supset \sim H)$

$(\sim E \vee F) \bullet (G \supset \sim H)$

SOLUTION

In this case the conclusion differs from the premise only in the fact that the conditional statement $(E \supset F)$ has been replaced, as first conjunct, by the disjunctive statement $(\sim E \vee F)$. The rule permitting such a replacement, Material Implication (Impl.), has the form

$(p \supset q) \overset{T}{\equiv} (\sim p \vee q)$

For each of the following one-step arguments, state the one rule of inference by which its conclusion follows from its premise.

1. $(A \supset B) \bullet (C \supset D)$

$\therefore (A \supset B) \bullet (\sim D \supset \sim C)$

2. $(E \supset F) \bullet (G \supset \sim H)$

$\therefore (\sim E \vee F) \bullet (G \supset \sim H)$

3. $[I \supset (J \supset K)] \bullet (J \supset \sim I)$

$\therefore [(I \bullet J) \supset K] \bullet (J \supset \sim I)$

4. $[L \supset (M \vee N)] \vee [L \supset (M \vee N)]$

$\therefore L \supset (M \vee N)$

***5.** $O \supset [(P \supset Q) \bullet (Q \supset P)]$

$\therefore O \supset (P \equiv Q)$

6. $\sim(R \vee S) \supset (\sim R \vee \sim S)$

$\therefore (\sim R \bullet \sim S) \supset (\sim R \vee \sim S)$

7. $(T \vee \sim U) \bullet [(W \bullet \sim V) \supset \sim T]$
$\therefore (T \vee \sim U) \bullet [W \supset (\sim V \supset \sim T)]$

8. $(X \vee Y) \bullet (\sim X \vee \sim Y)$
$\therefore [(X \vee Y) \bullet \sim X] \vee [(X \vee Y) \bullet \sim Y]$

9. $Z \supset (A \supset B)$
$\therefore Z \supset (\sim\sim A \supset B)$

***10.** $[C \bullet (D \bullet \sim E)] \bullet [(C \bullet D) \bullet \sim E]$
$\therefore [(C \bullet D) \bullet \sim E] \bullet [(C \bullet D) \bullet \sim E]$

11. $(\sim F \vee G) \bullet (F \supset G)$
$\therefore (F \supset G) \bullet (F \supset G)$

12. $(H \supset \sim I) \supset (\sim I \supset \sim J)$
$\therefore (H \supset \sim I) \supset (J \supset I)$

13. $(\sim K \supset L) \supset (\sim M \vee \sim N)$
$\therefore (\sim K \supset L) \supset \sim(M \bullet N)$

14. $[(\sim O \vee P) \vee \sim Q] \bullet [\sim O \vee (P \vee \sim Q)]$
$\therefore [\sim O \vee (P \vee \sim Q)] \bullet [\sim O \vee (P \vee \sim Q)]$

***15.** $[(R \vee \sim S) \bullet \sim T] \vee [(R \vee \sim S) \bullet U]$
$\therefore (R \vee \sim S) \bullet (\sim T \vee U)$

16. $[V \supset \sim(W \vee X)] \supset (Y \vee Z)$
$\therefore \{[V \supset \sim(W \vee X)] \bullet [V \supset \sim(W \vee X)]\} \supset (Y \vee Z)$

17. $[(\sim A \bullet B) \bullet (C \vee D)] \vee [\sim(\sim A \bullet B) \bullet \sim(C \vee D)]$
$\therefore (\sim A \bullet B) \equiv (C \vee D)$

18. $[\sim E \vee (\sim\sim F \supset G)] \bullet [\sim E \vee (F \supset G)]$
$\therefore [\sim E \vee (F \supset G)] \bullet [\sim E \vee (F \supset G)]$

19. $[H \bullet (I \vee J)] \vee [H \bullet (K \supset \sim L)]$
$\therefore H \bullet [(I \vee J) \vee (K \supset \sim L)]$

***20.** $(\sim M \vee \sim N) \supset (O \supset \sim\sim P)$
$\therefore \sim(M \bullet N) \supset (O \supset \sim\sim P)$

9.7 The System of Natural Deduction

The nineteen rules of inference that have been set forth (nine elementary argument forms and ten logical equivalences) are all the rules that are needed in truth-functional logic. It is a *complete* system of natural deduction.* This means that, using this set of rules, which is compact and readily mastered, one can construct a formal proof of validity for *any* valid truth functional argument.[1]

Two seeming flaws of this list of nineteen rules deserve attention. First. The set is somewhat *redundant*, in the sense that these nineteen do not constitute

*This kind of completeness of a set of rules can be proved. One method of proving such completeness may be found in I. M. Copi, *Symbolic Logic*, 5th ed. (New York: Macmillan, 1979), chap. 8.

the bare minimum that would suffice for the construction of formal proofs of validity for extended arguments. To illustrate this we might note that *Modus Tollens* could be dropped from the list without any real weakening of our proof apparatus, because any line that depends on that rule can be justified by appealing instead to other rules in our list. Suppose, for example, we know that $A \supset D$ is true, and that $\sim D$ is true, and suppose we want to deduce that $\sim A$ is true. *Modus Tollens* allows us to do that directly. But if *Modus Tollens* were not included in the list of rules, we would still have no trouble deducing $\sim A$ from $A \supset D$ and $\sim D$; we would simply need to insert the intermediate line, $\sim D \supset \sim A$, which follows from $A \supset D$ by Transposition (Trans.), then obtain $\sim A$ from $\sim D \supset \sim A$ by *Modus Ponens* (M.P.). We keep *Modus Tollens* in the list because it is such a commonly used and intuitively obvious rule of inference. Others among the nineteen rules are redundant in this same sense.

Second, the list of nineteen rules may also be said to be *deficient* in one sense. Because the set of rules is short, there are some arguments that, although they are simple and intuitively valid, require several steps to prove. To illustrate this point, consider the argument

$A \vee B$

$\sim B$

$/\therefore A$

which is obviously valid. Its form, equally valid, is

$p \vee q$

$\sim q$

$/\therefore p$

But this elementary argument form has not been included as a rule of inference. No single rule of inference will serve in this case, so we must construct the proof using two rules of inference, commuting the first premise, and then applying Disjunctive Syllogism, thus:

1. $A \vee B$

2. $\sim B$ $/\therefore A$

3. $B \vee A$ 1, Com.

4. A 3, 2, D.S.

One may complain that the system is in this way clumsy, at times obliging a slow and tortuous path to a proof that ought to be easy and direct. But, there is good reason for this clumsiness. We certainly want a set of rules that is complete, as this set is. But we also want a set of rules that is short and easily mastered. We could add rules to our set—additional equivalences, or additional valid argument forms—but with each such addition our logical toolbox would become more congested and more difficult to command. We could delete some rules (e.g., *Modus Tollens,* as noted above), but with each such deletion the set, although smaller, would become even more clumsy, requiring extended proofs for very simple arguments. Long experience has taught that this set of nineteen rules serves as an ideal compromise: a list of rules of inference that is short enough to master fully, yet long enough to do all that one may need to do with reasonable efficiency.

There is an important difference between the first nine and the last ten rules of inference. *The first nine rules can be applied only to whole lines of a proof.* Thus, in a formal proof of validity, the statement A can be inferred from the statement $A \bullet B$ by Simplification only if $A \bullet B$ constitutes a whole line. It is obvious that A cannot be inferred validly either from $(A \bullet B) \supset C$ or from $C \supset (A \bullet B)$, because the latter two statements can be true while A is false. And the statement $A \supset C$ does not follow from the statement $(A \bullet B) \supset C$ by Simplification or by any other rule of inference. It does not follow at all, for if A is true and B and C are both false, $(A \bullet B) \supset C$ is true but $A \supset C$ is false. Again, although $A \vee B$ follows from A by Addition, we cannot infer $(A \vee B) \supset C$ from $A \supset C$ by Addition or by any other rule of inference. For if A and C are both false and B is true, $A \supset C$ is true but $(A \vee B) \supset C$ is false. On the other hand, *any of the last ten rules can be applied either to whole lines or to parts of lines*. Not only can the statement $A \supset (B \supset C)$ be inferred from the whole line $(A \bullet B) \supset C$ by Exportation, but from the line $[(A \bullet B) \supset C] \vee D$ we can infer $[A \supset (B \supset C)] \vee D$ by Exportation. By replacement, logically equivalent expressions can replace each other wherever they occur, even where they do not constitute whole lines of a proof. But the first nine rules of inference can be used only with whole lines of a proof serving as premises.

The notion of *formal proof* is an *effective* notion, which means that it can be decided quite mechanically, in a finite number of steps, whether or not a given sequence of statements constitutes a formal proof (with reference to a given list of rules of inference). No thinking is required, either in the sense of thinking about what the statements in the sequence "mean" or in the sense of using logical intuition to check any step's validity. Only two things are required. The first is the ability to see that a statement occurring in one place is precisely the same as a statement occurring in another, for we must be able to check that some statements in the proof are premises of the argument being proved valid and that the last statement in the proof is the conclusion of that argument. The second thing that is required is the ability to see whether a given statement has a certain pattern—that is, to see if it is a substitution instance of a given statement form.

Thus, any question about whether the numbered sequence of statements on page 401 is a formal proof of validity can easily be settled in a completely mechanical fashion. That lines 1 and 2 are the premises and line 4 is the conclusion of the given argument is obvious on inspection. That 3 follows from preceding lines by one of the given rules of inference can be decided in a finite number of steps—even where the notation "1, Com." is not written at the side. The explanatory notation in the second column is a help and should always be included, but it is not, strictly speaking, a necessary part of the proof itself. At every line, there are only finitely many preceding lines and only finitely many rules of inference or reference forms to be consulted. Although it is time-consuming, it can be verified by inspection and comparison of shapes that 3 does not follow from 1 and 2 by *modus ponens*, or by *modus tollens*, or by a hypothetical syllogism, . . . and so on, until in following this procedure we come to the question of whether 3 follows from 1 by the principle of commutation, and there we see, simply by looking at the forms, that it does. In the same way, the legitimacy of *any* statement in *any* formal proof can be tested in a finite number of steps, none of which involves anything more than comparing forms or shapes.

To preserve this effectiveness, we require that only one step be taken at a time. One might be tempted to shorten a proof by combining steps, but the space and time saved are negligible. More important is the effectiveness we achieve by taking each step by means of one single rule of inference.

