Symbolic Logic

8

8.1 Modern Logic and Its Symbolic Language

We seek a full understanding of deductive reasoning. For this we need a general theory of deduction. A general theory of deduction will have two objectives: (1) to explain the relations between premises and conclusions in deductive arguments, and (2) to provide techniques for discriminating between valid and invalid deductions. Two great bodies of logical theory have sought to achieve these ends. The first, called classical (or Aristotelian) logic, was examined in Chapters 5 through 7. The second, called modern (or modern symbolic) logic, is the subject in this and the following two chapters.

Although these two great bodies of theory have similar aims, they proceed in very different ways. Modern logic does not build on the system of syllogisms discussed in preceding chapters. It does not begin with the analysis of categorical propositions. It does seek to discriminate valid from invalid arguments, although it does so using very different concepts and techniques. Therefore we must now begin afresh, developing a modern logical system that deals with some of the very same issues dealt with by traditional logic—and does so even more effectively.

Modern logic begins by first identifying the fundamental logical *connectives* on which deductive argument depends. Using these connectives, a general account of such arguments is given, and methods for testing the validity of arguments are developed.

This analysis of deduction requires an artificial symbolic language. In a natural language—English or any other—there are peculiarities that make exact logical analysis difficult: Words may be vague or equivocal, the construction of arguments may be ambiguous, metaphors and idioms may confuse or mislead, emotional appeals may distract—problems discussed in Part I of this book. These difficulties can be largely overcome with an artificial language in which logical relations can be formulated with precision. The most fundamental elements of this modern symbolic language will be introduced in this chapter.

Symbols greatly facilitate our thinking about arguments. They enable us to get to the heart of an argument, exhibiting its essential nature and putting aside what is not essential. Moreover, with symbols we can perform some logical operations almost mechanically, with the eye, which might otherwise demand great effort. It may seem paradoxical, but a symbolic language therefore helps us to accomplish some intellectual tasks without having to think too much.*

Classical logicians did understand the enormous value of symbols in analysis. Aristotle used symbols as variables in his own analyses, and the refined system of Aristotelian syllogistics uses symbols in very sophisticated ways, as the preceding chapters have shown. However, much real progress has been made, mainly during the twentieth century, in devising and using logical symbols more effectively.

The modern symbolism with which deduction is analyzed differs greatly from the classical. The relations of classes of things are not central for modern logicians as they were for Aristotle and his followers. Instead, logicians look now to the internal structure of propositions and arguments, and to the logical links—very few in number—that are critical in all deductive argument. Modern symbolic logic is therefore not encumbered, as Aristotelian logic was, by the need to transform deductive arguments into syllogistic form, an often laborious task explained in the immediately preceding chapter.

The system of modern logic we now begin to explore is in some ways less elegant than analytical syllogistics, but it is more powerful. There are forms of deductive argument that syllogistics cannot adequately address. Using the approach taken by modern logic with its more versatile symbolic language, we can pursue the aims of deductive analysis directly and we can penetrate more deeply. The logical symbols we shall now explore permit more complete and more efficient achievement of the central aim of deductive logic: discriminating between valid and invalid arguments.

*The Arabic numerals we use today (1, 2, 3, . . .) illustrate the advantages of an improved symbolic language. They replaced cumbersome Roman numerals (i, ii, iii, . . .), which are very difficult to manipulate. To multiply 113 by 9 is easy; to multiply CXIII by IX is not so easy. Even the Romans, some scholars contend, were obliged to find ways to symbolize numbers more efficiently.

8.2 The Symbols for Conjunction, Negation, and Disjunction

In this chapter we shall be concerned with relatively simple arguments such as:

> The blind prisoner has a red hat or the blind prisoner has a white hat.
>
> The blind prisoner does not have a red hat.
>
> Therefore the blind prisoner has a white hat.

and

> If Mr. Robinson is the brakeman's next-door neighbor, then Mr. Robinson lives halfway between Detroit and Chicago.
>
> Mr. Robinson does not live halfway between Detroit and Chicago.
>
> Therefore Mr. Robinson is not the brakeman's next-door neighbor.

Every argument of this general type contains at least one compound statement. In studying such arguments we divide all statements into two general categories, simple and compound. A **simple statement** does not contain any other statement as a component. For example, "Charlie's neat" is a simple statement. A **compound statement** does contain another statement as a component. For example, "Charlie's neat and Charlie's sweet" is a compound statement, because it contains two simple statements as components. Of course, the components of a compound statement may themselves be compound.*

---

*In formulating definitions and principles in logic, one must be very precise. What appears simple often proves more complicated than had been supposed. The notion of a "component of a statement" is a good illustration of this need for caution.

One might suppose that a *component* of a statement is simply a part of a statement that is itself a statement. But this account does not define the term with enough precision, because one statement may be a *part* of a larger statement and yet not be a *component* of it in the strict sense. For example, consider the statement: "The man who shot Lincoln was an actor." Plainly the last four words of this statement are a part of it, and could indeed be regarded as a statement; it is either true or it is false that Lincoln was an actor. But the statement that "Lincoln was an actor," although undoubtedly a part of the larger statement, is not a *component* of that larger statement.

We can explain this by noting that, for part of a statement to be a component of that statement, two conditions must be satisfied: (1) The part must be a statement in its own right; *and* (2) If the part is replaced in the larger statement by any other statement, the result of that replacement must be meaningful—it must make sense.

The first of these conditions is satisfied in the Lincoln example, but the second is not. Suppose the part "Lincoln was an actor" is replaced by "there are lions in Africa." The result of this replacement is nonsense: "The man who shot there are lions in Africa." The term *component* is not a difficult one to understand, but—like all logical terms—it must be defined accurately and applied carefully.

A. CONJUNCTION

There are several types of compound statements, each requiring its own logical notation. The first type of compound statement we consider is the *conjunction*. We can form the **conjunction** of two statements by placing the word "and" between them; the two statements so combined are called **conjuncts.** Thus the compound statement, "Charlie's neat and Charlie's sweet" is a conjunction whose first conjunct is "Charlie's neat" and whose second conjunct is "Charlie's sweet."

The word "and" is a short and convenient word, but it has other uses besides connecting statements. For example, the statement, "Lincoln and Grant were contemporaries" is *not* a conjunction, but a simple statement expressing a relationship. To have a unique symbol whose only function is to connect statements conjunctively, we introduce the **dot** "•" as our symbol for conjunction. Thus the previous conjunction can be written as "Charlie's neat • Charlie's sweet." More generally, where p and q are any two statements whatever, their conjunction is written $p \bullet q$.

We know that every statement is either true or false. Therefore we say that every statement has a **truth value,** where the truth value of a true statement is *true*, and the truth value of a false statement is *false*. Using this concept, we can divide compound statements into two distinct categories, according to whether the truth value of the compound statement is determined wholly by the truth values of its components, or is determined by anything other than the truth values of its components.

We apply this distinction to conjunctions. The truth value of the conjunction of two statements is determined wholly and entirely by the truth values of its two conjuncts. If both its conjuncts are true, the conjunction is true; otherwise it is false. For this reason a conjunction is said to be a **truth-functional compound statement,** and its conjuncts are said to be **truth-functional components** of it.

Not every compound statement is truth-functional. For example, the truth value of the compound statement, "Othello believes that Desdemona loves Cassio," is not in any way determined by the truth value of its component simple statement, "Desdemona loves Cassio," because it could be true that Othello believes that Desdemona loves Cassio, regardless of whether she does or not. So the component, "Desdemona loves Cassio," is not a truth-functional component of the statement, "Othello believes that Desdemona loves Cassio," and the statement itself is not a truth-functional compound statement.

For our present purposes we define a **component of a compound statement** as being a **truth-functional component** if, when the component is replaced in the compound by any different statements having the same truth value as each other, the different compound statements produced by those replacements also have the same truth values as each other. And now we define a **compound statement** as being a **truth-functional compound statement** if all of its components are truth-functional components of it.[1]

We shall be concerned only with those compound statements that are truth-functionally compound. In the remainder of this book, therefore, we shall use the term *simple statement* to refer to any statement that is not truth-functionally compound.

A conjunction is a truth-functional compound statement, so our dot symbol is a **truth-functional connective.** Given any two statements, p and q, there are only four possible sets of truth values they can have. These four possible cases, and the truth value of the conjunction in each, can be displayed as follows:

Where p is true and q is true, $p \bullet q$ is true.
Where p is true and q is false, $p \bullet q$ is false.
Where p is false and q is true, $p \bullet q$ is false.
Where p is false and q is false, $p \bullet q$ is false.

If we represent the truth values "true" and "false" by the capital letters **T** and **F,** the determination of the truth value of a conjunction by the truth values of its conjuncts can be represented more compactly and more clearly by means of a **truth table:**

p	q	$p \bullet q$
T	T	T
T	F	F
F	T	F
F	F	F

This truth table can be taken as defining the dot symbol, because it explains what truth values are assumed by $p \bullet q$ in every possible case.

We abbreviate simple statements by capital letters, generally using for this purpose a letter that will help us remember which statement it abbreviates. Thus we may abbreviate "Charlie's neat and Charlie's sweet" as $N \bullet S$. Some conjunctions, both of whose conjuncts have the same subject term—for example, "Byron was a great poet and Byron was a great adventurer"—are more briefly and perhaps more naturally stated in English by placing the "and" between the predicate terms and not repeating the subject term, as in "Byron was a great poet and a great adventurer." For our purposes, we regard the latter as formulating the same statement as the former and symbolize either one as $P \bullet A$. If both conjuncts of a conjunction have the same predicate term, as in "Lewis was a famous explorer and Clark was a famous explorer," again the conjunction is usually stated in English by placing the "and" between the subject terms and not repeating the predicate, as in "Lewis and Clark were famous explorers." Either formulation is symbolized as $L \bullet C$.

As shown by the truth table defining the dot symbol, a conjunction is true if and only if both of its conjuncts are true. The word "and" has another use in which it signifies not *mere* (truth-functional) conjunction but has the sense of "and subsequently," meaning temporal succession. Thus the statement, "Jones entered the country at New York and went straight to Chicago," is significant and might be true, whereas "Jones went straight to Chicago and entered the country at New York" is hardly intelligible. And there is quite a difference between "He took off his shoes and got into bed" and "He got into bed and took off his shoes."* Such examples show the desirability of having a special symbol with an exclusively truth-functional conjunctive use.

Note that the English words "but," "yet," "also," "still," "although," "however," "moreover," "nevertheless," and so on, and even the comma and the semicolon, can also be used to conjoin two statements into a single compound statement, and in their conjunctive sense they can all be represented by the dot symbol.

B. NEGATION

The **negation** (or *contradictory* or *denial*) of a statement in English is often formed by the insertion of a "not" in the original statement. Alternatively, one can express the negation of a statement in English by prefixing to it the phrase "it is false that" or "it is not the case that." It is customary to use the symbol "~," called a **curl** or a **tilde,** to form the negation of a statement. Thus, where M symbolizes the statement "All humans are mortal," the various statements "Not all humans are mortal," "Some humans are not mortal," "It is false that all humans are mortal," and "It is not the case that all humans are mortal" are all symbolized as $\sim M$. More generally, where p is any statement whatever, its negation is written $\sim p$. It is obvious that the curl is a truth-functional operator. The negation of any true statement is false, and the negation of any false

*In *The Victoria Advocate*, Victoria, Texas, 27 October 1990, appeared the following report: "Ramiro Ramirez Garza, of the 2700 block of Leary Lane, was arrested by police as he was threatening to commit suicide and flee to Mexico."

statement is true. This fact can be presented very simply and clearly by means of a truth table:

p	$\sim p$
T	F
F	T

This truth table may be regarded as the definition of the negation "~" symbol.

C. DISJUNCTION

The **disjunction** (or *alternation*) of two statements is formed in English by inserting the word "or" between them. The two component statements so combined are called *disjuncts* (or *alternatives*).

The English word "or" is ambiguous, having two related but distinguishable meanings. One of them is exemplified in the statement, "Premiums will be waived in the event of sickness or unemployment." The intention here is obviously that premiums are waived not only for sick persons and for unemployed persons, but also for persons who are *both* sick *and* unemployed. This sense of the word "or" is called *weak* or *inclusive*. An **inclusive disjunction** is true if one or the other or both disjuncts are true; only if both disjuncts are false is their inclusive disjunction false. The inclusive "or" has the sense of "either, possibly both." Where precision is at a premium, as in contracts and other legal documents, this sense is made explicit by the use of the phrase "and/or."

The word "or" is also used in a *strong* or *exclusive* sense, in which the meaning is not "at least one" but "at least one and at most one." Where a restaurant lists "salad or dessert" on its dinner menu, it is clearly meant that, for the stated price of the meal, the diner may have one or the other *but not both*. Where precision is at a premium and the exclusive sense of "or" is intended, the phrase "but not both" is often added.

We interpret the inclusive disjunction of two statements as an assertion that at least one of the statements is true, and we interpret their **exclusive disjunction** as an assertion that at least one of the statements is true but not both are true. Note that the two kinds of disjunction have a part of their meanings in common. This partial common meaning, that at least one of the disjuncts is true, is the *whole* meaning of the inclusive "or" and a *part* of the meaning of the exclusive "or."

Although disjunctions are stated ambiguously in English, they are unambiguous in Latin. Latin has two different words corresponding to the two different senses of the English word "or." The Latin word *vel* signifies weak or inclusive disjunction, and the Latin word *aut* corresponds to the word "or" in its strong or exclusive sense. It is customary to use the initial letter of the word *vel* to stand for "or" in its weak, inclusive sense. Where p and q are any two statements whatever, their weak or inclusive disjunction is written $p \vee q$. Our symbol for inclusive disjunction, called a **wedge** (or, less frequently, a *vee*) is also a truth-functional connective. A weak disjunction is false only if both of its disjuncts are false. We may regard the wedge as being defined by the following truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The first specimen argument presented in this section was a *disjunctive syllogism*.*

The blind prisoner has a red hat or the blind prisoner has a white hat.
The blind prisoner does not have a red hat.
Therefore the blind prisoner has a white hat.

