7.5 Orthogonality

Let V be an inner product space. The vectors $u, v \in V$ are said to be *orthogonal* and u is said to be *orthogonal* to v if

$$\langle u, v\rangle = 0$$

The relation is clearly symmetric—if u is orthogonal to v, then $\langle v, u\rangle = 0$, and so v is orthogonal to u. We note that $0 \in V$ is orthogonal to every $v \in V$, because

$$\langle 0, v\rangle = \langle 0v, v\rangle = 0\langle v, v\rangle = 0$$

Conversely, if u is orthogonal to every $v \in V$, then $\langle u, u\rangle = 0$ and hence $u = 0$ by $[I_3]$. Observe that u and v are orthogonal if and only if $\cos\theta = 0$, where θ is the angle between u and v. Also, this is true if and only if u and v are ''perpendicular''—that is, $\theta = \pi/2$ (or $\theta = 90^\circ$).

EXAMPLE 7.6

(a) Consider the vectors $u = (1, 1, 1)$, $v = (1, 2, -3)$, $w = (1, -4, 3)$ in $\mathbf{R}^3$. Then

$$\langle u, v\rangle = 1 + 2 - 3 = 0, \qquad \langle u, w\rangle = 1 - 4 + 3 = 0, \qquad \langle v, w\rangle = 1 - 8 - 9 = -16$$

Thus, u is orthogonal to v and w, but v and w are not orthogonal.

(b) Consider the functions $\sin t$ and $\cos t$ in the vector space $C[-\pi, \pi]$ of continuous functions on the closed interval $[-\pi, \pi]$. Then

$$\langle \sin t, \cos t\rangle = \int_{-\pi}^{\pi} \sin t \cos t\, dt = \tfrac{1}{2} \sin^2 t|_{-\pi}^{\pi} = 0 - 0 = 0$$

Thus, $\sin t$ and $\cos t$ are orthogonal functions in the vector space $C[-\pi, \pi]$.

Remark: A vector $w = (x_1, x_2, \ldots, x_n)$ is orthogonal to $u = (a_1, a_2, \ldots, a_n)$ in R^n if

$$\langle u, w\rangle = a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

That is, w is orthogonal to u if w satisfies a homogeneous equation whose coefficients are the elements of u.

EXAMPLE 7.7 Find a nonzero vector w that is orthogonal to $u_1 = (1, 2, 1)$ and $u_2 = (2, 5, 4)$ in R^3.

Let $w = (x, y, z)$. Then we want $\langle u_1, w\rangle = 0$ and $\langle u_2, w\rangle = 0$. This yields the homogeneous system

$$\begin{aligned} x + 2y + z &= 0 \\ 2x + 5y + 4z &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + z &= 0 \\ y + 2z &= 0 \end{aligned}$$

Here z is the only free variable in the echelon system. Set $z = 1$ to obtain $y = -2$ and $x = 3$. Thus, $w = (3, -2, 1)$ is a desired nonzero vector orthogonal to u_1 and u_2.

Any multiple of w will also be orthogonal to u_1 and u_2. Normalizing w, we obtain the following unit vector orthogonal to u_1 and u_2:

$$\hat{w} = \frac{w}{\|w\|} = \left(\frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$$

Orthogonal Complements

Let S be a subset of an inner product space V. The orthogonal complement of S, denoted by $S^\perp$ (read ''S perp'') consists of those vectors in V that are orthogonal to every vector $u \in S$; that is,

$$S^\perp = \{v \in V : \langle v, u\rangle = 0 \text{ for every } u \in S\}$$

In particular, for a given vector u in V, we have

$$u^{\perp} = \{v \in V : \langle v, u \rangle = 0\}$$

that is, $u^{\perp}$ consists of all vectors in V that are orthogonal to the given vector u.

We show that $S^{\perp}$ is a subspace of V. Clearly $0 \in S^{\perp}$, because 0 is orthogonal to every vector in V. Now suppose $v, w \in S^{\perp}$. Then, for any scalars a and b and any vector $u \in S$, we have

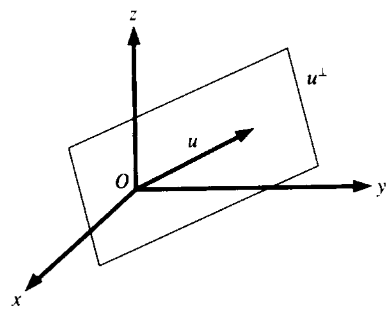
$$\langle av + bw, \ u \rangle = a\langle v, u \rangle + b\langle w, u \rangle = a \cdot 0 + b \cdot 0 = 0$$

Thus, $av + bw \in S^{\perp}$, and therefore $S^{\perp}$ is a subspace of V.

We state this result formally.

PROPOSITION 7.3: Let S be a subset of a vector space V. Then $S^{\perp}$ is a subspace of V.

Remark 1: Suppose u is a nonzero vector in $\mathbf{R}^3$. Then there is a geometrical description of $u^{\perp}$. Specifically, $u^{\perp}$ is the plane in $\mathbf{R}^3$ through the origin O and perpendicular to the vector u. This is shown in Fig. 7-2.



Orthogonal Complement $u^{\perp}$

Figure 7-2

Remark 2: Let W be the solution space of an $m \times n$ homogeneous system $AX = 0$, where $A = [a_{ij}]$ and $X = [x_i]$. Recall that W may be viewed as the kernel of the linear mapping $A: \mathbf{R}^n \to \mathbf{R}^m$. Now we can give another interpretation of W using the notion of orthogonality. Specifically, each solution vector $w = (x_1, x_2, \ldots, x_n)$ is orthogonal to each row of A; hence, W is the orthogonal complement of the row space of A.

EXAMPLE 7.8 Find a basis for the subspace $u^{\perp}$ of $\mathbf{R}^3$, where $u = (1, 3, -4)$.

Note that $u^{\perp}$ consists of all vectors $w = (x, y, z)$ such that $\langle u, w \rangle = 0$, or $x + 3y - 4z = 0$. The free variables are y and z.

(1) Set $y = 1$, $z = 0$ to obtain the solution $w_1 = (-3, 1, 0)$.

(2) Set $y = 0$, $z = 1$ to obtain the solution $w_1 = (4, 0, 1)$.

The vectors w_1 and w_2 form a basis for the solution space of the equation, and hence a basis for $u^{\perp}$.

Suppose W is a subspace of V. Then both W and $W^{\perp}$ are subspaces of V. The next theorem, whose proof (Problem 7.28) requires results of later sections, is a basic result in linear algebra.

THEOREM 7.4: Let W be a subspace of V. Then V is the direct sum of W and $W^{\perp}$; that is, $V = W \oplus W^{\perp}$.