Although a formal proof of validity is effective in the sense that it can be mechanically decided of any given sequence whether it is a proof, *constructing* a formal proof is not an effective procedure. In this respect, formal proofs differ from truth tables. The making of truth tables is completely mechanical: given any argument of the sort with which we are now concerned, we can always construct a truth table to test its validity by following the simple rules of procedure set forth in Chapter 8. But we have no effective or mechanical rules for the construction of formal proofs. Here we must think or "figure out" where to begin and how to proceed. Nevertheless, proving an argument valid by constructing a formal proof of its validity is much easier than the purely mechanical construction of a truth table with perhaps hundreds or even thousands of rows.

Although we have no purely mechanical rules for constructing formal proofs, some rough-and-ready rules of thumb or hints on procedure may be suggested. The first is simply to begin deducing conclusions from the given premises by the given rules of inference. As more and more of these subconclusions become available as premises for further deductions, the greater is the likelihood of being able to see how to deduce the conclusion of the argument to be proved valid. Another hint is to try to eliminate statements that occur in the premises but not in the conclusion. Such elimination can proceed, of course, only in accordance with the rules of inference, but the rules contain many techniques for eliminating statements. Simplification is such a rule, whereby the right-hand conjunct can be dropped from a whole line that is a conjunction. And Commutation is a rule that permits switching the left-hand conjunct of a conjunction over to the right-hand side, from which it can be dropped by Simplification. The "middle" term q can be eliminated by a Hypothetical Syllogism given two statements of the patterns $p \supset q$ and $q \supset r$. Distribution is a useful rule for transforming a disjunction of the pattern $p \vee (q \bullet r)$ into the conjunction $(p \vee q) \bullet (p \vee r)$, whose right-hand conjunct can then be eliminated by Simplification. Another rule of thumb is to introduce by means of Addition a statement that occurs in the conclusion but not in any premise. Yet another method, often very productive, is to work backward from the conclusion by looking for some statement or statements from which it can be deduced, and then trying to deduce those intermediate statements from the premises. There is, however, no substitute for practice as a method of acquiring facility in the construction of formal proofs.

9.8 Constructing Formal Proofs Using the Nineteen Rules of Inference

Having now a set of nineteen rules at our disposition, rather than just nine, the task of constructing formal proofs becomes somewhat more complicated. The objective remains the same, of course, but the process of devising the proof involves inspection of a larger intellectual toolbox. The unbroken logical chain that we devise, leading ultimately to the conclusion, may now include steps justified by either an elementary valid argument form or a logical equivalence. Any given proof is likely to employ rules of both kinds. The balance or order of their use is determined only by the logical need encountered as we implement the strategy that leads to the consummation of the proof.

Following is a set of flawless formal proofs, each of which relies on rules of both kinds. To become accustomed to the use of the full set of rules, we examine each of these proofs to determine what rule has been used to justify each step in that proof, noting that justification to right of each line. We begin with two examples.

EXAMPLE 1

1.

1. $A \supset B$

2. $C \supset {\sim}B$

$\therefore A \supset {\sim}C$

3. ${\sim}{\sim}B \supset {\sim}C$

4. $B \supset {\sim}C$

5. $A \supset {\sim}C$

OVERVIEW

The Rules of Inference

Nineteen rules of inference are specified for use in constructing formal proofs of validity. They are as follows:

Elementary Valid Argument Form	Logically Equivalent Expressions
1. ***Modus Ponens*** (M.P.) $p \supset q, p, \therefore q$	**10. De Morgan's theorems** (De M.) $\sim(p \bullet q) \overset{T}{\equiv} (\sim p \vee \sim q)$ $\sim(p \vee q) \overset{T}{\equiv} (\sim p \bullet \sim q)$
2. ***Modus Tollens*** (M.T.) $p \supset q, \sim q, \therefore \sim p$	**11. Commutation** (Com.) $(p \vee q) \overset{T}{\equiv} (q \vee p)$ $(p \bullet q) \overset{T}{\equiv} (q \bullet p)$
3. Hypothetical Syllogism (H.S.) $p \supset q, q \supset r, \therefore p \supset r$	**12. Association** (Assoc.) $[p \vee (q \vee r)] \overset{T}{\equiv} [(p \vee q) \vee r]$ $[p \bullet (q \bullet r)] \overset{T}{\equiv} [(p \bullet q) \bullet r]$
4. Disjunctive Syllogism (D.S.) $p \vee q, \sim p, \therefore q$	**13. Distribution** (Dist.) $[p \bullet (q \vee r)] \overset{T}{\equiv} [(p \bullet q) \vee (p \bullet r)]$ $[p \vee (q \bullet r)] \overset{T}{\equiv} [(p \vee q) \bullet (p \vee r)]$
5. Constructive Dilemma (C.D.) $(p \supset q) \bullet (r \supset s), p \vee r, \therefore q \vee s$	**14. Double Negation** (D.N.) $p \overset{T}{\equiv} \sim\sim p$
6. Absorption (Abs.) $p \supset q, \therefore p \supset (p \bullet q)$	**15. Transposition** (Trans.) $(p \supset q) \overset{T}{\equiv} (\sim q \supset \sim p)$
7. Simplification (Simp.) $p \bullet q, \therefore p$	**16. Material Implication** (Impl.) $(p \supset q) \overset{T}{\equiv} (\sim p \vee q)$
8. Conjunction (Conj.) $p, q, \therefore p \bullet q$	**17. Material Equivalence** (Equiv.) $(p \equiv q) \overset{T}{\equiv} [(p \supset q) \bullet (q \supset p)]$ $(p \equiv q) \overset{T}{\equiv} [(p \bullet q) \vee (\sim p \bullet \sim q)]$
9. Addition (Add.) $p, \therefore p \vee q$	**18. Exportation** (Exp.) $[(p \bullet q) \supset r] \overset{T}{\equiv} [p \supset (q \supset r)]$
	19. Tautology (Taut.) $p \overset{T}{\equiv} (p \vee p)$ $p \overset{T}{\equiv} (p \bullet p)$

SOLUTION

Line 3 is simply line 2 transposed; we write beside line 3: 2, Trans.
Line 4 is simply line 3 with ~~*B* replaced by *B*; so we write beside line 4: 3, D.N.
Line 5 applies the Hypothetical Syllogism argument form to lines 1 and 4. We write beside line 5: 1, 4, H. S.

EXAMPLE 2

2. **1.** $(D \bullet E) \supset F$
2. $(D \supset F) \supset G$
$\therefore E \supset G$
3. $(E \bullet D) \supset F$
4. $E \supset (D \supset F)$
5. $E \supset G$

SOLUTION

Line 3 merely commutes ($D \bullet E$) from line 1; we write: 1, Com.
Line 4 applies Exportation to line 3; we write: 3, Exp.
Line 5 applies Hypothetical Syllogism to lines 4 and 2; we write; 4, 2, H.S.

EXERCISES

A. For each numbered line (that is not a premise) in each of the formal proofs that follow below, state the rule of inference that justifies it.

1. **1.** $A \supset B$
2. $C \supset \sim B$
$\therefore A \supset \sim C$
3. $\sim\sim B \supset \sim C$
4. $B \supset \sim C$
5. $A \supset \sim C$

2. **1.** $(D \bullet E) \supset F$
2. $(D \supset F) \supset G$
$\therefore E \supset G$
3. $(E \bullet D) \supset F$
4. $E \supset (D \supset F)$
5. $E \supset G$

3. **1.** $(H \vee I) \supset [J \bullet (K \bullet L)]$
2. I
$\therefore J \bullet K$
3. $I \vee H$
4. $H \vee I$
5. $J \bullet (K \bullet L)$
6. $(J \bullet K) \bullet L$
7. $J \bullet K$

4. **1.** $(M \vee N) \supset (O \bullet P)$
2. $\sim O$
$\therefore \sim M$
3. $\sim O \vee \sim P$
4. $\sim(O \bullet P)$
5. $\sim(M \vee N)$
6. $\sim M \bullet \sim N$
7. $\sim M$

***5.**
1. $(Q \lor \sim R) \lor S$
2. $\sim Q \lor (R \cdot \sim Q)$
$\therefore R \supset S$
3. $(\sim Q \lor R) \cdot (\sim Q \lor \sim Q)$
4. $(\sim Q \lor \sim Q) \cdot (\sim Q \lor R)$
5. $\sim Q \lor \sim Q$
6. $\sim Q$
7. $Q \lor (\sim R \lor S)$
8. $\sim R \lor S$
9. $R \supset S$

6.
1. $T \cdot (U \lor V)$
2. $T \supset [U \supset (W \cdot X)]$
3. $(T \cdot V) \supset \sim(W \lor X)$
$\therefore W \equiv X$
4. $(T \cdot U) \supset (W \cdot X)$
5. $(T \cdot V) \supset (\sim W \cdot \sim X)$
6. $[(T \cdot U) \supset (W \cdot X)] \cdot$
$[(T \cdot V) \supset (\sim W \cdot \sim X)]$
7. $(T \cdot U) \lor (T \cdot V)$
8. $(W \cdot X) \lor (\sim W \cdot \sim X)$
9. $W \equiv X$

7.
1. $Y \supset Z$
2. $Z \supset [Y \supset (R \lor S)]$
3. $R \equiv S$
4. $\sim(R \cdot S)$
$\therefore \sim Y$
5. $(R \cdot S) \lor (\sim R \cdot \sim S)$
6. $\sim R \cdot \sim S$
7. $\sim(R \lor S)$
8. $Y \supset [Y \supset (R \lor S)]$
9. $(Y \cdot Y) \supset (R \lor S)$
10. $Y \supset (R \lor S)$
11. $\sim Y$

8.
1. $A \supset B$
2. $B \supset C$
3. $C \supset A$
4. $A \supset \sim C$
$\therefore \sim A \cdot \sim C$
5. $A \supset C$
6. $(A \supset C) \cdot (C \supset A)$
7. $A \equiv C$
8. $(A \cdot C) \lor (\sim A \cdot \sim C)$
9. $\sim A \lor \sim C$
10. $\sim(A \cdot C)$
11. $\sim A \cdot \sim C$

9.
1. $(D \cdot E) \supset \sim F$
2. $F \lor (G \cdot H)$
3. $D \equiv E$
$\therefore D \supset G$
4. $(D \supset E) \cdot (E \supset D)$
5. $D \supset E$
6. $D \supset (D \cdot E)$
7. $D \supset \sim F$
8. $(F \lor G) \cdot (F \lor H)$
9. $F \lor G$
10. $\sim\sim F \lor G$
11. $\sim F \supset G$
12. $D \supset G$

***10.**
1. $(I \lor \sim\sim J) \cdot K$
2. $[\sim L \supset \sim(K \cdot J)] \cdot$
$[K \supset (I \supset \sim M)]$
$\therefore \sim(M \cdot \sim L)$
3. $[(K \cdot J) \supset L] \cdot$
$[K \supset (I \supset \sim M)]$
4. $[(K \cdot J) \supset L] \cdot$
$[(K \cdot I) \supset \sim M]$
5. $(I \lor J) \cdot K$
6. $K \cdot (I \lor J)$
7. $(K \cdot I) \lor (K \cdot J)$
8. $(K \cdot J) \lor (K \cdot I)$
9. $L \lor \sim M$
10. $\sim M \lor L$
11. $\sim M \lor \sim\sim L$
12. $\sim(M \cdot \sim L)$

We now advance to the construction of formal proofs using the full set of rules of inference. We begin with simple arguments, whose proofs require only two statements added to the premises. Each of those statements, of course, may be justified by either an elementary valid argument form or by one of the rules of replacement. We begin with two examples, the first two exercises of Set **B,** immediately following.