Its form is characterized by saying that its first premise is a disjunction; its second premise is the negation of the first disjunct of the first premise; and its conclusion is the same as the second disjunct of the first premise. It is evident that the disjunctive syllogism, so defined, is valid on either interpretation of the word "or"; that is, regardless of whether an inclusive or exclusive disjunction is intended. The typical valid argument that has a disjunction for a premise is, like the disjunctive syllogism, valid on either interpretation of the word "or," so a simplification may be effected by translating the English word "or" into our logical symbol "$\vee$"—*regardless of which meaning of the English word "or" is intended.* In general, only a close examination of the context, or an explicit questioning of the speaker or writer, can reveal which sense of "or" is intended. This problem, often impossible to resolve, can be avoided if we agree to treat *any* occurrence of the word "or" as inclusive. On the other hand, if it is stated explicitly that the disjunction is intended to be exclusive, by means of the added phrase "but not both," for example, we have the symbolic machinery to formulate that additional sense, as will be shown directly.

Where both disjuncts have either the same subject term or the same predicate term, it is often natural to compress the formulation of their disjunction in English by placing the "or" so that there is no need to repeat the common part of the two disjuncts. Thus, "Either Smith is the owner or Smith is the manager" might equally well be stated as "Smith is either the owner or the manager,"

*A *syllogism* is a deductive argument consisting of two premises and a conclusion. The term *disjunctive syllogism* is being used in a narrower sense here than it was in Chapter 7.

and either one is properly symbolized as $O \vee M$. And "Either Red is guilty or Butch is guilty" may be stated as "Either Red or Butch is guilty"; either one may be symbolized as $R \vee B$.

The word "unless" is often used to form the disjunction of two statements. Thus, "You will do poorly on the exam unless you study" is correctly symbolized as $P \vee S$. The reason is that we use "unless" to mean that if one proposition is not true, the other is or will be true. The preceding sentence can be understood to mean, "If you don't study, you will do poorly on the exam"—and that is the thrust of the disjunction, because it asserts that one of the disjuncts is true, and hence that if one of them is false, the other must be true. Of course, you may study *and* do poorly on the exam.

The word "unless" is sometimes used to convey more information; it may mean (depending on context) that one or the other proposition is true but that not both are true. That is, "unless" may be intended as an exclusive disjunction. Thus it was noted by Ted Turner that global warming will put New York under water in one hundred years, and "will be the biggest catastrophe the world has ever seen—unless we have nuclear war."[2] Here the speaker did mean that at least one of the two disjuncts is true, but of course they cannot both be true. Other uses of "unless" are ambiguous. When we say, "The picnic will be held unless it rains," we surely do mean that the picnic will be held if it does not rain. But do we mean that it will not be held if it does rain? That may be uncertain. It is wise policy to treat every disjunction as weak or inclusive *unless* it is certain than an exclusive disjunction is meant. "Unless" is best symbolized simply with the wedge ($\vee$).

D. PUNCTUATION

In English, punctuation is absolutely required if complicated statements are to be clear. A great many different punctuation marks are used, without which many sentences would be highly ambiguous. For example, quite different meanings attach to "The teacher says John is a fool" when it is given different punctuations. Other sentences require punctuation for their very intelligibility, as, for example, "Jill where Jack had had had had had had had had had had the teacher's approval." Punctuation is equally necessary in mathematics. In the absence of a special convention, no number is uniquely denoted by $2 \times 3 + 5$, although when it is made clear how its constituents are to be grouped, it denotes either 11 or 16: the first when punctuated $(2 \times 3) + 5$, the second when punctuated $2 \times (3 + 5)$. To avoid ambiguity, and to make meaning clear, punctuation marks in mathematics appear in the form of parentheses, ( ), which are used to group individual symbols; brackets, [ ], which are used to group expressions that include parentheses; and braces, { }, which are used to group expressions that include brackets.

In the language of symbolic logic those same punctuation marks—parentheses, brackets, and braces—are equally essential, because in logic compound statements are themselves often compounded together into more complicated ones. Thus $p \bullet q \vee$ r is ambiguous: it might mean the conjunction of p with the disjunction of q with r, or it might mean the disjunction whose first disjunct is the conjunction of p and q and whose second disjunct is r. We distinguish between these two different senses by punctuating the given formula as p $\bullet$ $(q \vee r)$ or else as $(p \bullet q) \vee r$. That the different ways of punctuating the original formula do make a difference can be seen by considering the case in which p is false and q and r are both true. In this case the second punctuated formula is true (because its second disjunct is true), whereas the first one is false (because its first conjunct is false). Here the difference in punctuation makes all the difference between truth and falsehood, for different punctuations can assign different truth values to the ambiguous $p \bullet q \vee r$.

The word "either" has a variety of different meanings and uses in English. It has conjunctive force in the sentence, "There is danger on either side." More often it is used to introduce the first disjunct in a disjunction, as in "Either the blind prisoner has a red hat or the blind prisoner has a white hat." There it contributes to the rhetorical balance of the sentence, but it does not affect its meaning. Perhaps the most important use of the word "either" is to punctuate a compound statement. Thus the sentence

> The organization will meet on Thursday and Anand will be elected or the election will be postponed.

is ambiguous. This ambiguity can be resolved in one direction by placing the word "either" at its beginning, or in the other direction by inserting the word "either" before the name "Anand." Such punctuation is effected in our symbolic language by parentheses. The ambiguous formula $p \bullet q \vee r$ discussed in the preceding paragraph corresponds to the ambiguous sentence just examined. The two different punctuations of the formula correspond to the two different punctuations of the sentence effected by the two different insertions of the word "either."

The negation of a disjunction is often formed by use of the phrase "neither—nor." Thus the statement, "Either Fillmore or Harding was the greatest U.S. president," can be contradicted by the statement, "Neither Fillmore nor Harding was the greatest U.S. president." The disjunction would be symbolized as $F \vee H$, and its negation as either $\sim(F \vee H)$ or as $(\sim F) \bullet (\sim H)$. (The logical equivalence of these two symbolic formulas will be discussed in Section 8.9.) It should be clear that to deny a disjunction stating that one or another statement is true requires that both be stated to be false.

The word "both" in English has a very important role in logical punctuation, and it deserves the most careful attention. When we say "Both Jamal and Derek are not . . ." we are saying, as noted just above, that "Neither Jamal nor Derek is . . ."; we are applying the negation to each of them. But when we say "Jamal and Derek are not both . . .," we are saying something very different; we are applying the negation to the pair of them taken together, saying that "it is not the case that they are both . . .". This difference is very substantial. Entirely different meanings arise when the word "both" is placed differently in the English sentence. Consider the great difference between the meanings of

Jamal and Derek will not both be elected.

and

Jamal and Derek will both not be elected.

The first denies the conjunction $J \bullet D$ and may be symbolized as $\sim(J \bullet D)$. The second says that each one of the two will not be elected, and is symbolized as $\sim(J) \bullet \sim(D)$. Merely changing the *position* of the two words "both" and "not" alters the logical force of what is asserted.

Of course, the word "both" does not always have this role; sometimes we use it only to add emphasis. When we say that "Both Lewis and Clark were great explorers," we use the word only to state more emphatically what is said by "Lewis and Clark were great explorers." When the task is logical analysis, the punctuational role of "both" must be very carefully determined.

In the interest of brevity—that is, to decrease the number of parentheses required—it is convenient to establish the convention that *in any formula, the negation symbol will be understood to apply to the smallest statement that the punctuation permits*. Without this convention, the formula $\sim p \vee q$ is ambiguous, meaning either $(\sim p) \vee q$, or $\sim(p \vee q)$. By our convention we take it to mean the first of these alternatives, for the curl *can* (and therefore by our convention *does*) apply to the first component, p, rather than to the larger formula $p \vee q$.

Given a set of punctuation marks for our symbolic language, it is possible to write not just conjunctions, negations, and weak disjunctions in it, but exclusive disjunctions as well. The exclusive disjunction of p and q asserts that at least one of them is true but not both are true, which is written as $(p \vee q) \bullet \sim(p \bullet q)$.

The truth value of any compound statement constructed from simple statements using only the truth-functional connectives—dot, curl, and wedge—is completely determined by the truth or falsehood of its component simple statements. If we know the truth values of simple statements, the truth value of any truth-functional compound of them is easily calculated. In working with such compound statements we always begin with their inmost components and work outward. For example, if *A* and *B* are true statements and *X* and *Y* are false statements, we calculate the truth value of the compound statement ~[~(*A* • *X*) • (*Y* ∨ ~*B*)] as follows. Because *X* is false, the conjunction *A* • *X* is false, and so its negation ~(*A* • *X*) is true. *B* is true, so its negation ~*B* is false, and because *Y* is also false, the disjunction of *Y* with ~*B*, *Y* ∨ ~*B*, is false. The bracketed formula [~(*A* • *X*) • (*Y* ∨ ~*B*)] is the conjunction of a true with a false statement and is therefore false. Hence its negation, which is the entire statement, is true. Such a stepwise procedure always enables us to determine the truth value of a compound statement from the truth values of its components.

In some circumstances we may be able to determine the truth value of a truth-functional compound statement even if we cannot determine the truth or falsehood of one or more of its component simple statements. We do this by first calculating the truth value of the compound statement on the assumption that a given simple component is true, and then by calculating the truth value of the compound statement on the assumption that the same simple component is false, doing the same for each component whose truth value is unknown. If both calculations yield the *same* truth value for the compound statement in question, we have determined the truth value of the compound statement without having to determine the truth value of its components, because we know that the truth value of any component cannot be other than true or false.

OVERVIEW

Punctuation in Symbolic Notation

The statement

I will study hard and pass the exam or fail

is ambiguous. It could mean "I will study hard and pass the exam or I will fail the exam" or "I will study hard and I will either pass the exam or fail it."

The symbolic notation

S • *P* ∨ *F*

is similiarly ambiguous. Parentheses resolve the ambiguity. In place of "I will study hard and pass the exam or I will fail the exam," we get

(*S* • *P*) ∨ *F*

and in place of "I will study hard and I will either pass the exam or fail it," we get

S • (*P* ∨ *F*)

EXERCISES

A. Using the truth table definitions of the dot, the wedge, and the curl, determine which of the following statements are true.

*1. Rome is the capital of Italy ∨ Rome is the capital of Spain.

2. ~(London is the capital of England • Stockholm is the capital of Norway).

3. ~London is the capital of England • ~Stockholm is the capital of Norway.

4. ~(Rome is the capital of Spain ∨ Paris is the capital of France).

*5. ~Rome is the capital of Spain ∨ ~Paris is the capital of France.

6. London is the capital of England ∨ ~London is the capital of England.

7. Stockholm is the capital of Norway • ~Stockholm is the capital of Norway.

8. (Paris is the capital of France • Rome is the capital of Spain) ∨ (Paris is the capital of France • ~Rome is the capital of Spain).

9. (London is the capital of England ∨ Stockholm is the capital of Norway) • (~Rome is the capital of Italy • ~Stockholm is the capital of Norway).

*10. Rome is the capital of Spain ∨ ~(Paris is the capital of France • Rome is the capital of Spain).

11. Rome is the capital of Italy • ~(Paris is the capital of France ∨ Rome is the capital of Spain).

12. ~(~Paris is the capital of France • ~Stockholm is the capital of Norway).

13. ~[~(~Rome is the capital of Spain ∨ ~Paris is the capital of France) ∨ ~(~Paris is the capital of France ∨ Stockholm is the capital of Norway)].

14. ~[~(~London is the capital of England • Rome is the capital of Spain) • ~(Rome is the capital of Spain • ~Rome is the capital of Spain)].

*15. ~[~(Stockholm is the capital of Norway ∨ Paris is the capital of France) ∨ ~(~London is the capital of England • Rome is the capital of Spain)].

16. Rome is the capital of Spain ∨ (~London is the capital of England ∨ London is the capital of England).

17. Paris is the capital of France • ~(Paris is the capital of France • Rome is the capital of Spain).

18. London is the capital of England • ~(Rome is the capital of Italy • Rome is the capital of Italy).

19. (Stockholm is the capital of Norway v ~Paris is the capital of France) v ~(~Stockholm is the capital of Norway • ~London is the capital of England).

***20.** (Paris is the capital of France v ~Rome is the capital of Spain) v ~(~Paris is the capital of France • ~Rome is the capital of Spain).

21. ~[~(Rome is the capital of Spain • Stockholm is the capital of Norway) v ~(~Paris is the capital of France v ~Rome is the capital of Spain)].

22. ~[~(London is the capital of England • Paris is the capital of France) v ~(~Stockholm is the capital of Norway v ~Paris is the capital of France)].

23. ~[(~Paris is the capital of France v Rome is the capital of Italy) • ~(~Rome is the capital of Italy v Stockholm is the capital of Norway)].

24. ~[(~Rome is the capital of Spain v Stockholm is the capital of Norway) • ~(~Stockholm is the capital of Norway v Paris is the capital of France)].

***25.** ~[(~London is the capital of England • Paris is the capital of France) v ~(~Paris is the capital of France • Rome is the capital of Spain)].

B. If *A*, *B*, and *C* are true statements and *X*, *Y*, and *Z* are false statements, which of the following are true?

***1.** ~*A* v *B*

2. ~*B* v *X*

3. ~*Y* v *C*

4. ~*Z* v *X*

***5.** (*A* • *X*) v (*B* • *Y*)

6. (*B* • *C*) v (*Y* • *Z*)

7. ~(*C* • *Y*) v (*A* • *Z*)

8. ~(*A* • *B*) v (*X* • *Y*)

9. ~(*X* • *Z*) v (*B* • *C*)

***10.** ~(*X* • ~*Y*) v (*B* • ~*C*)

11. (*A* v *X*) • (*Y* v *B*)

12. (*B* v *C*) • (*Y* v *Z*)

13. (*X* v *Y*) • (*X* v *Z*)

14. ~(*A* v *Y*) • (*B* v *X*)

***15.** ~(*X* v *Z*) • (~*X* v *Z*)

16. ~(*A* v *C*) v ~(*X* • ~*Y*)

17. ~(*B* v *Z*) • ~(*X* v ~*Y*)

18. ~[(*A* v ~*C*) v (*C* v ~*A*)]

19. ~[(*B* • *C*) • ~(*C* • *B*)]

***20.** ~[(*A* • *B*) v ~(*B* • *A*)]

21. [*A* v (*B* v *C*)] • ~[(*A* v *B*) v *C*]

22. [*X* v (*Y* • *Z*)] v ~[(*X* v *Y*) • (*X* v *Z*)]

23. $[A \bullet (B \vee C)] \bullet \sim[(A \bullet B) \vee (A \bullet C)]$

24. $\sim\{[(\sim A \bullet B) \bullet (\sim X \bullet Z)] \bullet \sim[(A \bullet \sim B) \vee \sim(\sim Y \bullet \sim Z)]\}$

***25.** $\sim\{\sim[(B \bullet \sim C) \vee (Y \bullet \sim Z)] \bullet [(\sim B \vee X) \vee (B \vee \sim Y)]\}$

C. If *A* and *B* are known to be true and *X* and *Y* are known to be false, but the truth values of *P* and *Q* are not known, of which of the following statements can you determine the truth values?

