Linear Dependence and Echelon Matrices

Consider the following echelon matrix A, whose pivots have been circled:

$$A = \begin{bmatrix} 0 & \textcircled{2} & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & \textcircled{4} & 3 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & \textcircled{7} & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{6} & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that the rows R_2, R_3, R_4 have 0's in the second column below the nonzero pivot in R_1, and hence any linear combination of R_2, R_3, R_4 must have 0 as its second entry. Thus, R_1 cannot be a linear combination of the rows below it. Similarly, the rows R_3 and R_4 have 0's in the third column below the nonzero pivot in R_2, and hence R_2 cannot be a linear combination of the rows below it. Finally, R_3 cannot be a multiple of R_4, because R_4 has a 0 in the fifth column below the nonzero pivot in R_3. Viewing the nonzero rows from the bottom up, R_4, R_3, R_2, R_1, no row is a linear combination of the preceding rows. Thus, the rows are linearly independent by Lemma 4.10.

The argument used with the above echelon matrix A can be used for the nonzero rows of any echelon matrix. Thus, we have the following very useful result.

THEOREM 4.11: The nonzero rows of a matrix in echelon form are linearly independent.

4.8 Basis and Dimension

First we state two equivalent ways to define a basis of a vector space V. (The equivalence is proved in Problem 4.28.)

DEFINITION A: A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a *basis* of V if it has the following two properties: (1) S is linearly independent. (2) S spans V.

DEFINITION B: A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a *basis* of V if every $v \in V$ can be written uniquely as a linear combination of the basis vectors.

The following is a fundamental result in linear algebra.

THEOREM 4.12: Let V be a vector space such that one basis has m elements and another basis has n elements. Then $m = n$.

A vector space V is said to be of *finite dimension* n or n*-dimensional*, written

$$\dim V = n$$

if V has a basis with n elements. Theorem 4.12 tells us that all bases of V have the same number of elements, so this definition is well defined.

The vector space $\{0\}$ is defined to have dimension 0.

Suppose a vector space V does not have a finite basis. Then V is said to be of *infinite dimension* or to be *infinite-dimensional*.

The above fundamental Theorem 4.12 is a consequence of the following ''replacement lemma'' (proved in Problem 4.35).

LEMMA 4.13: Suppose $\{v_1, v_2, \ldots, v_n\}$ spans V, and suppose $\{w_1, w_2, \ldots, w_m\}$ is linearly independent. Then $m \le n$, and V is spanned by a set of the form

$$\{w_1, w_2, \ldots, w_m, \; v_{i_1}, v_{i_2}, \ldots, v_{i_{n-m}}\}$$

Thus, in particular, $n + 1$ or more vectors in V are linearly dependent.

Observe in the above lemma that we have replaced m of the vectors in the spanning set of V by the m independent vectors and still retained a spanning set.

Examples of Bases

This subsection presents important examples of bases of some of the main vector spaces appearing in this text.

(a) Vector space K^n: Consider the following n vectors in K^n:

$$e_1 = (1, 0, 0, 0, \dots, 0, 0), \; e_2 = (0, 1, 0, 0, \dots, 0, 0), \; \dots, \; e_n = (0, 0, 0, 0, \dots, 0, 1)$$

These vectors are linearly independent. (For example, they form a matrix in echelon form.) Furthermore, any vector $u = (a_1, a_2, \dots, a_n)$ in K^n can be written as a linear combination of the above vectors. Specifically,

$$v = a_1e_1 + a_2e_2 + \cdots + a_ne_n$$

Accordingly, the vectors form a basis of K^n called the *usual* or *standard* basis of K^n. Thus (as one might expect), K^n has dimension n. In particular, any other basis of K^n has n elements.

(b) Vector space $\mathbf{M} = \mathbf{M}_{r,s}$ of all $r \times s$ matrices: The following six matrices form a basis of the vector space $\mathbf{M}_{2,3}$ of all 2×3 matrices over K:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

More generally, in the vector space $\mathbf{M} = \mathbf{M}_{r,s}$ of all $r \times s$ matrices, let E_{ij} be the matrix with ij-entry 1 and 0's elsewhere. Then all such matrices form a basis of $\mathbf{M}_{r,s}$ called the *usual* or *standard* basis of $\mathbf{M}_{r,s}$. Accordingly, $\dim \mathbf{M}_{r,s} = rs$.

(c) Vector space $\mathbf{P}_n(t)$ of all polynomials of degree $\leq n$: The set $S = \{1, t, t^2, t^3, \dots, t^n\}$ of $n + 1$ polynomials is a basis of $\mathbf{P}_n(t)$. Specifically, any polynomial $f(t)$ of degree $\leq n$ can be expessed as a linear combination of these powers of t, and one can show that these polynomials are linearly independent. Therefore, $\dim \mathbf{P}_n(t) = n + 1$.