EXAMPLE 1

1. $A \supset \sim A$

   $\therefore \sim A$

SOLUTION

The first step in this proof, obviously, must manipulate the single premise. What can we do with it that will be helpful? If we apply Material Implication (Impl.), we will obtain a statement, $\sim A \vee \sim A$, to which we can apply the valid argument form Tautology (Taut.), and that will yield the conclusion we seek. So the proof is

1. $A \supset \sim A$

   $\therefore \sim A$
2. $\sim A \vee \sim A$ 1, Impl.
3. $\sim A$ 2, Taut.

EXAMPLE 2

2. $B \bullet (C \bullet D)$

   $\therefore C \bullet (D \bullet B)$

SOLUTION

In this proof we need only rearrange the statements, all of which are given as true. In the first step we can commute the main conjunction of the first premise, which will yield $(C \bullet D) \bullet B$. Then we need only regroup the three statements by Association. So the proof is

1. $B \bullet (C \bullet D)$

   $\therefore C \bullet (D \bullet B)$
2. $(C \bullet D) \bullet B$ 1, Com.
3. $C \bullet (D \bullet B)$ 2, Assoc.

In this proof, as in all formal proofs, the last line of the sequence we construct *is* the conclusion we are aiming to deduce.

EXERCISES

B. For each of the following arguments, adding just two statements to the premises will produce a formal proof of its validity. Construct a formal proof for each of these arguments.

In these formal proofs, and in all the proofs to follow in later sections, note to the right of each line the rule of inference that justifies that line of the proof. It is most convenient if the justification specifies first the number of the line (or lines) being used, and then the name (abbreviated) of the rule of inference that has been applied to those numbered lines.

1. $A \supset \sim A$
$\therefore \sim A$

2. $B \bullet (C \bullet D)$
$\therefore C \bullet (D \bullet B)$

3. E
$\therefore (E \vee F) \bullet (E \vee G)$

4. $H \vee (I \bullet J)$
$\therefore H \vee I$

***5.** $\sim K \vee (L \supset M)$
$\therefore (K \bullet L) \supset M$

6. $(N \bullet O) \supset P$
$\therefore (N \bullet O) \supset [N \bullet (O \bullet P)]$

7. $Q \supset [R \supset (S \supset T)]$
$Q \supset (Q \bullet R)$
$\therefore Q \supset (S \supset T)$

8. $U \supset \sim V$
V
$\therefore \sim U$

9. $W \supset X$
$\sim Y \supset \sim X$
$\therefore W \supset Y$

***10.** $Z \supset A$
$\sim A \vee B$
$\therefore Z \supset B$

11. $C \supset \sim D$
$\sim E \supset D$
$\therefore C \supset \sim\sim E$

12. $F \equiv G$
$\sim(F \bullet G)$
$\therefore \sim F \bullet \sim G$

13. $H \supset (I \bullet J)$
$I \supset (J \supset K)$
$\therefore H \supset K$

14. $(L \supset M) \bullet (N \supset M)$
$L \vee N$
$\therefore M$

***15.** $(O \vee P) \supset (Q \vee R)$
$P \vee O$
$\therefore Q \vee R$

16. $(S \bullet T) \vee (U \bullet V)$
$\sim S \vee \sim T$
$\therefore U \bullet V$

17. $(W \bullet X) \supset Y$
$(X \supset Y) \supset Z$
$\therefore W \supset Z$

18. $(A \vee B) \supset (C \vee D)$
$\sim C \bullet \sim D$
$\therefore \sim(A \vee B)$

19. $(E \bullet F) \supset (G \bullet H)$
$F \bullet E$
$\therefore G \bullet H$

***20.** $I \supset [J \vee (K \vee L)]$
$\sim[(J \vee K) \vee L]$
$\therefore \sim I$

21. $(M \supset N) \bullet (\sim O \vee P)$
$M \vee O$
$\therefore N \vee P$

22. $(\sim Q \supset \sim R) \bullet (\sim S \supset \sim T)$
$\sim\sim(\sim Q \vee \sim S)$
$\therefore \sim R \vee \sim T$

23. $\sim[(U \supset V) \bullet (V \supset U)]$
$(W \equiv X) \supset (U \equiv V)$
$\therefore \sim(W \equiv X)$

24. $(Y \supset Z) \bullet (Z \supset Y)$
$\therefore (Y \bullet Z) \vee (\sim Y \bullet \sim Z)$

***25.** $A \vee B$
$C \vee D$
$\therefore [(A \vee B) \bullet C] \vee [(A \vee B) \bullet D]$

26. $[(E \vee F) \bullet (G \vee H)] \supset (F \bullet I)$
$(G \vee H) \bullet (E \vee F)$
$\therefore F \bullet I$

27. $(J \bullet K) \supset [(L \bullet M) \vee (N \bullet O)]$
$\sim(L \bullet M) \bullet \sim(N \bullet O)$
$\therefore \sim(J \bullet K)$

28. $(P \supset Q) \supset [(R \vee S) \bullet (T \equiv U)]$
$(R \vee S) \supset [(T \equiv U) \supset Q]$
$\therefore (P \supset Q) \supset Q$

29. $[V \bullet (W \vee X)] \supset (Y \supset Z)$
$\sim(Y \supset Z) \vee (\sim W \equiv A)$
$\therefore [V \bullet (W \vee X)] \supset (\sim W \equiv A)$

***30.** $\sim[(B \supset \sim C) \bullet (\sim C \supset B)]$
$(D \bullet E) \supset (B \equiv \sim C)$
$\therefore \sim(D \bullet E)$

As we advance to arguments whose formal proofs require three lines added to the premises, it becomes important to devise a strategy for determining the needed sequence. Most such arguments remain fairly simple, but the path to the proof may sometimes be less than obvious. Again we begin with two examples, the first two exercises of Set **C,** which follows the examples.

EXAMPLE 1

1. $\sim A \supset A$

$\therefore A$

SOLUTION

We have only one premise with which to work. It is often fruitful to convert conditional statements into disjunctive statements. Doing that with line 1 (using Impl.) will yield $\sim\sim A$ as the first of the disjuncts; that component may be readily replaced with *A;* then, applying the argument form Tautology will give us what we aim for. The proof is

1. $\sim A \supset A$

$\therefore A$

2. $\sim\sim A \vee A$ 1, Impl.

3. $A \vee A$ 2, D.N.

4. A 3, Taut.

EXAMPLE 2

1. $\sim B \vee (C \bullet D)$

$B \supset C$

SOLUTION

The single premise in this argument contains the statement D. We need a proof whose conclusion is $B \supset C$, and therefore we must somehow eliminate that D. How can we do that? We can break apart the statement $(C \bullet D)$ by *distributing* the statement $\sim B$. Distribution asserts, in one of its variants, that $[p \vee (q \bullet r)] \overset{T}{\equiv} [(p \vee q) \bullet (p \vee r)]$. Applied to line 1, that replacement will yield $(\sim B \vee C) \bullet (\sim B \vee D)$. These two expressions are just conjoined, so by simplification we may extract $(\sim B \vee C)$. This statement may be replaced, using Impl., by $B \supset C$, which is the conclusion sought. The proof is

1. $\sim B \vee (C \bullet D)$

$\therefore B \supset C$

2. $(\sim B \vee C) \bullet (\sim B \vee D)$ — 1, Dist.

3. $\sim B \vee C$ — 2, Simp.

4. $B \supset C$ — 3, Impl.

EXERCISES

C. For each of the following arguments, a formal proof may be constructed by adding just three statements to the premises. Construct a formal proof of validity for each of them.

1. $\sim A \supset A$
$\therefore A$

2. $\sim B \vee (C \bullet D)$
$\therefore B \supset C$

3. $E \vee (F \bullet G)$
$\therefore E \vee G$

4. $H \bullet (I \bullet J)$
$\therefore J \bullet (I \bullet H)$

***5.** $[(K \vee L) \vee M] \vee N$
$\therefore (N \vee K) \vee (L \vee M)$

6. $O \supset P$
$P \supset \sim P$
$\therefore \sim O$

7. $Q \supset (R \supset S)$
$Q \supset R$
$\therefore Q \supset S$

8. $T \supset U$
$\sim(U \vee V)$
$\therefore \sim T$

9. $W \bullet (X \vee Y)$
$\sim W \vee \sim X$
$\therefore W \bullet Y$

***10.** $(Z \vee A) \vee B$
$\sim A$
$\therefore Z \vee B$

11. $(C \vee D) \supset (E \bullet F)$
$D \vee C$
$\therefore E$

12. $G \supset H$
$H \supset G$
$\therefore (G \bullet H) \vee (\sim G \bullet \sim H)$

13. $(I \supset J) \bullet (K \supset L)$
$I \vee (K \bullet M)$
$\therefore J \vee L$

14. $(N \bullet O) \supset P$
$(\sim P \supset \sim O) \supset Q$
$\therefore N \supset Q$

***15.** $[R \supset (S \supset T)] \bullet [(R \bullet T) \supset U]$
$R \bullet (S \vee T)$
$\therefore T \vee U$

Formal proofs of validity sometimes require many steps or lines in the needed sequence. We will find that certain *patterns* of inference are encountered repeatedly in longer proofs. It is wise to become familiar with these recurring patterns.