***1.** $A \vee P$

2. $Q \bullet X$

3. $Q \vee \sim X$

4. $\sim B \bullet P$

***5.** $P \vee \sim P$

6. $\sim P \vee (Q \vee P)$

7. $Q \bullet \sim Q$

8. $P \bullet (\sim P \vee X)$

9. $\sim(P \bullet Q) \vee P$

***10.** $\sim Q \bullet [(P \vee Q) \bullet \sim P]$

11. $(P \vee Q) \bullet \sim(Q \vee P)$

12. $(P \bullet Q) \bullet (\sim P \vee \sim Q)$

13. $\sim P \vee [\sim Q \vee (P \bullet Q)]$

14. $P \vee \sim(\sim A \vee X)$

***15.** $P \bullet [\sim(P \vee Q) \vee \sim P]$

16. $\sim(P \bullet Q) \vee (Q \bullet P)$

17. $\sim[\sim(\sim P \vee Q) \vee P] \vee P$

18. $(\sim P \vee Q) \bullet \sim[\sim P \vee (P \bullet Q)]$

19. $(\sim A \vee P) \bullet (\sim P \vee Y)$

***20.** $\sim[P \vee (B \bullet Y)] \vee [(P \vee B) \bullet (P \vee Y)]$

21. $[P \vee (Q \bullet A)] \bullet \sim[(P \vee Q) \bullet (P \vee A)]$

22. $[P \vee (Q \bullet X)] \bullet \sim[(P \vee Q) \bullet (P \vee X)]$

23. $\sim[\sim P \vee (\sim Q \vee X)] \vee [\sim(\sim P \vee Q) \vee (\sim P \vee X)]$

24. $\sim[\sim P \vee (\sim Q \vee A)] \vee [\sim(\sim P \vee Q) \vee (\sim P \vee A)]$

***25.** $\sim[(P \bullet Q) \vee (Q \bullet \sim P)] \bullet \sim[(P \bullet \sim Q) \vee (\sim Q \bullet \sim P)]$

D. Using the letters *E*, *I*, *J*, *L*, and *S* to abbreviate the simple statements, "Egypt's food shortage worsens," "Iran raises the price of oil," "Jordan requests more U.S. aid," "Libya raises the price of oil," and "Saudi Arabia buys five hundred more warplanes," symbolize these statements.

***1.** Iran raises the price of oil but Libya does not raise the price of oil.

2. Either Iran or Libya raises the price of oil.

3. Iran and Libya both raise the price of oil.

4. Iran and Libya do not both raise the price of oil.

*5. Iran and Libya both do not raise the price of oil.

6. Iran or Libya raises the price of oil but they do not both do so.

7. Saudi Arabia buys five hundred more warplanes and either Iran raises the price of oil or Jordan requests more U.S. aid.

8. Either Saudi Arabia buys five hundred more warplanes and Iran raises the price of oil or Jordan requests more U.S. aid.

9. It is not the case that Egypt's food shortage worsens, and Jordan requests more U.S. aid.

*10. It is not the case that either Egypt's food shortage worsens or Jordan requests more U.S. aid.

11. Either it is not the case that Egypt's food shortage worsens or Jordan requests more U.S. aid.

12. It is not the case that both Egypt's food shortage worsens and Jordan requests more U.S. aid.

13. Jordan requests more U.S. aid unless Saudi Arabia buys five hundred more warplanes.

14. Unless Egypt's food shortage worsens, Libya raises the price of oil.

*15. Iran won't raise the price of oil unless Libya does so.

16. Unless both Iran and Libya raise the price of oil neither of them does.

17. Libya raises the price of oil and Egypt's food shortage worsens.

18. It is not the case that neither Iran nor Libya raises the price of oil.

19. Egypt's food shortage worsens and Jordan requests more U.S. aid, unless both Iran and Libya do not raise the price of oil.

*20. Either Iran raises the price of oil and Egypt's food shortage worsens, or it is not the case both that Jordan requests more U.S. aid and that Saudi Arabia buys five hundred more warplanes.

21. Either Egypt's food shortage worsens and Saudi Arabia buys five hundred more warplanes, or either Jordan requests more U.S. aid or Libya raises the price of oil.

22. Saudi Arabia buys five hundred more warplanes, and either Jordan requests more U.S. aid or both Libya and Iran raise the price of oil.

23. Either Egypt's food shortage worsens or Jordan requests more U.S. aid, but neither Libya nor Iran raises the price of oil.

24. Egypt's food shortage worsens, but Saudi Arabia buys five hundred more warplanes and Libya raises the price of oil.

***25.** Libya raises the price of oil and Egypt's food shortage worsens; however, Saudi Arabia buys five hundred more warplanes and Jordan requests more U.S. aid.

8.3 Conditional Statements and Material Implication

Where two statements are combined by placing the word "if" before the first and inserting the word "then" between them, the resulting compound statement is a **conditional** statement (also called a *hypothetical*, an *implication*, or an *implicative statement*). In a conditional statement the component statement that follows the "if" is called the **antecedent** (or the *implicans* or—rarely—the *protasis*), and the component statement that follows the "then" is the **consequent** (or the *implicate* or—rarely—the *apodosis*). For example, "If Mr. Jones is the brakeman's next-door neighbor, then Mr. Jones earns exactly three times as much as the brakeman" is a conditional statement in which "Mr. Jones is the brakeman's next-door neighbor" is the antecedent and "Mr. Jones earns exactly three times as much as the brakeman" is the consequent.

A conditional statement asserts that in any case in which its antecedent is true, its consequent is also true. It does not assert that its antecedent is true, but only that *if* its antecedent is true, then its consequent is also true. It does not assert that its consequent is true, but only that its consequent is true *if* its antecedent is true. The essential meaning of a conditional statement is the *relationship* asserted to hold between the antecedent and the consequent, in that order. To understand the meaning of a conditional statement, then, we must understand what the relationship of implication is.

Implication plausibly appears to have more than one meaning. We found it useful to distinguish different senses of the word "or" before introducing a special logical symbol to correspond exactly to a single one of the meanings of the English word. Had we not done so, the ambiguity of the English would have infected our logical symbolism and prevented it from achieving the clarity and precision aimed at. It will be equally useful to distinguish the different senses of "implies" or "if–then" before we introduce a special logical symbol in this connection.

Consider the following four conditional statements, each of which seems to assert a different type of implication, and to each of which corresponds a different sense of "if–then":

A. If all humans are mortal and Socrates is a human, then Socrates is mortal.

B. If Leslie is a bachelor, then Leslie is unmarried.

C. If this piece of blue litmus paper is placed in acid, then this piece of blue litmus paper will turn red.

D. If State loses the homecoming game, then I'll eat my hat.

Even a casual inspection of these four conditional statements reveals that they are of quite different types. The consequent of **A** follows *logically* from its antecedent, whereas the consequent of **B** follows from its antecedent by the very *definition* of the term "bachelor," which means unmarried man. The consequent of **C** does not follow from its antecedent either by logic alone or by the definition of its terms; the connection must be discovered empirically, because the implication stated here is *causal*. Finally, the consequent of **D** does not follow from its antecedent either by logic or by definition, nor is there any causal law involved. Statement **D** reports a *decision* of the speaker to behave in the specified way under the specified circumstances.

These four conditional statements are different in that each asserts a different type of implication between its antecedent and its consequent. But they are not completely different; all assert types of implication. Is there any identifiable common meaning, any partial meaning that is common to these admittedly different types of implication, although perhaps not the whole or complete meaning of any one of them?

The search for a common partial meaning takes on added significance when we recall our procedure in working out a symbolic representation for the English word "or." In that case, we proceeded as follows. First, we emphasized the difference between the two senses of the word, contrasting inclusive with exclusive disjunction. The inclusive disjunction of two statements was observed to mean that at least one of the statements is true, and the exclusive disjunction of two statements was observed to mean that at least one of the statements is true but not both are true. Second, we noted that these two types of disjunction had a common *partial* meaning. This partial common meaning, that at least one of the disjuncts is true, was seen to be the *whole* meaning of the weak, inclusive "or," and a *part* of the meaning of the strong, exclusive "or." We then introduced the special symbol "$\vee$" to represent this common partial meaning (which is the entire meaning of "or" in its inclusive sense). Third, we noted that the symbol representing the common partial meaning is an adequate translation of either sense of the word "or" for the purpose of retaining the disjunctive syllogism as a valid form of argument. It was admitted that translating an exclusive "or" into the symbol "$\vee$" ignores and loses part of the word's meaning. But the part of its meaning that is preserved by this translation is all that is needed for the disjunctive syllogism to remain a valid form of argument. Because the disjunctive syllogism is typical of arguments involving disjunction, with which we are concerned here, this partial translation of the word "or," which may abstract from its "full" or "complete" meaning in some cases, is wholly adequate for our present purposes.

Now we wish to proceed in the same way, this time in connection with the English phrase "if–then." The first part is already accomplished: We have already emphasized the differences among four senses of the "if–then" phrase, corresponding to four different types of implication. We are now ready for the second step, which is to discover a sense that is at least a part of the meaning of all four types of implication.

We approach this problem by asking: What circumstances suffice to establish the falsehood of a given conditional statement? Under what circumstances should we agree that the conditional statement

> If this piece of blue litmus paper is placed in that acid solution, then this piece of blue litmus paper will turn red.

is *false?* It is important to realize that this conditional does not assert that any blue litmus paper is actually placed in the solution, or that any litmus paper actually turns red. It asserts merely that *if* this piece of blue litmus paper is placed in the solution, *then* this piece of blue litmus paper will turn red. It is proved false if this piece of blue litmus paper is actually placed in the solution and does not turn red. The acid test, so to speak, of the falsehood of a conditional statement is available when its antecedent is true, because if its consequent is false while its antecedent is true, the conditional itself is thereby proved false.

Any conditional statement, "If p, then q," is known to be false if the conjunction $p \bullet {\sim}q$ is known to be true—that is, if its antecedent is true and its consequent is false. For a conditional to be true, then, the indicated conjunction must be false; that is, its negation ${\sim}(p \bullet {\sim}q)$ must be true. In other words, for any conditional, "If p then q," to be true, ${\sim}(p \bullet {\sim}q)$, the negation of the conjunction of its antecedent with the negation of its consequent, must also be true. We may then regard ${\sim}(p \bullet {\sim}q)$ as a part of the meaning of "If p then q."

Every conditional statement means to deny that its antecedent is true and its consequent false, but this need not be the whole of its meaning. A conditional such as **A** on page 331 also asserts a logical connection between its antecedent and consequent, as **B** asserts a definitional connection, **C** a causal connection, and **D** a decisional connection. No matter what type of implication is asserted by a conditional statement, *part* of its meaning is the negation of the conjunction of its antecedent with the negation of its consequent.

We now introduce a special symbol to represent this common partial meaning of the "if–then" phrase. We define the new symbol "⊃," called a

horseshoe, by taking $p \supset q$ as an abbreviation of $\sim(p \bullet \sim q)$. The exact significance of the $\supset$ symbol can be indicated by means of a truth table:

p	q	$\sim q$	$p \bullet \sim q$	$\sim(p \bullet \sim q)$	$p \supset q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Here the first two columns are the guide columns; they simply lay out all possible combinations of truth and falsehood for p and q. The third column is filled in by reference to the second, the fourth by reference to the first and third, the fifth by reference to the fourth, and the sixth is identical to the fifth by definition.

The symbol $\supset$ is not to be regarded as denoting *the* meaning of "if–then," or standing for *the* relation of implication. That would be impossible, for there is no single meaning of "if–then"; there are several meanings. There is no unique relation of implication to be thus represented; there are several different implication relations. Nor is the symbol $\supset$ to be regarded as somehow standing for *all* the meanings of "if–then." These are all different, and any attempt to abbreviate all of them by a single logical symbol would render that symbol ambiguous—as ambiguous as the English phrase "if–then" or the English word "implication." The symbol $\supset$ is completely unambiguous. What $p \supset q$ abbreviates is $\sim(p \bullet \sim q)$, whose meaning is included in the meanings of each of the various kinds of implications considered but does not constitute the entire meaning of any of them.

We can regard the symbol $\supset$ as representing another kind of implication, and it will be expedient to do so, because a convenient way to read $p \supset q$ is "If p, then q." But it is not the same kind of implication as any of those mentioned earlier. It is called **material implication** by logicians. In giving it a special name, we admit that it is a special notion, not to be confused with other, more usual, types of implication.

Not all conditional statements in English need assert one of the four types of implication previously considered. Material implication constitutes a fifth type that may be asserted in ordinary discourse. Consider the remark, "If Hitler was a military genius, then I'm a monkey's uncle." It is quite clear that it does not assert logical, definitional, or causal implication. It cannot represent a decisional implication, because it scarcely lies in the speaker's power to make the consequent true. No "real connection," whether logical, definitional, or causal, obtains between antecedent and consequent here. A conditional of this sort is often used as an emphatic or humorous method of denying its antecedent. The consequent of such a conditional is usually a statement that is obviously or ludicrously false. And because no true conditional can have both its antecedent true and its consequent false, to affirm such a conditional amounts to denying that its antecedent is true. The full meaning of the present conditional seems to be the denial that "Hitler was a military genius" is true when "I'm a monkey's uncle" is false. And because the latter is so obviously false, the conditional must be understood to deny the former.

The point here is that no "real connection" between antecedent and consequent is suggested by a material implication. All it asserts is that, as a matter of fact, it is not the case that the antecedent is true when the consequent is false. Note that the material implication symbol is a truth-functional connective, like the symbols for conjunction and disjunction. As such, it is defined by the truth table:

p	q	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

As thus defined by the truth table, the symbol ⊃ has some features that may at first appear odd: The assertion that a false antecedent materially implies a true consequent is true; and the assertion that a false antecedent materially implies a false consequent is also true. This apparent strangeness can be dissipated in part by the following considerations. Because the number 2 is smaller than the number 4 (a fact notated symbolically as $2 < 4$), it follows that *any* number smaller than 2 is smaller than 4. The conditional formula

$$\text{If } x < 2, \text{ then } x < 4.$$

is true for any number x whatsoever. If we focus on the numbers 1, 3, and 4, and replace the number variable x in the preceding conditional formula by each of them in turn, we can make the following observations. In

$$\text{If } 1 < 2, \text{ then } 1 < 4.$$

both antecedent and consequent are true, and of course the conditional is true. In

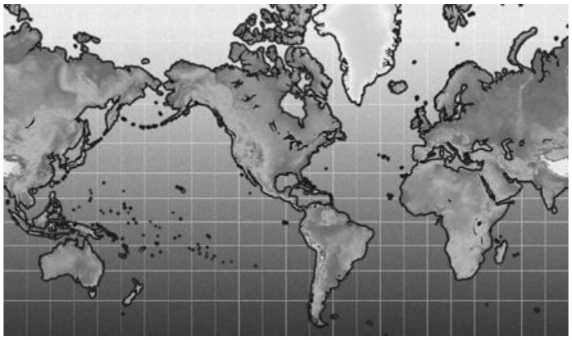
$$\text{If } 3 < 2, \text{ then } 3 < 4.$$

the antecedent is false and the consequent is true, and of course the conditional is again true. In

$$\text{If } 4 < 2, \text{ then } 4 < 4.$$

VISUAL LOGIC

Material Implication

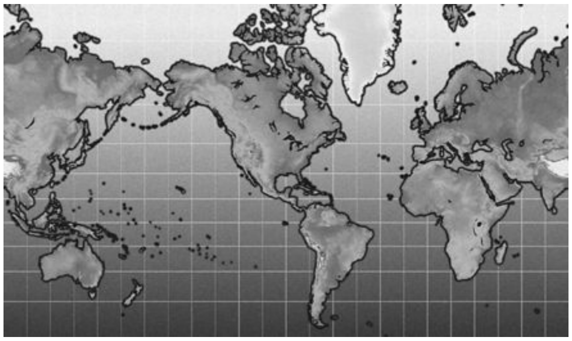


Source: Photodisc/Getty Images

Source: Photodisc/Getty Images

If the world is flat, then the moon is made of green cheese.

This proposition, in the form $F \supset G$, is a material implication. A material implication is true when the antecedent (the "if" clause) is false. Therefore a material implication is true when the antecedent is false and the consequent is also false, as in this illustrative proposition.



Source: Photodisc/Getty Images

Source: Photodisc/Getty Images

If the world is flat, the moon is round.

This proposition, in the similar form $F \supset R$, is also a material implication. A material implication is true when the antecedent (the "if" clause) is false. Therefore a material implication is true when the antecedent is false and the consequent is true, as in this illustrative proposition.

A material implication is false only if the antecedent is true and the consequent is false. Therefore a material implication is true whenever the antecedent is false, whether the consequent is false or true.

both antecedent and consequent are false, but the conditional remains true. These three cases correspond to the first, third, and fourth rows of the table defining the symbol ⊃. So it is not particularly remarkable or surprising that a conditional should be true when both antecedent and consequent are true, when the antecedent is false and the consequent is true, or when antecedent and consequent are both false. Of course, there is no number that is smaller than 2 but not smaller than 4; that is, there is no true conditional statement with a true antecedent and a false consequent. This is exactly what the defining truth table for ⊃ lays down.

Now we propose to translate any occurrence of the "if–then" phrase into our logical symbol ⊃. This proposal means that in translating conditional statements into our symbolism, we treat them all as merely material implications. Of course, most conditional statements assert more than that a merely material implication holds between their antecedents and consequents. So our proposal amounts to suggesting that we ignore, or put aside, or "abstract from," part of the meaning of a conditional statement when we translate it into our symbolic language. How can this proposal be justified?

The previous proposal to translate both inclusive and exclusive disjunctions by means of the symbol ∨ was justified on the grounds that the validity of the disjunctive syllogism was preserved even if the additional meaning that attaches to the exclusive "or" was ignored. Our present proposal to translate all conditional statements into the merely material implication symbolized by ⊃ may be justified in exactly the same way. Many arguments contain conditional statements of various different kinds, but the validity of all valid arguments of the general type with which we will be concerned is preserved even if the additional meanings of their conditional statements are ignored. This remains to be proved, of course, and will occupy our attention in the next section.

Conditional statements can be formulated in a variety of ways. The statement

> If he has a good lawyer, then he will be acquitted.

can equally well be stated without the use of the word "then" as

> If he has a good lawyer, he will be acquitted.

The order of the antecedent and consequent can be reversed, provided that the "if" still directly precedes the antecedent, as

> He will be acquitted if he has a good lawyer.

It should be clear that, in any of the examples just given, the word "if" can be replaced by such phrases as "in case," "provided that," "given that," or "on condition that," without any change in meaning. Minor adjustments in the phrasings of antecedent and consequent permit such alternative phrasings of the same conditional as

> That he has a good lawyer implies that he will be acquitted.

or

> His having a good lawyer entails his acquittal.

A shift from active to passive voice accompanies a reversal of order of antecedent and consequent, yielding the logically equivalent

> His being acquitted is implied (or entailed) by his having a good lawyer.

Any of these is symbolized as $L \supset A$.

The notions of necessary and sufficient conditions provide other formulations of conditional statements. For any specified event, many circumstances are necessary for it to occur. Thus, for a normal car to run, it is necessary that there be fuel in its tank, its spark plugs properly adjusted, its oil pump working, and so on. So if the event occurs, every one of the conditions necessary for its occurrence must have been fulfilled. Hence to say

> That there is fuel in its tank is a necessary condition for the car to run.

can equally well be stated as

> The car runs only if there is fuel in its tank.

which is another way of saying that

> If the car runs then there is fuel in its tank.

Any of these is symbolized as $R \supset F$. In general, "q is a *necessary condition* for p" is symbolized as $p \supset q$. And, likewise, "*p only if q*" is also symbolized a $p \supset q$.

For a specified situation there may be many alternative circumstances, any one of which is sufficient to produce that situation. For a purse to contain more than a dollar, for example, it is sufficient for it to contain five quarters, or eleven dimes, or twenty-one nickels, and so on. If any one of these circumstances obtains, the specified situation will be realized. Hence, to say "That the purse contains five quarters is a sufficient condition for it to contain more than a dollar" is to say "If the purse contains five quarters then it contains more than a dollar." In general, "p is a *sufficient condition* for q" is symbolized as $p \supset q$.

To illustrate, recruiters for the Wall Street investment firm Goldman, Sachs (where annual bonuses are commonly in the millions) grill potential employees repeatedly. Those who survive the grilling are invited to the firm's offices for a full day of interviews, culminating in a dinner with senior

Goldman executives. As reported recently, "Agile brains and near-perfect grades are necessary but not sufficient conditions for being hired. Just as important is fitting in."[3]

If p is a sufficient condition for q, we have $p \supset q$, and q must be a necessary condition for p. If p is a necessary condition for q, we have $q \supset p$, and q must be a sufficient condition for p. Hence, if p is necessary *and* sufficient for q, then q is sufficient *and* necessary for p.

Not every statement containing the word "if" is a conditional. None of the following statements is a conditional: "There is food in the refrigerator if you want some," "Your table is ready, if you please," "There is a message for you if you're interested," "The meeting will be held even if no permit is obtained." The presence or absence of particular words is never decisive. In every case, one must understand what a given sentence means, and then restate that meaning in a symbolic formula.

EXERCISES

A. If A, B, and C are true statements and X, Y, and Z are false statements, determine which of the following are true, using the truth tables for the horseshoe, the dot, the wedge, and the curl.

***1.** $A \supset B$

2. $A \supset X$

3. $B \supset Y$

4. $Y \supset Z$

***5.** $(A \supset B) \supset Z$

6. $(X \supset Y) \supset Z$

7. $(A \supset B) \supset C$

8. $(X \supset Y) \supset C$

9. $A \supset (B \supset Z)$

***10.** $X \supset (Y \supset Z)$

11. $[(A \supset B) \supset C] \supset Z$

12. $[(A \supset X) \supset Y] \supset Z$

13. $[A \supset (X \supset Y)] \supset C$

14. $[A \supset (B \supset Y)] \supset X$

***15.** $[(X \supset Z) \supset C] \supset Y$

16. $[(Y \supset B) \supset Y] \supset Y$

17. $[(A \supset Y) \supset B] \supset Z$

18. $[(A \bullet X) \supset C] \supset [(A \supset C) \supset X]$

19. $[(A \bullet X) \supset C] \supset [(A \supset X) \supset C]$

***20.** $[(A \bullet X) \supset Y] \supset [(X \supset A) \supset (A \supset Y)]$

21. $[(A \bullet X) \vee ({\sim}A \bullet {\sim}X)] \supset [(A \supset X) \bullet (X \supset A)]$

22. $\{[A \supset (B \supset C)] \supset [(A \bullet B) \supset C]\} \supset [(Y \supset B) \supset (C \supset Z)]$

23. $\{[(X \supset Y) \supset Z] \supset [Z \supset (X \supset Y)]\} \supset [(X \supset Z) \supset Y]$

24. $[(A \bullet X) \supset Y] \supset [(A \supset X) \bullet (A \supset Y)]$

***25.** $[A \supset (X \bullet Y)] \supset [(A \supset X) \vee (A \supset Y)]$

B. If A and B are known to be true, and X and Y are known to be false, but the truth values of P and Q are not known, of which of the following statements can you determine the truth values?

***1.** $P \supset A$

2. $X \supset Q$

3. $(Q \supset A) \supset X$

4. $(P \bullet A) \supset B$

***5.** $(P \supset P) \supset X$

6. $(X \supset Q) \supset X$

7. $X \supset (Q \supset X)$

8. $(P \bullet X) \supset Y$

9. $[P \supset (Q \supset P)] \supset Y$

***10.** $(Q \supset Q) \supset (A \supset X)$

11. $(P \supset X) \supset (X \supset P)$

12. $(P \supset A) \supset (B \supset X)$

13. $(X \supset P) \supset (B \supset Y)$

14. $[(P \supset B) \supset B] \supset B$

***15.** $[(X \supset Q) \supset Q] \supset Q$

16. $(P \supset X) \supset ({\sim}X \supset {\sim}P)$

17. $(X \supset P) \supset ({\sim}X \supset Y)$

18. $(P \supset A) \supset (A \supset {\sim}B)$

19. $(P \supset Q) \supset (P \supset Q)$

***20.** $(P \supset {\sim\sim}P) \supset (A \supset {\sim}B)$

21. ${\sim}(A \bullet P) \supset ({\sim}A \vee {\sim}P)$

22. ${\sim}(P \bullet X) \supset {\sim}(P \vee {\sim}X)$

23. ${\sim}(X \vee Q) \supset ({\sim}X \bullet {\sim}Q)$

24. $[P \supset (A \vee X)] \supset [(P \supset A) \supset X]$

***25.** $[Q \vee (B \bullet Y)] \supset [(Q \vee B) \bullet (Q \vee Y)]$

C. Symbolize the following, using capital letters to abbreviate the simple statements involved.

***1.** If Argentina mobilizes then if Brazil protests to the UN then Chile will call for a meeting of all the Latin American states.

2. If Argentina mobilizes then either Brazil will protest to the UN or Chile will call for a meeting of all the Latin American states.

3. If Argentina mobilizes then Brazil will protest to the UN and Chile will call for a meeting of all the Latin American states.

4. If Argentina mobilizes then Brazil will protest to the UN, and Chile will call for a meeting of all the Latin American states.

***5.** If Argentina mobilizes and Brazil protests to the UN then Chile will call for a meeting of all the Latin American states.

6. If either Argentina mobilizes or Brazil protests to the UN then Chile will call for a meeting of all the Latin American states.

7. Either Argentina will mobilize or if Brazil protests to the UN then Chile will call for a meeting of all the Latin American states.

8. If Argentina does not mobilize then either Brazil will not protest to the UN or Chile will not call for a meeting of all the Latin American states.

9. If Argentina does not mobilize then neither will Brazil protest to the UN nor will Chile call for a meeting of all the Latin American states.

***10.** It is not the case that if Argentina mobilizes then both Brazil will protest to the UN and Chile will call for a meeting of all the Latin American states.

11. If it is not the case that Argentina mobilizes then Brazil will not protest to the UN, and Chile will call for a meeting of all the Latin American states.

12. Brazil will protest to the UN if Argentina mobilizes.

13. Brazil will protest to the UN only if Argentina mobilizes.

14. Chile will call for a meeting of all the Latin American states only if both Argentina mobilizes and Brazil protests to the UN.

***15.** Brazil will protest to the UN only if either Argentina mobilizes or Chile calls for a meeting of all the Latin American states.

16. Argentina will mobilize if either Brazil protests to the UN or Chile calls for a meeting of all the Latin American states.

17. Brazil will protest to the UN unless Chile calls for a meeting of all the Latin American states.

18. If Argentina mobilizes, then Brazil will protest to the UN unless Chile calls for a meeting of all the Latin American states.

19. Brazil will not protest to the UN unless Argentina mobilizes.

***20.** Unless Chile calls for a meeting of all the Latin American states, Brazil will protest to the UN.

21. Argentina's mobilizing is a sufficient condition for Brazil to protest to the UN.

22. Argentina's mobilizing is a necessary condition for Chile to call for a meeting of all the Latin American states.

23. If Argentina mobilizes and Brazil protests to the UN, then both Chile and the Dominican Republic will call for a meeting of all the Latin American states.

24. If Argentina mobilizes and Brazil protests to the UN, then either Chile or the Dominican Republic will call for a meeting of all the Latin American states.

***25.** If neither Chile nor the Dominican Republic calls for a meeting of all the Latin American states, then Brazil will not protest to the UN unless Argentina mobilizes.

8.4 Argument Forms and Refutation by Logical Analogy

The central task of deductive logic, we have said, is discriminating valid arguments from invalid ones. If the premises of a valid argument are true (we explained in the very first chapter), its conclusion *must* be true. If the conclusion of a valid argument is false, at least one of the premises must be false. In short, the premises of a valid argument give *incontrovertible proof* of the conclusion drawn.