This may be nicely illustrated using the first two exercises of Set **D,** which follows immediately below. First, suppose that a given statement, *A,* is known to be false. The next stage of the proof may require that we prove that some different statement, say *B,* is implied by the truth of the statement that we know is false. This can be easily proved, and the pattern is not uncommon. Put formally, how may we infer $A \supset B$ from $\sim A$? Let us examine the argument.

EXAMPLE 1

1. $\sim A$

$\therefore A \supset B$

SOLUTION

If $\sim A$ is known to be true, as here, then *A* must be false. A false statement materially implies any other statement. So $A \supset B$ must be true, whatever *B* may assert, if we know that $\sim A$ is true. In this case, $\sim A$ is given as premise; we only need to add the desired *B* and then apply Implication. The proof of the argument (or the proof segment, when it is a part of some longer proof) is

1. $\sim A$

$\therefore A \supset B$

2. $\sim A \vee B$ 1, Add.

3. $A \supset B$ 2, Impl.

EXAMPLE 2

1. C

$\therefore D \supset C$

This pattern arises very frequently. The truth of some statement C is known. In this case it is given as a premise; in some longer proof we might have established its truth at some other point in the sequence. We know that a true statement is materially implied by any statement whatever. Therefore any statement we choose, D, must imply C. Put formally, how may we infer $D \supset C$ from C?

SOLUTION

D does not appear in the premise but it does appear in the conclusion, so we must somehow get D into the sequence of steps. We could simply add D, but that won't succeed—because after commuting that disjunction, and replacing it, using Impl., with a conditional, we wind up with $\sim D \supset C$, which is certainly not the conclusion we were after. We want $D \supset C$. To obtain this needed result we must, in the very first step, add $\sim D$ rather than D. This we certainly may do, because Addition permits us to add disjunctively any statement whatever to a statement we know to be true. Then, applying Com. and Impl. will give us what we seek. The formal proof of the argument in this case (or the proof segment, when it occurs as part of a longer proof) is

1. C
$\therefore D \supset C$
2. $C \vee \sim D$ 1, Add.
3. $\sim D \vee C$ 2, Com.
4. $D \supset C$ 3, Impl.

EXERCISES

D. Each of the following exercises immediately below exhibits a commonly recurring pattern. Constructing the formal proof in each case will take some ingenuity, and (in a few cases) the proof will require eight or nine lines. However, most of these proofs will present little difficulty, and devising the strategies needed to produce them is excellent practice. Construct a formal proof for each of the following arguments.

1. $\sim A$
$\therefore A \supset B$

2. C
$\therefore D \supset C$

3. $E \supset (F \supset G)$
$\therefore F \supset (E \supset G)$

4. $H \supset (I \bullet J)$
$\therefore H \supset I$

***5.** $K \supset L$
$\therefore K \supset (L \vee M)$

6. $N \supset O$
$\therefore (N \bullet P) \supset O$

7. $(Q \vee R) \supset S$
$\therefore Q \supset S$

8. $T \supset U$
$T \supset V$
$\therefore T \supset (U \bullet V)$

9. $W \supset X$
$Y \supset X$
$\therefore (W \vee Y) \supset X$

***10.** $Z \supset A$
$Z \vee A$
$\therefore A$

When, after substantial practice, one has become well familiar with the nineteen rules of inference, and is comfortable in applying them, it is time to tackle formal proofs that are longer and more convoluted. The three sets of exercises that follow will present some challenges, but devising these formal proofs will be a source of genuine satisfaction. The great mathematician, G. H. Hardy, long ago observed that there is a natural and widespread thirst for intellectual "kick"—and that "nothing else has quite the kick" that solving logical problems has.

Arguments in a natural language, as in the last two sets, need no further explanation. After translating them into symbolic notation, using the suggested abbreviations, the procedure for constructing the proofs is no different from that used when we begin with an argument formulated in symbols. Before adventuring further in the realm of logical proofs, it will be helpful to examine, from Exercise Set **E,** two examples of the kinds of formal proofs we will be dealing with from this point forward.

The arguments presented in all these sets of exercises are valid. Therefore, because the system of nineteen rules we have devised is known to be complete, we may be certain that a formal proof for each one of those arguments can be constructed. But the path from the premises to the conclusion may be far from obvious. In each case, some plan of action must be devised as one goes forward.

We illustrate the need for a plan of attack, and the way in which such a plan may be devised, by examining very closely two of the exercises—the first and the last—in Set **E,** which follows on pages 416–417.

EXAMPLE 1

1. $A \supset {\sim}B$
${\sim}(C \bullet {\sim}A)$
$\therefore C \supset {\sim}B$

SOLUTION

In this argument the conclusion unites a statement that appears in the second premise, C, with a statement that appears in the first premise, ${\sim}B$.

How shall we effect that unification? The first premise is a conditional whose consequent, $\sim B$, is also the consequent of the conclusion. The second premise contains the negation of the antecedent of the first premise, $\sim A$. If we can manipulate the second premise so as to emerge with $C \supset A$, we can achieve the needed unification with H.S. We can do that. If we apply De M. to the second premise we will get a disjunction that, when replaced by a conditional using Impl., will be one short step away from the conditional needed. The formal proof is

1.	$A \supset \sim B$	
2.	$\sim(C \bullet \sim A)$	
	$\therefore C \supset \sim B$	
3.	$\sim C \lor \sim\sim A$	2, De M.
4.	$C \supset \sim\sim A$	3, Impl.
5.	$C \supset A$	4, D.N.
6.	$C \supset \sim B$	5, 1, H.S.

Note that in this proof, as in many, a somewhat different sequence can be devised that leads to the same successful result. Line 3 is a needed first step. But we could have kept the disjunction on line 4, at that point only replacing $\sim\sim A$ by A:

4.	$\sim C \lor A$	3, D.N.	Replacement of this by a conditional is then needed.
5.	$C \supset A$	4, Impl.	H.S. will then again concludes the proof:
6.	$C \supset \sim B$	5, 1, H.S.	

The difference between the two sequences, in this case, is chiefly one of order. Sometimes there are alternative proofs using quite different strategies altogether.

Let us examine, as our final explication of the detail of formal proofs, one of the longer arguments in Set **E**, exercise 20, in which devising the strategy needed is more challenging.

EXAMPLE 2

20. $(R \lor S) \supset (T \bullet U)$
$\sim R \supset (V \supset \sim V)$
$\sim T$
$\therefore \sim V$

The conclusion we seek, $\sim V$, appears only in the second of three premises, and even there it is buried in a longer compound statement. How may we prove it? We notice that the consequent of the second premise $(V \supset \sim V)$ is a conditional that, if replaced by a disjunction, yields $\sim V \lor \sim V$, which in turn yields $\sim V$ independently, by Taut. Might we obtain $(V \supset \sim V)$ by M.P.? For that we need $\sim R$. R appears in the first premise, as part of a disjunction; if we can obtain the negation of that disjunction, we may derive ~*R*. To obtain the negation of that disjunction we need the negation of the consequent of the first premise, so M.T. may be applied. It can be seen that the negation of that consequent (*T* • *U*) should be available, because the third premise asserts ~*T*, and if ~*T* is true, than (*T* • *U*) surely is false. How may we show this? We look at the negation that we seek: ~(*T* • *U*). This is logically equivalent to ~*T* ∨ ~*U*. We can establish ~*T* ∨ ~*U* simply by adding ~*U* to ~*T*. All the elements of the plan are before us; we need only put them into a logical sequence that is watertight. This is not at all difficult once the strategy has been devised. We begin by building the negation of the consequent of the first premise, then derive the negation of the antecedent of that premise, then obtain ~*R*. With ~*R* we establish (*V* ⊃ ~*V*) by M.P., and the conclusion we want to prove is at hand. The actual lines of the formal proof are

1. (*R* ∨ *S*) ⊃ (*T* • *U*)		
2. ~*R* ⊃ (*V* ⊃ ~*V*)		
3. ~*T*		
∴ ~*V*		
4. ~*T* ∨ ~*U*	3, Add.	
5. ~(*T* • *U*)	4, De M.	
6. ~(*R* ∨ *S*)	1, 5, M. T.	
7. ~*R* • ~*S*	6, De M.	
8. ~*R*	7, Simp.	
9. *V* ⊃ ~*V*	2, 8, M.P.	
10. ~*V* ∨ ~*V*	9, Impl.	
11. ~*V*	10, Taut.	Q.E.D.

At the conclusion of a proof it is traditional practice to place the letters Q.E.D.—a minor exhibition of pride in the form of an acronym for the Latin expression, *Quod erat demonstrandum*—What was to be demonstrated.