This informal account of validity must now be made more precise. To do this we introduce the concept of an *argument form.* Consider the following two arguments, which plainly have the *same* logical form. Suppose we are presented with the first of these arguments:

> If Bacon wrote the plays attributed to Shakespeare, then Bacon was a great writer.
>
> Bacon was a great writer.
>
> Therefore Bacon wrote the plays attributed to Shakespeare.

We may agree with the premises but disagree with the conclusion, judging the argument to be invalid. One way of proving invalidity is by the method of logical analogy. "You might as well argue," we could retort, "that

> If Washington was assassinated, then Washington is dead.
>
> Washington is dead.
>
> Therefore Washington was assassinated.

And you cannot seriously defend this argument," we would continue, "because here the premises are known to be true and the conclusion is known to be false. This argument is obviously invalid; your argument is of the *same form,* so yours is also invalid." This type of refutation is very effective.

This method of **refutation by logical analogy** points the way to an excellent general technique for testing arguments. To prove the invalidity of an argument, it suffices to formulate another argument that (1) has exactly the same form as the first and (2) has true premises and a false conclusion. This method is based on the fact that validity and invalidity are purely *formal* characteristics of arguments, which is to say that any two arguments that have exactly the same form are either both valid or both invalid, regardless of any differences in the subject matter with which they are concerned.*

---

*Here we assume that the simple statements involved are neither logically true (e.g., "All chairs are chairs") nor logically false (e.g., "Some chairs are nonchairs"). We also assume that the only logical relations among the simple statements involved are those asserted or entailed by the premises. The point of these restrictions is to limit our considerations, in this chapter and the next, to truth-functional arguments alone, and to exclude other kinds of arguments whose validity turns on more complex logical considerations that are not appropriately introduced at this point.

A given argument exhibits its form very clearly when the simple statements that appear in it are abbreviated by capital letters. Thus we may abbreviate the statements, "Bacon wrote the plays attributed to Shakespeare," "Bacon was a great writer," "Washington was assassinated," and "Washington is dead," by the letters *B*, *G*, *A*, and *D*, respectively, and using the familiar three-dot symbol "∴" for "therefore," we may symbolize the two preceding arguments as

$B \supset G$		$A \supset D$
G	and	D
$\therefore B$		$\therefore A$

So written, their common form is easily seen.

To discuss forms of arguments rather than particular arguments having those forms, we need some method of symbolizing argument forms themselves. To achieve such a method, we introduce the notion of a **variable.** In the preceding sections we used capital letters to symbolize particular simple statements. To avoid confusion, we use small, or lowercase, letters from the middle part of the alphabet, *p*, *q*, *r*, s, . . ., as *statement variables.* A **statement variable,** as we shall use the term, is simply a letter for which, or in place of which, a statement may be substituted. Compound statements as well as simple statements may be substituted for statement variables.

We define an **argument form** as any array of symbols containing statement variables but no statements, such that when statements are substituted for the statement variables—the same statement being substituted for the same statement variable throughout—the result is an argument. For definiteness, we establish the convention that in any argument form, *p* shall be the first statement variable that occurs in it, *q* shall be the second, *r* the third, and so on. Thus the expression

$p \supset q$

q

$\therefore p$

is an argument form, for when the statements *B* and *G* are substituted for the statement variables *p* and *q*, respectively, the result is the first argument in this section. If the statements *A* and *D* are substituted for the variables *p* and *q*, the result is the second argument. Any argument that results from the substitution of statements for statement variables in an argument form is called a **substitution instance** of that argument form. It is clear that any substitution instance of an argument form may be said to have that form, and that any argument that has a certain form is a substitution instance of that form.

For any argument there are usually several argument forms that have the given argument as a substitution instance. For example, the first argument of this section,

$B \supset G$

G

$\therefore B$

is a substitution instance of each of the four argument forms

$p \supset q$	$p \supset q$	$p \supset q$	p
q	r	r	q
$\therefore p$	$\therefore p$	$\therefore s$	$\therefore r$

Thus we obtain the given argument by substituting B for p and G for q in the first argument form; by substituting B for p and G for both q and r in the second; B for both p and s and G for both q and r in the third; and $B \supset G$ for p, G for q, and B for r in the fourth. Of these four argument forms, the first corresponds more closely to the structure of the given argument than do the others. It does so because the given argument results from the first argument form by substituting a different simple statement for each different statement variable in it. We call the first argument form the *specific form* of the given argument. Our definition of the specific form of a given argument is the following: If an argument is produced by substituting consistently a different simple statement for each different statement variable in an argument form, that argument form is the **specific form of that argument.** For any given argument, there is a unique argument form that is the specific form of that argument.

EXERCISES

Here follows a group of arguments (Group **A,** lettered **a–o**) and a group of argument forms (Group **B,** numbered **1–24**). For each of the arguments (in Group **A**), indicate which of the argument forms (in Group **B**), if any, have the given argument as a *substitution instance.* In addition, for each given argument (in Group **A**), indicate which of the argument forms (in Group **B**), if any, is the *specific form* of that argument.

EXAMPLES

Argument **a** in Group **A:** Examining all the argument forms in Group **B,** we find that the only one of which Argument **a** is a *substitution instance* is Number **3.** Number **3** *is* also the *specific form* of Argument **a.**

Argument **j** in Group **A:** Examining all the argument forms in Group **B,** we find that Argument **j** is a *substitution instance* of *both* Number **6** and Number **23.** But *only* Number **23** is the *specific form* of Argument **j.**

Argument **m** in Group **A:** Examining all the argument forms in Group **B,** we find that Argument **m** is a *substitution instance* of *both* Number **3** and Number **24.** But there is *no* argument form in Group **B** that is the *specific form* of Argument **m.**

Group A—Arguments

a. $A \bullet B$
$\therefore A$

b. $C \supset D$
$\therefore C \supset (C \bullet D)$

c. E
$\therefore E \vee F$

d. $G \supset H$
$\sim H$
$\therefore \sim G$

***e.** I
J
$\therefore I \bullet J$

f. $(K \supset L) \bullet (M \supset N)$
$K \vee M$
$\therefore L \vee N$

g. $O \supset P$
$\sim O$
$\therefore \sim P$

h. $Q \supset R$
$Q \supset S$
$\therefore R \vee S$

i. $T \supset U$
$U \supset V$
$\therefore V \supset T$

j. $(W \bullet X) \supset (Y \bullet Z)$
$\therefore (W \bullet X) \supset [(W \bullet X) \bullet (Y \bullet Z)]$

k. $A \supset B$
$\therefore (A \supset B) \vee C$

l. $(D \vee E) \bullet \sim F$
$\therefore D \vee E$

m. $[G \supset (G \bullet H)] \bullet [H \supset (H \bullet G)]$
$\therefore G \supset (G \bullet H)$

n. $(I \vee J) \supset (I \bullet J)$
$\sim(I \vee J)$
$\therefore \sim(I \bullet J)$

***o.** $(K \supset L) \bullet (M \supset N)$
$\therefore K \supset L$

Group B—Argument Forms

***1.** $p \supset q$
$\therefore \sim q \supset \sim p$

2. $p \supset q$
$\therefore \sim p \supset \sim q$

3. $p \bullet q$
$\therefore p$

4. p
$\therefore p \vee q$

***5.** p
$\therefore p \supset q$

6. $p \supset q$
$\therefore p \supset (p \bullet q)$

7. $(p \vee q) \supset (p \bullet q)$
$\therefore (p \supset q) \bullet (q \supset p)$

8. $p \supset q$
$\sim p$
$\therefore \sim q$

9. $p \supset q$
$\sim q$
$\therefore \sim p$

***10.** p
q
$\therefore p \bullet q$

11. $p \supset q$
$p \supset r$
$\therefore q \vee r$

12. $p \supset q$
$q \supset r$
$\therefore r \supset p$

13. $p \supset (q \supset r)$
$p \supset q$
$\therefore p \supset r$

14. $p \supset (q \bullet r)$
$(q \vee r) \supset \sim p$
$\therefore \sim p$

***15.** $p \supset (q \supset r)$
$q \supset (p \supset r)$
$\therefore (p \vee q) \supset r$

16. $(p \supset q) \cdot (r \supset s)$
$p \vee r$
$\therefore q \vee s$

17. $(p \supset q) \cdot (r \supset s)$
$\sim q \vee \sim s$
$\therefore \sim p \vee \sim s$

18. $p \supset (q \supset r)$
$q \supset (r \supset s)$
$\therefore p \supset s$

19. $p \supset (q \supset r)$
$(q \supset r) \supset s$
$\therefore p \supset s$

***20.** $(p \supset q) \cdot [(p \cdot q) \supset r]$
$p \supset (r \supset s)$
$\therefore p \supset s$

21. $(p \vee q) \supset (p \cdot q)$
$\sim (p \vee q)$
$\therefore \sim (p \cdot q)$

22. $(p \vee q) \supset (p \cdot q)$
$(p \cdot q)$
$\therefore p \vee q$

23. $(p \cdot q) \supset (r \cdot s)$
$\therefore (p \cdot q) \supset [(p \cdot q) \cdot (r \cdot s)]$

24. $(p \supset q) \cdot (r \supset s)$
$\therefore p \supset q$

8.5 The Precise Meaning of "Invalid" and "Valid"

We are now in a position to address with precision the central questions of deductive logic:

1. *What precisely is meant* by saying that an argument form is invalid, or valid?
2. *How do we decide* whether a deductive argument form is invalid, or valid?

The first of these questions is answered in this section, the second in the following section.

We proceed by using the technique of refutation by logical analogy.* If the specific form of a given argument has any substitution instance whose premises are true and whose conclusion is false, then the given argument is invalid. We may define the term **invalid** as applied to argument forms as follows: *An argument form is invalid if and only if it has at least one substitution instance with true premises and a false conclusion*. Refutation by logical analogy is based on the fact that any argument whose specific form is an *invalid argument form* is an invalid argument. Any argument form that is not invalid must be **valid.** Hence *an argument form is valid if and only if it has no substitution instances with true premises and a false conclusion*. And because validity is a formal notion, an argument is valid if and only if the specific form of that argument is a *valid argument form*.

A given argument is proved invalid if a refuting analogy can be found for it, but "thinking up" such refuting analogies may not always be easy. Happily, it is not necessary, because for arguments of this type there is a simpler, purely

*Just as in analyzing the categorical syllogism; we discuss refutation by logical analogy in Section 6.2.

mechanical test based on the same principle. Given any argument, we test the specific form of that argument, because its validity or invalidity determines the validity or invalidity of the argument.

8.6 Testing Argument Validity Using Truth Tables

Knowing exactly what it means to say that an argument is valid, or invalid, we can now devise a method for *testing* the validity of every truth-functional argument. Our method, using a truth table, is very simple and very powerful. It is simply an application of the analysis of argument forms just given.

To test an argument form, we examine all possible substitution instances of it to see if any one of them has true premises and a false conclusion. Of course any argument form has an infinite number of substitution instances, but we need not worry about having to examine them one at a time. We are interested only in the truth or falsehood of their premises and conclusions, so we need consider only the truth values involved. The arguments that concern us here contain only simple statements and compound statements that are built up out of simple statements using the truth-functional connectives symbolized by the dot, curl, wedge, and horseshoe. Hence we obtain all possible substitution instances whose premises and conclusions have different truth values by examining all possible different arrangements of truth values for the statements that can be substituted for the different statement variables in the argument form to be tested.

When an argument form contains just two different statement variables, *p* and *q*, all of its substitution instances are the result of either substituting true statements for both *p* and *q*, or a true statement for *p* and a false one for *q*, or a false one for *p* and a true one for *q*, or false statements for both *p* and *q*. These different cases are assembled most conveniently in the form of a truth table. To decide the validity of the argument form

$p \supset q$

q

$\therefore p$

we construct the following truth table:

p	*q*	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

Each row of this table represents a whole class of substitution instances. The **T**'s and **F**'s in the two initial or guide columns represent the truth values of the statements substituted for the variables *p* and *q* in the argument form. We fill in the third column by referring back to the initial or guide columns and the definition of the horseshoe symbol. The third column heading is the first "premise" of the argument form, the second column is the second "premise" and the first column is the "conclusion." In examining this truth table, we find that in the third row there are **T**'s under both premises and an **F** under the conclusion, which indicates that there is at least one substitution instance of this argument form that has true premises and a false conclusion. This row suffices to show that the argument form is invalid. Any argument of this specific form (that is, any argument the specific argument form of which is the given argument form) is said to commit the fallacy of affirming the consequent, since its second premise affirms the consequent of its conditional first premise.

Truth tables, although simple in concept, are powerful tools. In using them to establish the validity or the invalidity of an argument form, it is critically important that the table first be constructed correctly. To construct the truth table correctly, there must be a guide column for each statement variable in the argument form, *p*, *q*, *r*, and so on. The array must exhibit all the possible combinations of the truth and falsity of all these variables, so there must be a number of horizontal rows sufficient to do this: four rows if there are two variables, eight rows if there are three variables, and so on. And there must be an additional vertical column for each of the premises and for the conclusion, and also a column for each of the symbolic expressions out of which the premises and conclusion are built. The construction of a truth table in this fashion is essentially a mechanical task; it requires only careful counting and the careful placement of **T**'s and **F**'s in the appropriate columns, all governed by our understanding of the several truth-functional connectives—the dot, the wedge, the horseshoe—and the circumstances under which each truth-functional compound is true and the circumstances under which it is false.

Once the table has been constructed and the completed array is before us, it is essential to *read* it correctly, that is, to use it correctly to make the appraisal of the argument form in question. We must note carefully which columns are those representing the premises of the argument being tested, and which column represents the conclusion of that argument. In testing the argument just above, which we found to be invalid, we noted that it was the second and third columns of the truth table that represent the premises, while the conclusion was represented by the first (leftmost) column. Depending on which argument form we are testing, and the order in which we have placed the columns as the table was built, it is possible for the premises and the conclusion to appear in any order at the top of the table. Their position to the right or to the left is not significant; we, who use the table, must understand which column represents what, and we must understand what we are in search of. *Is there any one case*, we ask ourselves, *any single row in which all the premises are true and the conclusion false*? If there is such a row, the argument form is invalid; if there is no such row, the argument form must be valid. After the full array has been neatly and accurately set forth, great care in reading the truth table accurately is of the utmost importance.