EXERCISES

E. Construct a formal proof of validity for each of the following arguments.

1. *A* ⊃ ~*B*
~(*C* • ~*A*)
∴ *C* ⊃ ~*B*

2. (*D* • ~*E*) ⊃ *F*
~(*E* ∨ *F*)
∴ ~*D*

3. (*G* ⊃ ~*H*) ⊃ *I*
~(*G* • *H*)
∴ *I* ∨ ~*H*

4. (*J* ∨ *K*) ⊃ ~*L*
L
∴ ~*J*

***5.** $[(M \bullet N) \bullet O] \supset P$
$Q \supset [(O \bullet M) \bullet N]$
$\therefore \sim Q \vee P$

6. $R \vee (S \bullet \sim T)$
$(R \vee S) \supset (U \vee \sim T)$
$\therefore T \supset U$

7. $(\sim V \supset W) \bullet (X \supset W)$
$\sim(\sim X \bullet V)$
$\therefore W$

8. $[(Y \bullet Z) \supset A] \bullet [(Y \bullet B) \supset C]$
$(B \vee Z) \bullet Y$
$\therefore A \vee C$

9. $\sim D \supset (\sim E \supset \sim F)$
$\sim(F \bullet \sim D) \supset \sim G$
$\therefore G \supset E$

***10.** $[H \vee (I \vee J)] \supset (K \supset J)$
$L \supset [I \vee (J \vee H)]$
$\therefore (L \bullet K) \supset J$

11. $M \supset N$
$M \supset (N \supset O)$
$\therefore M \supset O$

12. $(P \supset Q) \bullet (P \vee R)$
$(R \supset S) \bullet (R \vee P)$
$\therefore Q \vee S$

13. $T \supset (U \bullet V)$
$(U \vee V) \supset W$
$\therefore T \supset W$

14. $(X \vee Y) \supset (X \bullet Y)$
$\sim(X \vee Y)$
$\therefore \sim(X \bullet Y)$

***15.** $(Z \supset Z) \supset (A \supset A)$
$(A \supset A) \supset (Z \supset Z)$
$\therefore A \supset A$

16. $\sim B \vee [(C \supset D) \bullet (E \supset D)]$
$B \bullet (C \vee E)$
$\therefore D$

17. $\sim F \vee \sim[\sim(G \bullet H) \bullet (G \vee H)]$
$(G \supset H) \supset [(H \supset G) \supset I]$
$\therefore F \supset (F \bullet I)$

18. $J \vee (\sim J \bullet K)$
$J \supset L$
$\therefore (L \bullet J) \equiv J$

19. $(M \supset N) \bullet (O \supset P)$
$\sim N \vee \sim P$
$\sim(M \bullet O) \supset Q$
$\therefore Q$

***20.** $(R \vee S) \supset (T \bullet U)$
$\sim R \supset (V \supset \sim V)$
$\sim T$
$\therefore \sim V$

F. Construct a formal proof of validity for each of the following arguments, in each case using the suggested notation.

***1.** Either the manager didn't notice the change or else he approves of it. He noticed it all right. So he must approve of it. *(N, A)*

2. The oxygen in the tube either combined with the filament to form an oxide or else it vanished completely. The oxygen in the tube could not have vanished completely. Therefore the oxygen in the tube combined with the filament to form an oxide. *(C, V)*

3. If a political leader who sees her former opinions to be wrong does not alter her course, she is guilty of deceit; and if she does alter her course, she is open to a charge of inconsistency. She either alters her course or she doesn't. Therefore either she is guilty of deceit or else she is open to a charge of inconsistency. *(A, D, I)*

4. It is not the case that she either forgot or wasn't able to finish. Therefore she was able to finish. *(F, A)*

***5.** If the litmus paper turns red, then the solution is acid. Hence if the litmus paper turns red, then either the solution is acid or something is wrong somewhere. *(R, A, W)*

6. She can have many friends only if she respects them as individuals. If she respects them as individuals, then she cannot expect them all to behave alike. She does have many friends. Therefore she does not expect them all to behave alike. *(F, R, E)*

7. If the victim had money in his pockets, then robbery wasn't the motive for the crime. But robbery or vengeance was the motive for the crime. The victim had money in his pockets. Therefore vengeance must have been the motive for the crime. *(M, R, V)*

8. Napoleon is to be condemned if he usurped power that was not rightfully his own. Either Napoleon was a legitimate monarch or else he usurped power that was not rightfully his own. Napoleon was not a legitimate monarch. So Napoleon is to be condemned. *(C, U, L)*

9. If we extend further credit on the Wilkins account, they will have a moral obligation to accept our bid on their next project. We can figure a more generous margin of profit in preparing our estimates if they have a moral obligation to accept our bid on their next project. Figuring a more generous margin of profit in preparing our estimates will cause our general financial condition to improve considerably. Hence a considerable improvement in our general financial condition will follow from our extension of further credit on the Wilkins account. *(C, M, P, I)*

***10.** If the laws are good and their enforcement is strict, then crime will diminish. If strict enforcement of laws will make crime diminish, then our problem is a practical one. The laws are good. Therefore our problem is a practical one. *(G, S, D, P)*

11. Had Roman citizenship guaranteed civil liberties, then Roman citizens would have enjoyed religious freedom. Had Roman citizens enjoyed religious freedom, there would have been no persecution of the early Christians. But the early Christians were persecuted. Hence Roman citizenship could not have guaranteed civil liberties. *(G, F, P)*

12. If the first disjunct of a disjunction is true, the disjunction as a whole is true. Therefore if both the first and second disjuncts of the disjunction are true, then the disjunction as a whole is true. *(F, W, S)*

13. If the new courthouse is to be conveniently located, it will have to be situated in the heart of the city; and if it is to be adequate to its function, it will have to be built large enough to house all the city offices. If the new courthouse is situated in the heart of the city and is built large enough to house all the city offices, then its cost will run to over $10 million. Its cost cannot exceed $10 million. Therefore either the new courthouse will have an inconvenient location or it will be inadequate to its function. (*C*, *H*, *A*, *L*, *O*)

14. Jalana will come if she gets the message, provided that she is still interested. Although she didn't come, she is still interested. Therefore she didn't get the message. (*C*, *M*, *I*)

***15.** If the Mosaic account of the cosmogony (the account of the creation in Genesis) is strictly correct, the sun was not created until the fourth day. And if the sun was not created until the fourth day, it could not have been the cause of the alternation of day and night for the first three days. But either the word "day" is used in Scripture in a different sense from that in which it is commonly accepted now or else the sun must have been the cause of the alternation of day and night for the first three days. Hence it follows that either the Mosaic account of the cosmogony is not strictly correct or else the word "day" is used in Scripture in a different sense from that in which it is commonly accepted now. (*M*, *C*, *A*, *D*)

16. If the teller or the cashier had pushed the alarm button, the vault would have locked automatically, and the police would have arrived within three minutes. Had the police arrived within three minutes, the robbers' car would have been overtaken. But the robbers' car was not overtaken. Therefore the teller did not push the alarm button. (*T*, *C*, *V*, *P*, *O*)

17. If people are always guided by their sense of duty, they must forgo the enjoyment of many pleasures; and if they are always guided by their desire for pleasure, they must often neglect their duty. People are either always guided by their sense of duty or always guided by their desire for pleasure. If people are always guided by their sense of duty, they do not often neglect their duty; and if they are always guided by their desire for pleasure, they do not forgo the enjoyment of many pleasures. Therefore people must forgo the enjoyment of many pleasures if and only if they do not often neglect their duty. (*D*, *F*, *P*, *N*)

18. Although world population is increasing, agricultural production is declining and manufacturing output remains constant. If agricultural production declines and world population increases, then either new food sources will become available or else there will be a radical redistribution of food resources in the world unless human nutritional requirements diminish. No new food sources will become available, yet neither will family planning be encouraged nor will human nutritional requirements diminish. Therefore there will be a radical redistribution of food resources in the world. (*W, A, M, N, R, H, P*)

19. Either the robber came in the door, or else the crime was an inside one and one of the servants is implicated. The robber could come in the door only if the latch had been raised from the inside; but one of the servants is surely implicated if the latch was raised from the inside. Therefore one of the servants is implicated. (*D, I, S, L*)

***20.** If I pay my tuition, I won't have any money left. I'll buy a computer only if I have money. I won't learn to program computers unless I buy a computer. But if I don't pay tuition, I can't enroll in classes; and if I don't enroll in classes I certainly won't buy a computer. I must either pay my tuition or not pay my tuition. So I surely will not learn to program computers! (*P, M, C, L, E*)

G. The five arguments that follow are also valid, and a proof of the validity of each of them is called for. However, these proofs will be somewhat more difficult to construct than those in earlier exercises, and students who find themselves stymied from time to time ought not become discouraged. What may appear difficult on first appraisal may come to seem much less difficult with continuing efforts. Familiarity with the nineteen rules of inference, and repeated practice in applying those rules, are the keys to the construction of these proofs.

1. If you study the humanities, then you will develop an understanding of people, and if you study the sciences, then you will develop an understanding of the world about you. So if you study either the humanities or the sciences, then you will develop an understanding either of people or of the world about you. (*H, P, S, W*)

2. If you study the humanities then you will develop an understanding of people, and if you study the sciences then you will develop an understanding of the world about you. So if you study both the humanities and the sciences, you will develop an understanding both of people and of the world about you. (*H, P, S, W*)

3. If you have free will, then your actions are not determined by any antecedent events. If you have free will, then if your actions are not determined by any antecedent events, then your actions cannot be predicted. If your actions are not determined by any antecedent events, then if your actions cannot be predicted then the consequences of your actions cannot be predicted. Therefore if you have free will, then the consequences of your actions cannot be predicted. (*F*, *A*, *P*, *C*)

4. Socrates was a great philosopher. Therefore either Socrates was happily married or else he wasn't. (*G*, *H*)

***5.** If either Socrates was happily married or else he wasn't, then Socrates was a great philosopher. Therefore Socrates was a great philosopher. (*H*, *G*)

9.9 Proof of Invalidity

For an invalid argument there is, of course, no formal proof of validity. But if we fail to discover a formal proof of validity for a given argument, this failure does not prove that the argument is invalid and that no such proof can be constructed. It may mean only that we have not tried hard enough. Our inability to find a proof of validity may be caused by the fact that the argument is not valid, but it may be caused instead by our own lack of ingenuity—as a consequence of the noneffective character of the process of proof construction. Not being able to construct a formal proof of its validity does not prove an argument to be invalid. What *does* constitute a proof that a given argument is invalid?

The method about to be described is closely related to the truth-table method, although it is a great deal shorter. It will be helpful to recall how an invalid argument form is proved invalid by a truth table. If a single case (row) can be found in which truth values are assigned to the statement variables in such a way that the premises are made true and the conclusion false, then the argument form is invalid. If we can somehow make an assignment of truth values to the simple component statements of an argument that will make its premises true and its conclusion false, then making that assignment will suffice to prove the argument invalid. To make such an assignment is, in effect, what the truth table does. If we can make such an assignment of truth values without actually constructing the whole truth table, much work will be eliminated.

Consider this argument:

If the governor favors public housing, then she is in favor of restricting the scope of private enterprise.

If the governor were a socialist, then she would be in favor of restricting the scope of private enterprise.

Therefore if the governor favors public housing, then she is a socialist.