8.7 Some Common Argument Forms

A. COMMON VALID FORMS

Some valid argument forms are exceedingly common and may be intuitively understood. These may now be precisely identified. They should be recognized wherever they appear, and they may be called by their widely accepted names: (1) **Disjunctive Syllogism,** (2) ***Modus Ponens*,** (3) ***Modus Tollens*,** and, (4) **Hypothetical Syllogism.**

Disjunctive Syllogism

One of the simplest argument forms relies on the fact that in every true disjunction, at least one of the disjuncts must be true. Therefore, if one of them is false, the other must be true. Arguments in this form are exceedingly common. When a candidate for a high appointed office was forced to withdraw her candidacy because of a tax violation involving one of her employees, a critic wrote: "In trying to cover up her illegal alien peccadillo, or stonewall her way out of it, she was driven either by stupidity or arrogance. She's obviously not stupid; her plight must result, then, from her arrogance."[4]

We symbolize the Disjunctive Syllogism as

$p \vee q$

$\sim p$

$\therefore q$

And to show its validity we construct the following truth table:

p	q	$p \vee q$	$\sim p$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

Here, too, the initial or guide columns exhibit all possible different truth values of statements that may be substituted for the variables *p* and *q*. We fill in the third column by referring back to the first two, and the fourth by reference to the first alone. Now the third row is the only one in which T's appear under both premises (the third and fourth columns), and there a T also appears under the conclusion (the second column). The truth table thus shows that the argument form has no substitution instance having true premises and a false conclusion, and thereby proves the validity of the argument form being tested.*

Here, as always, it is essential that the truth table be *read* accurately; the column representing the conclusion (second from the left) and the columns representing the premises (third and fourth from the left) must be carefully identified. Only by using those three columns correctly can we reliably determine the validity (or invalidity) of the argument form in question. Note that the very same truth table could be used to test the validity of a very different argument form, one whose premises are represented by the second and third columns and whose conclusion is represented by the fourth column. That argument form, as we can see from the top row of the table, is invalid. The truth-table technique provides a completely mechanical method for testing the validity of any argument of the general type considered here.

We are now in a position to justify our proposal to translate any occurrence of the "if–then" phrase into our material implication symbol "⊃". In section 8.3, the claim was made that all valid arguments of the general type with which we are concerned here that involve "if–then" statements remain valid when those statements are interpreted as affirming merely material implications. Truth tables can be used to substantiate this claim, and will justify our translation of "if–then" into the horseshoe symbol.

Modus Ponens

The simplest type of intuitively valid argument involving a conditional statement is illustrated by the argument:

> If the second native told the truth, then only one native is a politician.
>
> The second native told the truth.
>
> Therefore only one native is a politician.

---

*As used in this chapter, the term *Disjunctive Syllogism* is the name of an elementary argument form, here proved valid. This form is always valid, of course, and therefore, in modern logic, "Disjunctive Syllogism" always refers to an elementary argument form that is valid. In traditional logic, however, the term "disjunctive syllogism" is used more broadly, to refer to any syllogism that contains a disjunctive premise; some such syllogisms may of course be invalid. One must be clear whether the expression is being used in the broader or the narrower sense. Here we use it in the narrower sense.

The specific form of this argument, known as *Modus Ponens* ("the method of putting, or affirming") is

$p \supset q$

p

$\therefore q$

and is proved valid by the following truth table:

p	q	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

Here the two premises are represented by the third and first columns, and the conclusion is represented by the second. Only the first row represents substitution instances in which both premises are true, and the **T** in the second column shows that in these arguments the conclusion is true also. This truth table establishes the validity of any argument of the form *modus ponens*.

Modus Tollens

If a conditional statement is true, then if the consequent is false, the antecedent must also be false. The argument form that relies on this is very commonly used to establish the falsehood of some proposition under attack. To illustrate, a distinguished rabbi, insisting that the Book of Genesis was never meant to be a scientific treatise, presented this crisp argument:

> A literal reading of Genesis would lead one to conclude that the world is less than 6,000 years old and that the Grand Canyon could have been carved by the global flood 4,500 years ago. Since this is impossible, a literal reading of Genesis must be wrong.[5]

The argument may be symbolized as

$p \supset q$

$\sim q$

$\therefore \sim p$

The validity of this argument form, called *Modus Tollens* ("the method of taking away or denying"), may be shown by the following truth table:

p	q	$p \supset q$	$\sim q$	$\sim p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Here again, there is no substitution instance, no line, on which the premises, $p \supset q$ and $\sim q$, are both true and the conclusion, $\sim p$, is false.

Hypothetical Syllogism

Another common type of intuitively valid argument contains only conditional statements. Here is an example:

If the first native is a politician, then the first native lies.

If the first native lies, then the first native denies being a politician.

Therefore if the first native is a politician, then the first native denies being a politician.

The specific form of this argument is

$p \supset q$

$q \supset r$

$\therefore p \supset r$

This argument, called a **Hypothetical Syllogism,*** contains three distinct statement variables, so the truth table must have three initial (or guide) columns and requires eight rows to list all possible substitution instances. Besides the initial columns, three additional columns are needed: two for the premises, the third for the conclusion. The table is

p	q	r	$p \supset q$	$q \supset r$	$p \supset r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

*Called a *pure hypothetical syllogism* in Chapter 7.

In constructing it, we fill in the fourth column by referring back to the first and second, the fifth by reference to the second and third, and the sixth by reference to the first and third. Examining the completed table, we observe that the premises are true only in the first, fifth, seventh, and eighth rows, and that in all of these the conclusion is also true. This truth table establishes the validity of the argument form and proves that the Hypothetical Syllogism remains valid when its conditional statements are translated by means of the horseshoe symbol.

Enough examples have been provided to illustrate the proper use of the truth-table technique for testing arguments. And perhaps enough have been given to show that the validity of any valid argument involving conditional statements is preserved when its conditionals are translated into merely material implications. Any doubts that remain can be allayed by the reader's translating and testing similar examples.

As more complicated argument forms are considered, larger truth tables are required to test them, because a separate initial or guide column is required for each different statement variable in the argument form. Only two are required for a form with just two variables, and that table will have four rows. But three initial columns are required for a form with three variables, such as the hypothetical syllogism, and such truth tables have eight rows. To test the validity of an argument form such as that of the *Constructive Dilemma*,

$(p \supset q) \bullet (r \supset s)$

$p \vee r$

$\therefore q \vee s$

which contains four distinct statement variables, a truth table with four initial columns and sixteen rows is required. In general, to test an argument form containing n distinct statement variables we need a truth table with n initial columns and 2^n rows.

B. COMMON INVALID FORMS

Two invalid argument forms deserve special notice because they superficially resemble valid forms and therefore often tempt careless writers or readers. The *fallacy of affirming the consequent*, discussed also in Section 7.7, is symbolized as

$p \supset q$

q

$\therefore p$

Although the shape of this form is something like that of *modus ponens*, the two argument forms are very different, and this form is not valid. It is well illustrated in a "bogus syllogism" about the dictatorial president of Iraq, the late Saddam Hussein. Here is that syllogism (as recounted in 2005 by Orlando Patterson), whose invalidity does indeed render it bogus: "If one is a terrorist one is a tyrant who hates freedom. Saddam Hussein is a tyrant who hates freedom. Therefore Saddam Hussein is a terrorist."[6] Let us suppose that the hypothetical first premise is true, and that the second premise describing Saddam Hussein is also true. But that second premise affirms (about Saddam Hussein as one tyrant) only the *consequent* of the preceding hypothetical. The argument plainly commits the fallacy of affirming the consequent.

Another invalid form, called the *fallacy of denying the antecedent*, has a shape somewhat like that of *modus tollens* and may be symbolized as

$p \supset q$

$\sim p$

$\therefore \sim q$

An example of this fallacy is the campaign slogan used by a candidate for mayor of New York City some years ago: "If you don't know the buck, you don't know the job—and Abe knows the buck." The unstated conclusion to which the voter was deliberately tempted was that "Abe knows the job"—a proposition that does not follow from the stated premises.

Both of these common fallacies may readily be shown to be invalid by means of truth tables. In each case there is one line of the truth table in which the premises of these fallacious arguments are all true, but the conclusion is false.

C. SUBSTITUTION INSTANCES AND SPECIFIC FORMS

A given argument can be a substitution instance of several different argument forms, as we noted earlier when defining argument form. Hence the valid disjunctive syllogism examined on page 317, which may be symbolized as

$R \vee W$

$\sim R$

$\therefore W$

is a substitution instance of the valid argument form

$p \vee q$

$\sim p$

$\therefore q$

and is *also* a substitution instance of the *in*valid argument form

p

q

$\therefore r$

It is obvious, in this last form, that from two premises, *p* and *q*, we cannot validly infer *r*. So it is clear that an invalid argument form can have a valid argument *or* an invalid argument as a substitution instance. Therefore, in determining whether any given argument is valid, *we must look to the specific form of the argument* in question. Only the specific form of the argument reveals the full logical structure of that argument, and because it does, we know that if the specific form of an argument is valid, the argument itself must be valid.

In the illustration just given, we see an argument ($R \vee W$, $\sim R$, $\therefore W$), and two argument forms of which that argument could be a substitution instance. The first of these argument forms ($p \vee q$, $\sim p$, $\therefore q$) is valid, and because that form is the *specific* form of the given argument, its validity establishes that the given argument is valid. The second of these argument forms is invalid, but because it is *not* the specific form of the given argument, it cannot be used to show that the given argument is invalid.

This point should be emphasized: An argument form that is valid can have only valid arguments as substitution instances. That is, all of the substitution instances of a valid form *must* be valid. This is proved by the truth-table proof of validity for the valid argument form, which shows that there is no possible substitution instance of a valid form that has true premises and a false conclusion.

EXERCISES

A. Use truth tables to prove the validity or invalidity of each of the argument forms in section 8.4, Group B, pages 345–346.

B. Use truth tables to determine the validity or invalidity of each of the following arguments.

***1.** $(A \vee B) \supset (A \bullet B)$
$A \vee B$
$\therefore A \bullet B$

2. $(C \vee D) \supset (C \bullet D)$
$C \bullet D$
$\therefore C \vee D$

3. $E \supset F$
$F \supset E$
$\therefore E \vee F$

4. $(G \vee H) \supset (G \bullet H)$
$\sim(G \bullet H)$
$\therefore \sim(G \vee H)$

***5.** $(I \vee J) \supset (I \bullet J)$
$\sim(I \vee J)$
$\therefore \sim(I \bullet J)$

6. $K \vee L$
K
$\therefore \sim L$

7. $M \vee (N \bullet \sim N)$
M
$\therefore \sim(N \bullet \sim N)$

8. $(O \vee P) \supset Q$
$Q \supset (O \bullet P)$
$\therefore (O \vee P) \supset (O \bullet P)$

9. $(R \vee S) \supset T$
$T \supset (R \bullet S)$
$\therefore (R \bullet S) \supset (R \vee S)$

***10.** $U \supset (V \vee W)$
$(V \bullet W) \supset \sim U$
$\therefore \sim U$

C. Use truth tables to determine the validity or invalidity of the following arguments.

***1.** If Angola achieves stability, then both Botswana and Chad will adopt more liberal policies. But Botswana will not adopt a more liberal policy. Therefore Angola will not achieve stability.

2. If Denmark refuses to join the European Community, then, if Estonia remains in the Russian sphere of influence, then Finland will reject a free trade policy. Estonia will remain in the Russian sphere of influence. So if Denmark refuses to join the European community, then Finland will reject a free trade policy.

3. If Greece strengthens its democratic institutions, then Hungary will pursue a more independent policy. If Greece strengthens its democratic institutions, then the Italian government will feel less threatened. Hence, if Hungary pursues a more independent policy, the Italian government will feel less threatened.

4. If Japan continues to increase the export of automobiles, then either Korea or Laos will suffer economic decline. Korea will not suffer economic decline. It follows that if Japan continues to increase the export of automobiles, then Laos will suffer economic decline.

***5.** If Montana suffers a severe drought, then, if Nevada has its normal light rainfall, Oregon's water supply will be greatly reduced. Nevada does have its normal light rainfall. So if Oregon's water supply is greatly reduced, then Montana suffers a severe drought.

6. If equality of opportunity is to be achieved, then those people previously disadvantaged should now be given special opportunities. If those people previously disadvantaged should now be given special opportunities, then some people receive preferential treatment. If some people receive preferential treatment, then equality of opportunity is not to be achieved. Therefore equality of opportunity is not to be achieved.

7. If terrorists' demands are met, then lawlessness will be rewarded. If terrorists' demands are not met, then innocent hostages will be murdered. So either lawlessness will be rewarded or innocent hostages will be murdered.

8. If people are entirely rational, then either all of a person's actions can be predicted in advance or the universe is essentially deterministic. Not all of a person's actions can be predicted in advance. Thus, if the universe is not essentially deterministic, then people are not entirely rational.

9. If oil consumption continues to grow, then either oil imports will increase or domestic oil reserves will be depleted. If oil imports increase and domestic oil reserves are depleted, then the nation eventually will go bankrupt. Therefore, if oil consumption continues to grow, then the nation eventually will go bankrupt.

***10.** If oil consumption continues to grow, then oil imports will increase and domestic oil reserves will be depleted. If either oil imports increase or domestic oil reserves are depleted, then the nation will soon be bankrupt. Therefore, if oil consumption continues to grow, then the nation will soon be bankrupt.

8.8 Statement Forms and Material Equivalence

A. STATEMENT FORMS AND STATEMENTS

We now make explicit a notion that was tacitly assumed in the preceding section, the notion of a *statement form*. There is an exact parallel between the relation of argument to argument form, on the one hand, and the relation of statement to statement form, on the other. The definition of a statement form makes this evident: A **statement form** is any sequence of symbols containing statement variables but no statements, such that when statements are substituted for the statement variables—the same statement being substituted for the same statement variable throughout—the result is a statement. Thus $p \vee q$ is a statement form, because when statements are substituted for the variables p and q, a statement results. The resulting statement is a disjunction, so $p \vee q$ is called a **disjunctive statement form.** Analogously, $p \bullet q$ and $p \supset q$ are called *conjunctive* and *conditional statement forms*, and $\sim p$ is called a *negation form* or *denial form*. Just as any argument of a certain form is said to be a substitution instance of that argument form, so any statement of a certain form is said to be a *substitution instance* of that statement form. And just as we distinguished the *specific form* of a given argument, so we distinguish the **specific form** of a given statement as that statement form from which the statement results by substituting consistently a different simple statement for each different statement variable. Thus $p \vee q$ is the *specific form* of the statement, "The blind prisoner has a red hat or the blind prisoner has a white hat."