It is symbolized as

$F \supset R$

$S \supset R$

$\therefore F \supset S$

and we can prove it invalid without having to construct a complete truth table. First we ask, "What assignment of truth values is required to make the conclusion false?" It is clear that a conditional is false only if its antecedent is true and its consequent false. Hence assigning the truth value "true" to F and "false" to S will make the conclusion $F \supset S$ false. Now if the truth value "true" is assigned to R, both premises are made true, because a conditional is true whenever its consequent is true. We can say, then, that if the truth value "true" is assigned to F and to R, and the truth value "false" is assigned to S, the argument will have true premises and a false conclusion and is thus proved to be invalid.

This method of proving invalidity is an alternative to the truth-table method of proof. The two methods are closely related, however, and the essential connection between them should be noticed. In effect, what we did when we made the indicated assignment of truth values was to construct one row of the given argument's truth table. The relationship can perhaps be seen more clearly when the truth-value assignments are written out horizontally:

F	R	S	$F \supset R$	$S \supset R$	$F \supset S$
True	True	False	True	True	False

in which configuration they constitute one row (the second) of the truth table for the given argument. *An argument is proved invalid by displaying at least one row of its truth table in which all its premises are true but its conclusion is false*. Consequently, we need not examine *all* rows of its truth table to discover an argument's invalidity: discovering a single row in which its premises are all true and its conclusion false will suffice. This method of proving invalidity is a method of constructing such a row without having to construct the entire truth table.*

The present method is shorter than writing out an entire truth table, and the amount of time and work saved is proportionally greater for arguments involving a greater number of component simple statements. For arguments

*The whole truth table (were we to construct it) would of course test the validity of the *specific form* of the argument in question. If it can be shown that the specific form of an argument is invalid, we may infer that the argument having that specific form is an invalid argument. The method described here differs only in that truth values here are assigned directly to premises and conclusion; nonetheless, the relation between this method and the truth-table method applied in Chapter 8 is very close.

with a considerable number of premises, or with premises of considerable complexity, the needed assignment of truth values may not be so easy to make. There is no mechanical method of proceeding, but some hints may prove helpful.

It is most efficient to proceed by assigning those values seen immediately to be essential if invalidity is to be proved. Thus, any premise that simply asserts the truth of some statement *S* suggests the immediate assignment of **T** to *S* (or **F** if the falsehood of *S* was asserted as premise), because we know that all the premises must be made true. The same principle applies to the statements in the conclusion, except that the assignments of truth values there must make the conclusion false. Thus a conclusion of the form $A \supset B$ suggests the immediate assignment of **T** to *A* and **F** to *B*, and a conclusion in the form $A \vee B$ suggests the immediate assignment of **F** to *A* and **F** to *B*, because only those assignments could result in a proof of invalidity.

Whether one ought to begin by seeking to make the premises true or by seeking to make the conclusion false depends on the structure of the propositions; usually it is best to begin wherever assignments can be made with greatest confidence. Of course, there will be many circumstances in which the first assignments have to be arbitrary and tentative. A certain amount of trial and error is likely to be needed. Even so, this method of proving invalidity is almost always shorter and easier than writing out a complete truth table.

EXERCISES

Prove the invalidity of each of the following by the method of assigning truth values.

***1.** $A \supset B$
$C \supset D$
$A \vee D$
$\therefore B \vee C$

2. $\sim(E \bullet F)$
$(\sim E \bullet \sim F) \supset (G \bullet H)$
$H \supset G$
$\therefore G$

3. $I \vee \sim J$
$\sim(\sim K \bullet L)$
$\sim(\sim I \bullet \sim L)$
$\therefore \sim J \supset K$

4. $M \supset (N \vee O)$
$N \supset (P \vee Q)$
$Q \supset R$
$\sim(R \vee P)$
$\therefore \sim M$

***5.** $S \supset (T \supset U)$
$V \supset (W \supset X)$
$T \supset (V \bullet W)$
$\sim(T \bullet X)$
$\therefore S \equiv U$

6. $A \equiv (B \vee C)$
$B \equiv (C \vee A)$
$C \equiv (A \vee B)$
$\sim A$
$\therefore B \vee C$

7. $D \supset (E \vee F)$
$G \supset (H \vee I)$
$\sim E \supset (I \vee J)$
$(I \supset G) \bullet (\sim H \supset \sim G)$
$\sim J$
$\therefore D \supset (G \vee I)$

8. $K \supset (L \bullet M)$
$(L \supset N) \vee \sim K$
$O \supset (P \vee \sim N)$
$(\sim P \vee Q) \bullet \sim Q$
$(R \vee \sim P) \vee \sim M$
$\therefore K \supset R$

9. $(S \supset T) \bullet (T \supset S)$
$(U \bullet T) \vee (\sim T \bullet \sim U)$
$(U \vee V) \vee (S \vee T)$
$\sim U \supset (W \bullet X)$
$(V \supset \sim S) \bullet (\sim V \supset \sim Y)$
$X \supset (\sim Y \supset \sim X)$
$(U \vee S) \bullet (V \vee Z)$
$\therefore X \bullet Z$

***10.** $A \supset (B \supset \sim C)$
$(D \supset B) \bullet (E \supset A)$
$F \vee C$
$G \supset \sim H$
$(I \supset G) \bullet (H \supset J)$
$I \equiv \sim D$
$(B \supset H) \bullet (\sim H \supset D)$
$\therefore E \equiv F$

9.10 Inconsistency

An argument is proved invalid if truth values can be assigned to make all of its premises true and its conclusion false. If a deductive argument is not invalid, it must be valid. So, if truth values *cannot* be assigned to make the premises true and the conclusion false, then the argument must be valid. This follows from the definition of validity, but it has this curious consequence: Any argument whose premises are inconsistent must be valid.

In the following argument, for example, the premises appear to be totally irrelevant to the conclusion:

> If the airplane had engine trouble, it would have landed at Bend.
>
> If the airplane did not have engine trouble, it would have landed at Creswell.
>
> The airplane did not land at either Bend or Creswell.
>
> Therefore the airplane must have landed in Denver.

Here is its symbolic translation:

$A \supset B$
$\sim A \supset C$
$\sim(B \vee C)$
$\therefore D$

Any attempt to assign truth values to its component simple statements in such a way as to make the conclusion false and the premises all true is doomed to failure. Even if we ignore the conclusion and attend only to the premises, we find that there is no way to assign truth values to their components such that the premises will all be true. No truth-value assignment can make them all true because they are inconsistent with one another. Their conjunction is self-contradictory, being a substitution instance of a self-contradictory statement form. If we were to construct a truth table for this argument, we would find that in every row at least one of the premises is false. Because there is no row in which the premises are all true, there is no row in which the premises are all true and the conclusion false. Hence the truth table for this argument would establish that it is in fact valid. And, of course, we can also provide a formal proof of its validity:

1.	$A \supset B$	
2.	$\sim A \supset C$	
3.	$\sim(B \vee C)$ / $\therefore D$	
4.	$\sim B \bullet \sim C$	3, De. M.
5.	$\sim B$	4, Simp.
6.	$\sim A$	1,5, M.T.
7.	C	2,6, M.P.
8.	$\sim C \bullet \sim B$	4, Com
9.	$\sim C$	8, Simp.
10.	$C \vee D$	7, Add.
11.	D	10, 9, D.S.

In this proof, lines 1 though 9 are devoted to making explicit the inconsistency that is implicitly contained in the premises. That inconsistency emerges clearly in line 7 (which asserts *C*) and line 9 (which asserts ~*C*). Once this explicit contradiction has been expressed, the conclusion follows swiftly using Add. and D.S.

Thus we see that if a set of premises is inconsistent, those premises will validly yield any conclusion, no matter how irrelevant. The essence of the matter is more simply shown with the following outrageous argument, whose openly inconsistent premises allow us to infer—validly!—an irrelevant and absurd conclusion.

Today is Sunday.

Today is not Sunday.

Therefore the moon is made of green cheese.

In symbols we have

1. S

2. $\sim S$ / $\therefore M$

And the formal proof of its validity is almost immediately obvious:

3. $S \vee M$ 1, Add.

4. M 3, 2, D.S.

Of course, an argument that is valid because its premises are inconsistent cannot possibly be *sound*—for if the premises are inconsistent with each other, they cannot possibly be all true. By such an argument, therefore, it is not possible to establish any conclusion to be true, because we know that at least one of the premises must be false.

How can such meager premises make any argument in which they occur valid? The premises of a valid argument imply its conclusion not merely in the sense of "material" implication, but *logically*, or strictly. In a valid argument it is logically impossible for the premises to be true when the conclusion is false—and this is the situation that obtains when it is logically impossible for the premises to be true, putting the conclusion aside. What we have shown is this: *Any argument with inconsistent premises is valid, regardless of what its conclusion may be*. Its validity may be established by a truth table, or as we saw above by a formal proof, in which the contradiction is first formally expressed (for example, S and $\sim S$), the desired conclusion is then added to one side of the contradiction (for example, $S \vee M$), and that desired conclusion (for example, M) is then inferred by Disjunctive Syllogism using the other side of the contradiction (for example, $\sim S$).

This discussion helps to explain why consistency is prized so highly. One reason is that inconsistent statements cannot both be true. In a courtroom, therefore, cross-examination often aims to bring a hostile witness to contradict himself. If a witness makes inconsistent assertions, all that he says cannot be true, and his credibility is seriously undermined. When it has been once established that a witness has lied under oath (or is perhaps thoroughly confused), no testimony of that witness can be fully trusted. Lawyers say: *Falsus in unum, falsus in omnibus*; untrustworthy in one thing, untrustworthy in all.

Another, deeper reason why inconsistency is so repugnant is that—as we have seen—any and every conclusion follows logically from inconsistent statements taken as premises. Inconsistent statements are not "meaningless"; their trouble is just the opposite. They mean too much. They mean everything, in the sense of *implying* everything. And if *everything* is asserted, half of what is asserted is surely *false*, because every statement has a denial.

We are thus provided with an answer to the old riddle: What happens when an irresistible force meets an immovable object? The situation described by the riddle involves a contradiction. An irresistible force can meet an immovable object only if both exist; there must be an irresistible force and there must also be an immovable object. But if there is an irresistible force, there can be no immovable object. Let us make the contradiction explicit: There is an immovable object, and there is no immovable object. From these inconsistent premises, *any* conclusion may validly be inferred. So the correct answer to the question, "What happens when an irresistible force meets an immovable object?" is "Everything!"