B. TAUTOLOGOUS, CONTRADICTORY, AND CONTINGENT STATEMENT FORMS

The statement, "Lincoln was assassinated" (symbolized as L), and the statement, "Either Lincoln was assassinated or else he wasn't" (symbolized as $L \vee \sim L$), are both obviously true. But, we would say, they are true "in different ways" or have "different kinds" of truth. Similarly, the statement, "Washington was assassinated" (symbolized as W), and the statement "Washington was both assassinated and not assassinated" (symbolized as $W \bullet \sim W$), are both plainly false—but they also are false "in different ways" or have "different kinds" of falsehood. These differences in the "kinds" of truth or of falsehood are important and very great.

That the statement L is true, and that the statement W is false, are historical facts—facts about the way events did happen. There is no logical necessity about them. Events might have occurred differently, and therefore the truth values of such statements as L and W must be discovered by an empirical study of history. But the statement $L \vee \sim L$, although true, is not a truth of history. There is logical necessity here: Events could not have been such as to make it false, and its truth can be known independently of any particular empirical investigation. The statement $L \vee \sim L$ is a logical truth, a formal truth, true in virtue of its form alone. It is a substitution instance of a statement form all of whose substitution instances are true statements.

A statement form that has only true substitution instances is called a *tautologous statement form*, or a **tautology.** To show that the statement form $p \vee \sim p$ is a tautology, we construct the following truth table:

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

There is only one initial or guide column to this truth table, because the form we are considering contains only one statement variable. Consequently, there are only two rows, which represent all possible substitution instances. There are only **T**'s in the column under the statement form in question, and this fact shows that all of its substitution instances are true. Any statement that is a substitution instance of a tautologous statement form is true in virtue of its form, and is itself said to be tautologous, or a tautology.

A statement form that has only false substitution instances is said to be **self-contradictory,** or a **contradiction,** and is logically false. The statement form $p \bullet \sim p$

is self-contradictory, because only **F**'s occur under it in its truth table, signifying that all of its substitution instances are false. Any statement, such as $W \bullet \sim W$, which is a substitution instance of a self-contradictory statement form, is false in virtue of its form and is itself said to be self-contradictory, or a contradiction.

Statement forms that have both true and false statements among their substitution instances are called **contingent statement forms.** Any statement whose specific form is contingent is called a "*contingent statement*."* Thus p, $\sim p$, $p \bullet q$, $p \vee q$, and $p \supset q$ are all contingent statement forms. And such statements as L, $\sim L$, $L \bullet W$, $L \vee W$, and $L \supset W$ are contingent statements, because their truth values are dependent or contingent on their contents rather than on their forms alone.

Not all statement forms are so obviously tautological or self-contradictory or contingent as the simple examples cited. For example, the statement form $[(p \supset q) \supset p] \supset p$ is not at all obvious, though its truth table will show it to be a tautology. It even has a special name, *Peirce's law*.

C. MATERIAL EQUIVALENCE

Material equivalence is a truth-functional connective, just as disjunction and material implication are truth-functional connectives. The truth value of any truth-functional connective, as explained earlier, depends on (is a function of) the truth or falsity of the statements it connects. Thus, we say that the disjunction of A and B is true if either A is true or B is true or if they are both true. **Material equivalence** is the truth-functional connective that asserts that the statements it connects have the *same* truth value. Two statements that are equivalent in truth value, therefore, are materially equivalent. One straightforward definition is this: Two statements are *materially equivalent* when they are both true, or both false.

Just as the symbol for disjunction is the wedge, and the symbol for material implication is the horseshoe, there is also a special symbol for material equivalence, the three-bar sign "≡." And just as we gave truth-table definitions for the wedge and the horseshoe, we can do so for the three-bar sign, or **tribar.** Here is the truth table for material equivalence, "≡":

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

*It will be recalled that we are assuming here that no simple statements are either logically true or logically false. Only contingent simple statements are admitted here. See footnote, on page 342.

Any two true statements materially imply one another; that is a consequence of the meaning of material implication. And any two false statements also materially imply one another. Therefore any two statements that are materially equivalent must imply one another, because they are either both true or both false.

Any two statements, *A* and *B*, that are materially equivalent imply one another, so we may infer from their material equivalence that *B* is true *if A* is true, and also that *B* is true *only if A* is true. Because both of these relations are entailed by material equivalence, we can read the three-bar sign, ≡, to say "*if and only if*."

In everyday discourse we use this logical relation only occasionally. I will go to the championship game, one may say, if and only if I can acquire a ticket. I will go *if* I do acquire a ticket, but I can go *only if* I acquire a ticket. So my going to the game, and my acquiring a ticket to the game, are materially equivalent.

Every implication is a *conditional* statement, as we noted earlier. Two statements, *A* and *B*, that are materially equivalent entail the truth of the conditional $A \supset B$, and also entail the truth of the conditional $B \supset A$. Because the implication goes both ways when material equivalence holds, a statement of the form $A \equiv B$ is often called a **biconditional.**

There are four truth-functional connectives on which deductive arguments commonly depend: *conjunction*, *disjunction*, *material implication*, and *material equivalence*. Our discussion of the four is now complete.

OVERVIEW

The Four Truth-Functional Connectives

Truth-Functional Connective	Symbol (Name of Symbol)	Proposition Type	Names of Components of Propositions of That Type	Example
And	• (dot)	Conjunction	Conjuncts	Carol is mean and Bob sings the blues. $C \bullet B$
Or	∨ (wedge)	Disjunction	Disjuncts	Carol is mean **or** Tyrell is a music lover. $C \vee T$
If . . . then	⊃ (horseshoe)	Conditional	Antecedent, Consequent	**If** Bob sings the blues, **then** Myrna gets moody. $B \supset M$
If and only if	≡ (tribar)	Biconditional	Components	Myrna gets moody **if and only if** Bob sings the blues. $M \equiv B$

Note: "Not" is not a connective, but is a truth-function operator, so it is omitted here.

D. ARGUMENTS, CONDITIONAL STATEMENTS, AND TAUTOLOGIES

To every argument there corresponds a conditional statement whose antecedent is the conjunction of the argument's premises and whose consequent is the argument's conclusion. Thus, an argument having the form of *Modus Ponens*,

$p \supset q$

p

$\therefore q$

may be expressed as a conditional statement of the form $[(p \supset q) \bullet p] \supset q$. If the argument expressed as a conditional has a valid argument form, then its conclusion must in every case follow from its premises, and therefore the conditional statement of it may be shown on a truth table to be a tautology. That is, the statement that the conjunction of the premises implies the conclusion will (if the argument is valid) have all and only true instances.

Truth tables are powerful devices for the evaluation of arguments. An argument form is valid if and only if its truth table has a **T** under the conclusion in every row in which there are **T**'s under all of its premises. This follows from the precise meaning of *validity*. Now, if the conditional statement expressing that argument form is made the heading of one column of the truth table, an **F** can occur in that column only in a row in which there are **T**'s under all the premises and an **F** under the conclusion. But there will be no such row if the argument is valid. Hence only **T**'s will occur under a conditional statement that corresponds to a valid argument, and that conditional statement *must* be a tautology. We may therefore say that an argument form is valid if, and only if, its expression in the form of a conditional statement (of which the antecedent is the conjunction of the premises of the given argument form, and the consequent is the conclusion of the given argument form) is a tautology.

For every *invalid* argument of the truth-functional variety, however, the corresponding conditional statement will not be a tautology. The statement that the conjunction of its premises implies its conclusion is (for an invalid argument) either contingent or contradictory.

EXERCISES

A. For each statement in the left-hand column, indicate which, if any, of the statement forms in the right-hand column have the given statement as a substitution instance, and indicate which, if any, is the specific form of the given statement.

1. $A \vee B$
2. $C \bullet {\sim}D$
3. ${\sim}E \supset (F \bullet G)$
4. $H \supset (I \bullet J)$
*5. $(K \bullet L) \vee (M \bullet N)$
6. $(O \vee P) \supset (P \bullet Q)$
7. $(R \supset S) \vee (T \bullet {\sim}U)$
8. $V \supset (W \vee {\sim}W)$
9. $[(X \supset Y) \supset X] \supset X$
*10. $Z \equiv {\sim}{\sim}Z$

a. $p \bullet q$
b. $p \supset q$
c. $p \vee q$
d. $p \bullet {\sim}q$
e. $p \equiv q$
f. $(p \supset q) \vee (r \bullet s)$
g. $[(p \supset q) \supset r] \supset s$
h. $[(p \supset q) \supset p] \supset p$
i. $(p \bullet q) \vee (r \bullet s)$
j. $p \supset (q \vee {\sim}r)$

B. Use truth tables to characterize the following statement forms as tautologous, self-contradictory, or contingent.

*1. $[p \supset (p \supset q)] \supset q$
2. $p \supset [(p \supset q) \supset q]$
3. $(p \bullet q) \bullet (p \supset {\sim}q)$
4. $p \supset [{\sim}p \supset (q \vee {\sim}q)]$
*5. $p \supset [p \supset (q \bullet {\sim}q)]$
6. $(p \supset p) \supset (q \bullet {\sim}q)$
7. $[p \supset (q \supset r)] \supset [(p \supset q) \supset (p \supset r)]$
8. $[p \supset (q \supset p)] \supset [(q \supset q) \supset {\sim}(r \supset r)]$
9. $\{[(p \supset q) \bullet (r \supset s)] \bullet (p \vee r)\} \supset (q \vee s)$
10. $\{[(p \supset q) \bullet (r \supset s)] \bullet (q \vee s)\} \supset (p \vee r)$

C. Use truth tables to decide which of the following biconditionals are tautologies.

*1. $(p \supset q) \equiv ({\sim}q \supset {\sim}p)$
2. $(p \supset q) \equiv ({\sim}p \supset {\sim}q)$
3. $[(p \supset q) \supset r] \equiv [(q \supset p) \supset r]$
4. $[p \supset (q \supset r)] \equiv [q \supset (p \supset r)]$
*5. $p \equiv [p \bullet (p \vee q)]$
6. $p \equiv [p \vee (p \bullet q)]$
7. $p \equiv [p \bullet (p \supset q)]$
8. $p \equiv [p \bullet (q \supset p)]$
9. $p \equiv [p \vee (p \supset q)]$
*10. $(p \supset q) \equiv [(p \vee q) \equiv q]$
11. $p \equiv [p \vee (q \bullet {\sim}q)]$
12. $p \equiv [p \bullet (q \bullet {\sim}q)]$
13. $p \equiv [p \bullet (q \vee {\sim}q)]$
14. $p \equiv [p \vee (q \vee {\sim}q)]$
*15. $[p \bullet (q \vee r)] \equiv [(p \bullet q) \vee (p \bullet r)]$
16. $[p \bullet (q \vee r)] \equiv [(p \vee q) \bullet (p \vee r)]$
17. $[p \vee (q \bullet r)] \equiv [(p \bullet q) \vee (p \bullet r)]$
18. $[p \vee (q \bullet r)] \equiv [(p \vee q) \bullet (p \vee r)]$
19. $[(p \bullet q) \supset r] \equiv [p \supset (q \supset r)]$
*20. $[(p \supset q) \bullet (q \supset p)] \equiv [(p \bullet q) \vee ({\sim}p \bullet {\sim}q)]$

8.9 Logical Equivalence

At this point we introduce a new relation, important and very useful, but not a connective, and somewhat more complicated than any of the truth-functional connectives just discussed.

Statements are materially equivalent when they have the same truth value. Because two materially equivalent statements are either both true, or both false, we can readily see that they must (materially) imply one another, because a false antecedent (materially) implies any statement, and a true consequent is (materially) implied by any statement. We may therefore read the three-bar sign, $\equiv$, as "if and only if."

However, statements that are materially equivalent most certainly cannot be substituted for one another. Knowing that they are materially equivalent, we know only that their truth values are the same. The statements, "Jupiter is larger than the Earth" and "Tokyo is the capital of Japan," are materially equivalent because they are both true, but we obviously cannot replace one with the other. Similarly, the statements, "All spiders are poisonous" and "No spiders are poisonous," are materially equivalent simply because they are both false, and they certainly cannot replace one another!

There are many circumstances, however, in which we must express the relationship that does permit mutual replacement. Two statements can be equivalent in a sense much stronger than that of material equivalence; they may be equivalent in *meaning* as well as having the same truth value. If they do have the same meaning, any proposition that incorporates one of them can just as well incorporate the other; there will not be—there cannot be—any case in which one of these statements is true while the other is false. Statements that are equivalent in this very strong sense are called *logically equivalent*.

Of course, any two statements that are logically equivalent are materially equivalent as well, for they obviously have the same truth value. Indeed, if two statements are logically equivalent, they are materially equivalent under *all* circumstances—and this explains the short but powerful definition of **logical equivalence:** *Two statements are logically equivalent if the statement of their material equivalence is a tautology.* That is, the statement that they have the same truth value is itself necessarily true. And this is why, to express this very strong logical relationship, we use the three-bar symbol with a small **T** immediately above it, $\overset{T}{\equiv}$, indicating that the logical relationship is of such a nature that the material equivalence of the two statements is a tautology. And because material equivalence is a biconditional (the two statements implying one another), we may think of this symbol of logical equivalence, $\overset{T}{\equiv}$, as expressing a tautological biconditional.

Some simple logical equivalences that are very commonly used will make this relation, and its great power, very clear. It is a commonplace that *p* and ~~*p* mean the same thing; "he is aware of that difficulty" and "he is not unaware of that difficulty" are two statements with the same content. In substance, either of these expressions may be replaced by the other because they both say the same thing. This principle of **double negation,** whose truth is obvious to all, may be exhibited in a truth table, where the material equivalence of two statement forms is shown to be a tautology:

p	$\sim p$	$\sim\sim p$	$p \overset{T}{\equiv} \sim\sim p$
T	F	T	T
F	T	F	T

This truth table proves that *p* and ~~*p* are *logically equivalent*. This very useful logical equivalence, double negation, is symbolized as

$$p \overset{T}{\equiv} \sim\sim p$$

The difference between *material equivalence* on the one hand and *logical equivalence* on the other hand is very great and very important. The former is a truth-functional connective, ≡, which may be true or false depending only on the truth or falsity of the elements it connects. But the latter, logical equivalence, $\overset{T}{\equiv}$, is not a mere connective, and it expresses a relation between two statements that is not truth-functional. Two statements are logically equivalent only when it is absolutely impossible for them to have different truth values. But if they *always* have the same truth value, logically equivalent statements must have the same meaning, and in that case they may be substituted for one another in any truth-functional context without changing the truth value of that context. By contrast, two statements are materially equivalent if they merely *happen* to have the same truth value, even if there are no factual connections between them. Statements that are merely materially equivalent certainly may not be substituted for one another!