Inconsistency, devastating when found among the premises of an argument, can be highly amusing. Everett Dirksen, leader of the Republican Party in the U.S. Senate for a decade in the twentieth century, enjoyed describing himself as "a man of fixed and unbending principles, the first of which is to be flexible at all times."[2] When an internal contradiction, not recognized by a speaker, yields unseen absurdity, we call the statement an "Irish Bull." Writes the schoolboy, for example, "The climate of the Australian interior is so bad that the inhabitants don't live there any more." Yogi Berra, famous for his Irish Bulls, observed that a certain restaurant, once very popular, had become "so crowded that nobody goes there any more." And, said he, "When you see a fork in the road, take it."

Sets of propositions that are internally inconsistent cannot all be true, as a matter of logic. But human beings are not always logical and do utter, and sometimes may even believe, two propositions that contradict one another. This may seem difficult to do, but we are told by Lewis Carroll, a very reliable authority in such matters, that the White Queen in *Alice in Wonderland* made a regular practice of believing six impossible things before breakfast!

EXERCISES

A. For each of the following, either construct a formal proof of validity or prove invalidity by the method of assigning truth values to the simple statements involved.

***1.** $(A \supset B) \cdot (C \supset D)$
$\therefore (A \cdot C) \supset (B \vee D)$

2. $(E \supset F) \cdot (G \supset H)$
$\therefore (E \vee G) \supset (F \cdot H)$

3. $I \supset (J \vee K)$
$(J \cdot K) \supset L$
$\therefore I \supset L$

4. $M \supset (N \cdot O)$
$(N \vee O) \supset P$
$\therefore M \supset P$

***5.** $[(X \cdot Y) \cdot Z] \supset A$
$(Z \supset A) \supset (B \supset C)$
B
$\therefore X \supset C$

6. $[(D \vee E) \cdot F] \supset G$
$(F \supset G) \supset (H \supset I)$
H
$\therefore D \supset I$

7. $(J \cdot K) \supset (L \supset M)$
$N \supset {\sim}M$
${\sim}(K \supset {\sim}N)$
${\sim}(J \supset {\sim}L)$
$\therefore {\sim}J$

8. $(O \cdot P) \supset (Q \supset R)$
$S \supset {\sim}R$
${\sim}(P \supset {\sim}S)$
${\sim}(O \supset Q)$
$\therefore {\sim}O$

9. $T \supset (U \bullet V)$
$U \supset (W \bullet X)$
$(T \supset W) \supset (Y \equiv Z)$
$(T \supset U) \supset {\sim}Y$
${\sim}Y \supset ({\sim}Z \supset X)$
$\therefore X$

***10.** $A \supset (B \bullet C)$
$B \supset (D \bullet E)$
$(A \supset D) \supset (F \equiv G)$
$A \supset (B \supset {\sim}F)$
${\sim}F \supset ({\sim}G \supset E)$
$\therefore E$

B. For each of the following, either construct a formal proof of validity or prove invalidity by the method of assigning truth values to the simple statements involved. In each case, use the notation in parentheses.

***1.** If the linguistics investigators are correct, then if more than one dialect was present in ancient Greece, then different tribes came down at different times from the north. If different tribes came down at different times from the north, they must have come from the Danube River valley. But archaeological excavations would have revealed traces of different tribes there if different tribes had come down at different times from the north, and archaeological excavations have revealed no such traces there. Hence if more than one dialect was present in ancient Greece, then the linguistics investigators are not correct. *(C, M, D, V, A)*

2. If there are the ordinary symptoms of a cold and the patient has a high temperature, then if there are tiny spots on his skin, he has measles. Of course the patient cannot have measles if his record shows that he has had them before. The patient does have a high temperature and his record shows that he has had measles before. Besides the ordinary symptoms of a cold, there are tiny spots on his skin. I conclude that the patient has a viral infection. *(O, T, S, M, R, V)*

3. If God were willing to prevent evil, but unable to do so, he would be impotent; if he were able to prevent evil, but unwilling to do so, he would be malevolent. Evil can exist only if God is either unwilling or unable to prevent it. There is evil. If God exists, he is neither impotent nor malevolent. Therefore God does not exist. *(W, A, I, M, E, G)*

4. If I buy a new car this spring or have my old car fixed, then I'll get up to Canada this summer and stop off in Duluth. I'll visit my parents if I stop off in Duluth. If I visit my parents, they'll insist on my spending the summer with them. If they insist on my spending the summer with them, I'll be there till autumn. But if I stay there till autumn, then I won't get to Canada after all! So I won't have my old car fixed. *(N, F, C, D, V, I, A)*

***5.** If Salome is intelligent and studies hard, then she will get good grades and pass her courses. If Salome studies hard but lacks intelligence, then her efforts will be appreciated; and if her efforts are appreciated, then she will pass her courses. If Salome is intelligent, then she studies hard. Therefore Salome will pass her courses. *(I, S, G, P, A)*

6. If there is a single norm for greatness of poetry, then Milton and Edgar Guest cannot both be great poets. If either Pope or Dryden is regarded as a great poet, then Wordsworth is certainly no great poet; but if Wordsworth is no great poet, then neither is Keats nor Shelley. But after all, even though Edgar Guest is not, Dryden and Keats are both great poets. Hence there is no single norm for greatness of poetry. (*N, M, G, P, D, W, K, S*)

7. If the butler were present, he would have been seen; and if he had been seen, he would have been questioned. If he had been questioned, he would have replied; and if he had replied, he would have been heard. But the butler was not heard. If the butler was neither seen nor heard, then he must have been on duty; and if he was on duty, he must have been present. Therefore the butler was questioned. *(P, S, Q, R, H, D)*

8. If the butler told the truth, then the window was closed when he entered the room; and if the gardener told the truth, then the automatic sprinkler system was not operating on the evening of the murder. If the butler and the gardener are both lying, then a conspiracy must exist to protect someone in the house and there would have been a little pool of water on the floor just inside the window. We know that the window could not have been closed when the butler entered the room. There was a little pool of water on the floor just inside the window. So if there is a conspiracy to protect someone in the house, then the gardener did not tell the truth. *(B, W, G, S, C, P)*

9. Their chief would leave the country if she feared capture, and she would not leave the country unless she feared capture. If she feared capture and left the country, then the enemy's espionage network would be demoralized and powerless to harm us. If she did not fear capture and remained in the country, it would mean that she was ignorant of our own agents' work. If she is really ignorant of our agents' work, then our agents can consolidate their positions within the enemy's organization; and if our agents can consolidate their positions there, they will render the enemy's espionage network powerless to harm us. Therefore the enemy's espionage network will be powerless to harm us. *(L, F, D, P, I, C)*

***10.** If the investigators of extrasensory perception are regarded as honest, then considerable evidence for extrasensory perception must be admitted; and the doctrine of clairvoyance must be considered seriously if extrasensory perception is tentatively accepted as a fact. If considerable evidence for extrasensory perception is admitted, then it must be tentatively accepted as a fact and an effort must be made to explain it. The doctrine of clairvoyance must be considered seriously if we are prepared to take seriously that class of phenomena called occult; and if we are prepared to take seriously that class of phenomena called occult, a new respect must be paid to mediums. If we pursue the matter further, then if a new respect must be paid to mediums, we must take seriously their claims to communicate with the dead. We do pursue the matter further, but still we are practically committed to believing in ghosts if we take seriously the mediums' claims to communicate with the dead. Hence if the investigators of extrasensory perception are regarded as honest, we are practically committed to believing in ghosts. *(H, A, C, F, E, O, M, P, D, G)*

11. If we buy a lot, then we will build a house. If we buy a lot, then if we build a house we will buy furniture. If we build a house, then if we buy furniture we will buy dishes. Therefore if we buy a lot, we will buy dishes. *(L, H, F, D)*

12. If your prices are low, then your sales will be high, and if you sell quality merchandise, then your customers will be satisfied. So if your prices are low and you sell quality merchandise, then your sales will be high and your customers satisfied. *(L, H, Q, S)*

13. If your prices are low, then your sales will be high, and if you sell quality merchandise, then your customers will be satisfied. So if either your prices are low or you sell quality merchandise, then either your sales will be high or your customers will be satisfied. *(L, H, Q, S)*

14. If Jordan joins the alliance, then either Algeria or Syria boycotts it. If Kuwait joins the alliance, then either Syria or Iraq boycotts it. Syria does not boycott it. Therefore if neither Algeria nor Iraq boycotts it, then neither Jordan nor Kuwait joins the alliance. *(J, A, S, K, I)*

***15.** If either Jordan or Algeria joins the alliance, then if either Syria or Kuwait boycotts it, then although Iraq does not boycott it, Yemen boycotts it. If either Iraq or Morocco does not boycott it, then Egypt will join the alliance. Therefore if Jordan joins the alliance, then if Syria boycotts it, then Egypt will join the alliance. *(J, A, S, K, I, Y, M, E)*

C. If any truth-functional argument is valid, we have the tools to prove it valid; and if it is invalid, we have the tools to prove it invalid. Prove each of the following arguments valid or invalid. These proofs will be more difficult to construct than in preceding exercises, but they will offer greater satisfaction.

1. If the president cuts Social Security benefit payments, he will lose the support of the senior citizens; and if he cuts defense spending, he will lose the support of the conservatives. If the president loses the support of either the senior citizens or the conservatives, then his influence in the Senate will diminish. But his influence in the Senate will not diminish. Therefore the president will not cut either Social Security benefits or defense spending. *(B, S, D, C, I)*

2. If inflation continues, then interest rates will remain high. If inflation continues, then if interest rates remain high then business activity will decrease. If interest rates remain high, then if business activity decreases then unemployment will rise. So if unemployment rises, then inflation will continue. *(I, H, D, U)*

3. If taxes are reduced, then inflation will rise, but if the budget is balanced, then unemployment will increase. If the president keeps his campaign promises, then either taxes are reduced or the budget is balanced. Therefore if the president keeps his campaign promises, then either inflation will rise or unemployment will increase. *(T, I, B, U, K)*

4. Weather predicting is an exact science. Therefore either it will rain tomorrow or it won't. *(W, R)*

***5.** If either it will rain tomorrow or it won't rain tomorrow, then weather predicting is an exact science. Therefore weather predicting is an exact science. *(R, W)*

9.11 Indirect Proof of Validity

Contradictory statements cannot both be true. Therefore, a statement added to the premises that makes it possible to deduce a contradiction must entail a falsehood. This gives rise to another method of proving validity. Suppose we *assume* (for the purposes of the proof only) the *denial* of what is to be proved. And suppose, using that assumption, we can derive a contradiction. That contradiction will show that when we denied what was to be proved we were brought to absurdity. We will have established the desired conclusion *indirectly*, with a proof by *reductio ad absurdum*.