There are two well-known logical equivalences (that is, logically true biconditionals) of great importance because they express the interrelations among conjunction and disjunction, and their negations. Let us examine these two logical equivalences more closely.

First, what will serve to deny that a disjunction is true? Any disjunction $p \vee q$ asserts no more than that at least one of its two disjuncts is true. One cannot contradict it by asserting that at least one is false; we must (to deny it) assert that both disjuncts are false. Therefore, asserting the *negation of the disjunction* ($p \vee q$) is logically equivalent to asserting the *conjunction of the negations* *of p and of q*. To show this in a truth table, we may formulate the biconditional, $\sim(p \vee q) \equiv (\sim p \bullet \sim q)$, place it at the top of its own column, and examine its truth value under all circumstances, that is, in each row.

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \bullet \sim q$	$\sim(p \vee q) \equiv (\sim p \bullet \sim q)$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Of course we see that, whatever the truth values of p and of q, this biconditional must always be true. It is a tautology. Because the statement of that material equivalence *is* a tautology, we conclude that the two statements are logically equivalent. We have proved that

$$\sim(p \vee q) \overset{T}{\equiv} (\sim p \bullet \sim q)$$

Similarly, asserting the conjunction of p and q asserts that both are true, so to contradict this assertion we need merely assert that at least one is false. Thus, asserting the negation of the conjunction, $(p \bullet q)$, is logically equivalent to asserting the disjunction of the negations of p and of q. In symbols, the biconditional, $\sim(p \bullet q) \equiv (\sim p \vee \sim q)$ may be shown, in a truth table, to be a tautology. Such a table proves that

$$\sim(p \bullet q) \overset{T}{\equiv} (\sim p \vee \sim q)$$

These two tautologous biconditionals, or logical equivalences, are known as De Morgan's theorems, because they were formally stated by the mathematician and logician Augustus De Morgan (1806–1871). **De Morgan's theorems** can be formulated in English thus:

a. The negation of the disjunction of two statements is logically equivalent to the conjunction of the negations of the two statements;

and

b. The negation of the conjunction of two statements is logically equivalent to the disjunction of the negations of the two statements.

These theorems of De Morgan are exceedingly useful.

Another important logical equivalence is very helpful when we seek to manipulate truth-functional connectives. Material implication, $\supset$, was defined in Section 8.3) as an abbreviated way of saying $\sim(p \bullet \sim q)$. That is, "p materially implies q" simply means, by definition, that it is not the case that p is true while q is false. In this definition we see that the *definiens*, $\sim(p \bullet \sim q)$, is the denial of a conjunction. And by De Morgan's theorem we know that any such denial is logically equivalent to the disjunction of the denials of the conjuncts; that is, we know that $\sim(p \bullet \sim q)$ is logically equivalent to $(\sim p \vee \sim\sim q)$; and this expression in turn, applying the principle of double negation, is logically equivalent to $\sim p \vee q$. Logically equivalent expressions mean the same thing, and therefore the original *definiens* of the horseshoe, $\sim(p \bullet \sim q)$, may be replaced with no change of meaning by the simpler expression $\sim p \vee q$. This gives us a very useful definition of material implication: *$p \supset q$ is logically equivalent to $\sim p \vee q$*. In symbols we write:

$$(p \supset q) \overset{T}{\equiv} (\sim p \vee q)$$

This definition of material implication is widely relied on in the formulation of logical statements and the analysis of arguments. Manipulation is often essential, and manipulation is more efficient when the statements we are working with have the same central connective. With the simple definition of the horseshoe we have just established, $(p \supset q) \overset{T}{\equiv} (\sim p \vee q)$, statements in which the horseshoe is the connective can be conveniently replaced by statements in which the wedge is the connective; and likewise, statements in disjunctive form may be readily replaced by statements in implicative form. When we seek to present a formal proof of the validity of deductive arguments, replacements of this kind are very useful indeed.

Before going on to the methods of testing for validity and invalidity in the next section, it is worthwhile to pause for a more thorough consideration of the meaning of material implication. Implication is central in argument but, as we noted earlier, the word "implies" is highly ambiguous. *Material* implication, on which we rely in this analysis, is only one sense of that word, although it is a very important sense, of course. The definition of material implication explained just above makes it clear that when we say, in this important sense, that "p implies q," we are saying no more than that "either q is true or p is false."

Asserting the "if–then" relation in this sense has consequences that may *seem* paradoxical. For in this sense we can say, *correctly*, "If a statement is true, then it is implied by any statement whatever." Because it is true that the earth is round, it follows that "The moon is made of green cheese implies that the earth is round." This appears to be very curious, especially because it also follows that "The moon is *not* made of green cheese implies that the earth is round." Our precise understanding of material implication also entitles us to say, *correctly*, "If a statement is false, then it implies any statement whatever." Because it is false that the moon is made of green cheese, it follows that "The moon is made of green cheese implies that the earth is round;" and this is the more curious when we realize that it also follows that "The moon is made of green cheese implies that the earth is *not* round."

Why do these true statements seem so curious? It is because we recognize that the shape of the earth and the cheesiness of the moon are utterly irrelevant to each other. As we normally use the word "implies," a statement cannot imply some other statement, false or true, to which it is utterly irrelevant. That is the case when "implies" is used in most of its everyday senses. And yet those "paradoxical" statements in the preceding paragraph are indeed true, and not really problematic at all, because they use the word "implies" in the logical sense of "material implication." The precise meaning of material implication we have made very clear; we understand that to say p materially implies q is only to say that either p is false or q is true.

What needs to be borne in mind is this: *Meaning*—subject matter—is strictly irrelevant to material implication. *Material implication is a truth function*. Only the truth and falsity of the antecedent and the consequent, not their content, are relevant here. There is nothing paradoxical in stating that any disjunction is true that contains one true disjunct. Well, when we say that "The moon is made of green cheese (materially) implies that the earth is round," we know that to be logically equivalent to saying "Either the moon is not made of green cheese or the earth is round"—a disjunction that is most certainly true. And any disjunction we may confront in which "The moon is not made of green cheese" is the first disjunct will certainly be true, no matter what the second disjunct asserts. So, yes, "The moon is made of green cheese (materially) implies that the earth is square" because that is logically equivalent to "The moon is not made of green cheese or the earth is square." A false statement materially implies any statement whatever. A true statement is materially implied by any statement whatever.

Every occurrence of "if–then" should be treated, we have said, as a material implication, and represented with the horseshoe, $\supset$. The justification of this practice, its logical expediency, is the fact that doing so preserves the validity of all valid arguments of the type with which we are concerned in this part of our logical studies. Other symbolizations have been proposed, adequate to other types of implication, but they belong to more advanced parts of logic, beyond the scope of this book.

8.10 The Three "Laws of Thought"

Some early thinkers, after having defined logic as "the science of the laws of thought," went on to assert that there are exactly three *basic* laws of thought, laws so fundamental that obedience to them is both the necessary and the sufficient condition of correct thinking. These three have traditionally been called:

- The **principle of identity.** This principle asserts that *if any statement is true, then it is true.* Using our notation we may rephrase it by saying that the principle of identity asserts that every statement of the form $p \supset p$ must be true, that every such statement is a tautology.
- The **principle of noncontradiction.** This principle asserts that *no statement can be both true and false.* Using our notation we may rephrase it by saying that the principle of noncontradiction asserts that every statement of the form $p \bullet {\sim}p$ must be false, that every such statement is self-contradictory.
- The **principle of excluded middle.** This principle asserts that *every statement is either true or false.* Using our notation we may rephrase it by saying that the principle of excluded middle asserts that every statement of the form $p \vee {\sim}p$ must be true, that every such statement is a tautology.

It is obvious that these three principles are indeed true—logically true—but the claim that they deserve privileged status as the most fundamental laws of thought is doubtful. The first (identity) and the third (excluded middle) are tautologies, but there are many other tautologous forms whose truth is equally certain. And the second (noncontradiction) is by no means the only self-contradictory form of statement.

We do use these principles in completing truth tables. In the initial columns of each row of a table we place either a **T** or an **F,** being guided by the principle of excluded middle. Nowhere do we put both **T** and **F,** being guided by the principle of noncontradiction. And once having put a **T** under a symbol in a given row, then (being guided by the principle of identity) when we encounter that symbol in other columns of that row, we regard it as still being assigned a **T.** So we could regard the three laws of thought as principles governing the construction of truth tables.

Nevertheless, in regarding the entire system of deductive logic, these three principles are no more important or fruitful than many others. Indeed, there are tautologies that are more fruitful than they for purposes of deduction, and in that sense more important than these three. A more extended treatment of this point lies beyond the scope of this book.[7]

Some thinkers, believing themselves to have devised a new and different logic, have claimed that these three principles are in fact not true, and that obedience to them has been needlessly confining. But these criticisms have been based on misunderstandings.

The principle of identity has been attacked on the ground that things change, and are always changing. Thus, for example, statements that were true of the United States when it consisted of the thirteen original states are no longer true of the United States today, which has fifty states. But this does not undermine the principle of identity. The sentence, "There are only thirteen states in the United States," is incomplete, an elliptical formulation of the statement that "There were only thirteen states in the United States *in 1790*"—and that statement is as true today as it was in 1790. When we confine our attention to complete, nonelliptical formulations of propositions, we see that their truth (or falsity) does not change over time. The principle of identity is true, and it does not interfere with our recognition of continuing change.

The principle of noncontradiction has been attacked by Hegelians and Marxists on the grounds that genuine contradiction is everywhere pervasive, that the world is replete with the inevitable conflict of contradictory forces. That there are conflicting forces in the real word is true, of course—but to call these conflicting forces "contradictory" is a loose and misleading use of that term. Labor unions and the private owners of industrial plants may indeed find themselves in conflict—but neither the owner nor the union is the "negation" or the "denial" or the "contradictory" of the other. The principle of noncontradiction, understood in the straightforward sense in which it is intended by logicians, is unobjectionable and perfectly true.

The principle of excluded middle has been the object of much criticism, on the grounds that it leads to a "two-valued orientation," which implies that things in the world must be either "white or black," and which thereby hinders the realization of compromise and less than absolute gradations. This objection also arises from misunderstanding. Of course the statement "This is black" cannot be jointly true with the statement "This is white"—where "this" refers to exactly the same thing. But although these two statements cannot both be true, they can both be false. "This" may be neither black nor white; the two statements are *contraries*, not contradictory. The contradictory of the statement "This is white" is the statement "It is not the case that this is white" and (if "white" is used in precisely the same sense in both of these statements) one of them must be true and the other false. The principle of excluded middle is inescapable.

All three of these "laws of thought" are unobjectionable—so long as they are applied to statements containing unambiguous, nonelliptical, and precise terms. They may not deserve the honorific status assigned to them by some philosophers,* but they are indubitably true.

---

*Plato appealed explicitly to the principle of noncontradiction in Book IV of his *Republic* (at nos. 436 and 439); Aristotle discussed all three of these principles in Books IV and XI of his *Metaphysics*. Of the principle of noncontradiction, Aristotle wrote: "That the same attribute cannot at the same time belong and not belong to the same subject and in the same respect" is a principle "which everyone must have who understands anything that is," and which "everyone must already have when he comes to a special study." It is, he concluded, "the most certain of all principles."

SUMMARY

This chapter has presented the fundamental concepts of modern symbolic logic.

In Section 8.1, we explained the general approach of modern symbolic logic, and its need for an artificial symbolic language.

In Section 8.2, we introduced and defined the symbols for negation (the curl: ~); and for the truth-functional connectives of conjunction (the dot: •) and disjunction (the wedge: ∨). We also explained logical punctuation.

In Section 8.3, we discussed the different senses of implication, and defined the truth-functional connective material implication (the horseshoe: ⊃).

In Section 8.4, we explained the formal structure of arguments, defined argument forms, and explained other concepts essential in analyzing deductive arguments.

In Section 8.5, we gave a precise account of valid and invalid argument forms.

In Section 8.6, we explained the truth-table method of testing the validity of argument forms.

In Section 8.7, we identified and described a few very common argument forms, some valid and some invalid.

In Section 8.8, we explained the formal structure of statements and defined essential terms for dealing with statement forms. We introduced tautologous, contradictory, and contingent statement forms, and defined a fourth truth-functional connective, material equivalence (three bars: ≡).

In Section 8.9, we introduced and defined a powerful new relation, logical equivalence, using the symbol $\overset{T}{\equiv}$. We explained why statements that are logically equivalent may be substituted for one another, while statements that are merely materially equivalent cannot replace one another. We introduced several logical equivalences of special importance: De Morgan's theorems, the principle of double negation, and the definition of material implication.

In Section 8.10, we discussed certain logical equivalences that have been thought by many to be fundamental in all reasoning: the principle of identity, the principle of noncontradiction, and the principle of the excluded middle.

End Notes

[1]Somewhat more complicated definitions have been proposed by David H. Sanford in "What Is a Truth Functional Component?" *Logique et Analyse* 14 (1970): 483–486.

[2]Ted Turner, quoted in *The New Yorker*, 30 April 2001.

[3]"The Firm" *The New Yorker*, 8 March 1999.

[4]Peter J. Bertocci, "Chavez' Plight Must Come from Arrogance," *The New York Times*, 19 January 2001.

[5]Rabbi Ammiel Hirsch, "Grand Canyon," *The New York Times*, 10 October 2005.

[6]Orlando Patterson, "The Speech Misheard Round the World," *The New York Times*, 22 January 2005. Mr. Patterson's wording of the syllogism is very slightly different but has exactly the same logical force.

[7]For further discussion of these matters, the interested reader can consult I. M. Copi and J. A. Gould, eds., *Readings on Logic*, 2d ed. (New York: Macmillan, 1972), part 2; and I. M. Copi and J. A. Gould, eds., *Contemporary Philosophical Logic* (New York: St. Martin's Press, 1978), part 8.