An *indirect proof of validity* is written out by stating as an additional assumed premise the negation of the conclusion. If we can derive an explicit contradiction from the set of premises thus augmented, the argument with which we began must be valid. The method is illustrated with the following argument:

1. $A \supset (B \bullet C)$

2. $(B \vee D) \supset E$

3. $D \vee A$

$\therefore E$

In the very next line we make explicit our assumption (for the purpose of the indirect proof) of the denial of the conclusion.

4. $\sim E$ I.P. (Indirect proof)

With the now enlarged set of premises we can, using the established rules of inference, bring out an explicit contradiction, thus:

5.	$\sim(B \vee D)$	2, 4, M.T.
6.	$\sim B \bullet \sim D$	5, De M.
7.	$\sim D \bullet \sim B$	6, Com.
8.	$\sim D$	7, Simp.
9.	A	3, 8, D.S.
10.	$B \bullet C$	1, 9, M.P.
11.	B	10, Simp.
12.	$\sim B$	6, Simp.
13.	$B \bullet \sim B$	11, 12, Conj.

The last line of the proof is an explicit contradiction, which is a demonstration of the absurdity to which we were led by assuming $\sim E$ in line 4. This contradiction, formally and explicitly expressed in the last line, exhibits the absurdity and completes the proof.

This method of indirect proof strengthens our machinery for testing arguments by making it possible, in some circumstances, to prove validity more quickly than would be possible without it. We can illustrate this by first constructing a direct formal proof of the validity of an argument, and then demonstrating the validity of that same argument using an indirect proof. In the following example, the proof without the *reductio ad absurdum* is on the left and requires fifteen steps; the proof using the *reductio ad absurdum* is on the right and requires only eight steps. An exclamation point (!) is used to indicate that a given step is derived after the assumption advancing the indirect proof had been made.

1. $(H \supset I) \bullet (J \supset K)$

2. $(I \vee K) \supset L$

3. $\sim L$

$\therefore \sim(H \vee J)$

4.	$\sim(I \vee K)$	2, 3, M. T.	4! $\sim\sim(H \vee J)$	I. P. (Indirect Proof)
5.	$\sim I \bullet \sim K$	4, De M.	5! $H \vee J$	4, D. N.
6.	$\sim I$	5, Simp.	6! $I \vee K$	1, 5, C. D.
7.	$H \supset I$	1, Simp.	7! L	2, 6, M. P.
8.	$\sim H$	7, 6, M. T.	8! $L \bullet \sim L$	7, 3, Conj.
9.	$(J \supset K) \bullet (H \supset I)$	1, Com.		
10.	$J \supset K$	9, Simp.		
11.	$\sim K \bullet \sim I$	5, Com.		
12.	$\sim K$	11, Simp.		
13.	$\sim J$	10, 12, M. T.		
14.	$\sim H \bullet \sim J$	8, 13, Conj.		
15.	$\sim(H \vee J)$	14, De M.		

EXERCISES

A. For each of the following arguments, construct an indirect proof of validity.

1. **1.** $A \vee (B \bullet C)$
2. $A \supset C$
$\therefore C$

2. **1.** $(G \vee H) \supset \sim G$
$\therefore \sim G$

3. **1.** $(D \vee E) \supset (F \supset G)$
2. $(\sim G \vee H) \supset (D \bullet F)$
$\therefore G$

4. **1.** $(M \vee N) \supset (O \bullet P)$
2. $(O \vee Q) \supset (\sim R \bullet S)$
3. $(R \vee T) \supset (M \bullet U)$
$\therefore \sim R$

***5.** **1.** $D \supset (Z \supset Y)$
2. $Z \supset (Y \supset \sim Z)$
$\therefore \sim D \vee \sim Z$

6. **1.** $(O \vee P) \supset (D \bullet E)$
2. $(E \vee L) \supset (Q \vee \sim D)$
3. $(Q \vee Z) \supset \sim(O \bullet E)$
$\therefore \sim O$

7. **1.** $(F \vee G) \supset (D \bullet E)$
2. $(E \vee H) \supset Q$
3. $(F \vee H)$
$\therefore Q$

8. **1.** $B \supset [(O \vee \sim O) \supset (T \vee U)]$
2. $U \supset \sim(G \vee \sim G)$
$\therefore B \supset T$

B. For each of the following two arguments, construct an indirect proof of validity.

1. If a sharp fall in the prime rate of interest produces a rally in the stock market, then inflation is sure to come soon. But if a drop in the money supply produces a sharp fall in the prime rate of interest, then early inflation is equally certain. So inflation will soon be upon us. *(F, R, I, D)*

2. If precipitation levels remain unchanged and global warming intensifies, ocean levels will rise and some ocean ports will be inundated. But ocean ports will not be inundated if global warming intensifies. Therefore either precipitation levels will not remain unchanged or global warming will not intensify. *(L, G, O, P, D)*

C. For the following argument, construct both (a) a direct formal proof of validity and (b) an indirect proof of validity. Compare the lengths of the two proofs.

1. $(V \supset {\sim}W) \bullet (X \supset Y)$
2. $({\sim}W \supset Z) \bullet (Y \supset {\sim}A)$
3. $(Z \supset {\sim}B) \bullet ({\sim}A \supset C)$
4. $V \bullet X$

$\therefore {\sim}B \bullet C$

9.12 Shorter Truth-Table Technique

There is still another method of testing the validity of arguments. We have seen how an argument may be proved *invalid* by assigning truth values to its component simple statements in such a way as to make all its premises true and its conclusion false. It is of course impossible to make such assignments if the argument is valid. So we can prove the *validity* of an argument by showing that *no such set of truth values can be assigned*. We do this by showing that its premises can be made true, and its conclusion false, only by assigning truth values *inconsistently*—that is, only with an assignment of values such that some component statement is assigned both a **T** and an **F.** In other words, if the truth value **T** is assigned to each premise of a valid argument, and the truth value **F** is assigned to its conclusion, this will necessitate assigning both **T** and **F** to some component statement—which is, of course, a contradiction. Here again we use the general method of *reductio ad absurdum*.

For example, we can very quickly prove the validity of the argument

$(A \vee B) \supset (C \bullet D)$

$(D \vee E) \supset G$

$\therefore A \supset G$

by first assigning **T** to each premise and **F** to the conclusion. But assigning **F** to the conclusion requires that **T** be assigned to *A* and **F** be assigned to *G*. Because **T** is assigned to *A*, the antecedent of the first premise is true, and because the premise as a whole has been assigned **T,** its consequent must be true also—so **T** must be assigned to both *C* and *D*. Because **T** is assigned to *D*, the antecedent of the second premise is true, and because the premise as a whole has been assigned **T,** its consequent must also be true, so **T** must be assigned to *G*. But we have already been forced to assign **F** to *G*, in order to make the conclusion false. Hence the argument would be invalid only if the statement *G* were both false and true, which is obviously impossible. Proving the validity of an argument with this "shorter truth-table technique" is one version of the use of *reductio ad absurdum*, reducing to the absurd—but instead of using the rules of inference, it uses truth-value assignments.

This *reductio ad absurdum* method of assigning truth values is often the quickest method of testing arguments, but it is more readily applied in some arguments than in others, depending on the kinds of propositions involved. Its easiest application is when **F** is assigned to a disjunction (in which case both of the disjuncts must be assigned **F**) or **T** to a conjunction (in which case both of the conjuncts must be assigned **T**). When assignments to simple statements are thus forced, the absurdity (if there is one) is quickly exposed. But where the method calls for **T** to be assigned to a disjunction, we cannot be sure which disjunct is true; and where **F** must be assigned to a conjunction, we cannot be sure which conjunct is false; in such cases we must make various "trial assignments," which slows the process and diminishes the advantage of this method. However, it remains the case that the *reductio ad absurdum* method of proof is often the most efficient means in testing the validity of a deductive argument.

EXERCISES

A. Use the *reductio ad absurdum* method of assigning truth values (the shorter truth-table technique) to determine the validity or invalidity of the arguments in Exercise Set B, on pages 355–356.

B. Do the same for the arguments in Exercise Set C, on pages 356–357.

SUMMARY

In this chapter we explained various methods with which the validity, and the invalidity, of deductive arguments may be proved.

In Section 9.1 we introduced and explained the notion of a formal proof of validity, and we listed the first nine rules of inference with which formal proofs may be constructed.

In Section 9.2 we examined in detail the elementary valid argument forms that constitute the first nine rules of inference, and illustrated their use in simple arguments.

In Section 9.3 we illustrated the ways in which the elementary valid argument forms can be used to build formal proofs of validty.

In Section 9.4 we began the process of constructing formal proofs of validity, using only the first nine rules of inference.

In Section 9.5 we illustrated the ways in which the first nine rules of inference can be used to construct more extended formal proofs of validity.

In Section 9.6 we introduced the general rule of replacement, and expanded the rules of inference by adding ten logical equivalences, each of which permits the replacement of one logical expression by another having exactly the same meaning.

In Section 9.7 we discussed the features of the system of natural deduction that contains nineteen rules of inference.

In Section 9.8 we began the enterprise of building formal proofs of validity using all nineteen rules of inference: nine elementary valid argument forms, and ten logical equivalences permitting replacement.

In Section 9.9 we explained the method of proving invalidity when deductive arguments are not valid.

In Section 9.10 we discussed inconsistency, explaining why any argument with inconsistent premises cannot be sound, but will be valid.

In Section 9.11 we explained and illustrated indirect proof of validity.

In Section 9.12 we explained and illustrated the shorter truth-table technique for proving validity.

End Notes

[1]See also John A. Winnie, "The Completeness of Copi's System of Natural Deduction," *Notre Dame Journal of Formal Logic* 11 (July 1970), pp. 379–382.

[2]Recounted by George Will, in *Newsweek*, 27 October 2003.