(d) Vector space $\mathbf{P}(t)$ of all polynomials: Consider any finite set $S = \{f_1(t), f_2(t), \dots, f_m(t)\}$ of polynomials in $\mathbf{P}(t)$, and let m denote the largest of the degrees of the polynomials. Then any polynomial $g(t)$ of degree exceeding m cannot be expressed as a linear combination of the elements of S. Thus, S cannot be a basis of $\mathbf{P}(t)$. This means that the dimension of $\mathbf{P}(t)$ is infinite. We note that the infinite set $S' = \{1, t, t^2, t^3, \dots\}$, consisting of all the powers of t, spans $\mathbf{P}(t)$ and is linearly independent. Accordingly, S' is an infinite basis of $\mathbf{P}(t)$.

Theorems on Bases

The following three theorems (proved in Problems 4.37, 4.38, and 4.39) will be used frequently.

THEOREM 4.14: Let V be a vector space of finite dimension n. Then:

(i) Any $n + 1$ or more vectors in V are linearly dependent.

(ii) Any linearly independent set $S = \{u_1, u_2, \dots, u_n\}$ with n elements is a basis of V.

(iii) Any spanning set $T = \{v_1, v_2, \dots, v_n\}$ of V with n elements is a basis of V.

THEOREM 4.15: Suppose S spans a vector space V. Then:

(i) Any maximum number of linearly independent vectors in S form a basis of V.

(ii) Suppose one deletes from S every vector that is a linear combination of preceding vectors in S. Then the remaining vectors form a basis of V.

THEOREM 4.16: Let V be a vector space of finite dimension and let $S = \{u_1, u_2, \ldots, u_r\}$ be a set of linearly independent vectors in V. Then S is part of a basis of V; that is, S may be extended to a basis of V.

EXAMPLE 4.11

(a) The following four vectors in $\mathbf{R}^4$ form a matrix in echelon form:

$$(1,1,1,1),\ (0,1,1,1),\ (0,0,1,1),\ (0,0,0,1)$$

Thus, the vectors are linearly independent, and, because $\dim \mathbf{R}^4 = 4$, the four vectors form a basis of $\mathbf{R}^4$.

(b) The following $n+1$ polynomials in $\mathbf{P}_n(t)$ are of increasing degree:

$$1,\ t-1,\ (t-1)^2,\ \ldots,\ (t-1)^n$$

Therefore, no polynomial is a linear combination of preceding polynomials; hence, the polynomials are linear independent. Furthermore, they form a basis of $\mathbf{P}_n(t)$, because $\dim \mathbf{P}_n(t) = n+1$.

(c) Consider any four vectors in $\mathbf{R}^3$, say

$$(257, -132, 58), \qquad (43, 0, -17), \qquad (521, -317, 94), \qquad (328, -512, -731)$$

By Theorem 4.14(i), the four vectors must be linearly dependent, because they come from the three-dimensional vector space $\mathbf{R}^3$.

Dimension and Subspaces

The following theorem (proved in Problem 4.40) gives the basic relationship between the dimension of a vector space and the dimension of a subspace.

THEOREM 4.17: Let W be a subspace of an n-dimensional vector space V. Then $\dim W \le n$. In particular, if $\dim W = n$, then $W = V$.

EXAMPLE 4.12 Let W be a subspace of the real space $\mathbf{R}^3$. Note that $\dim \mathbf{R}^3 = 3$. Theorem 4.17 tells us that the dimension of W can only be 0, 1, 2, or 3. The following cases apply:

(a) If $\dim W = 0$, then $W = \{0\}$, a point.

(b) If $\dim W = 1$, then W is a line through the origin 0.

(c) If $\dim W = 2$, then W is a plane through the origin 0.

(d) If $\dim W = 3$, then W is the entire space $\mathbf{R}^3$.

4.9 Application to Matrices, Rank of a Matrix

Let A be any $m \times n$ matrix over a field K. Recall that the rows of A may be viewed as vectors in K^n and that the row space of A, written rowsp(A), is the subspace of K^n spanned by the rows of A. The following definition applies.

DEFINITION: The *rank* of a matrix A, written rank(A), is equal to the maximum number of linearly independent rows of A or, equivalently, the dimension of the row space of A.

Recall, on the other hand, that the columns of an $m \times n$ matrix A may be viewed as vectors in K^m and that the column space of A, written colsp(A), is the subspace of K^m spanned by the columns of A. Although m may not be equal to n—that is, the rows and columns of A may belong to different vector spaces—we have the following fundamental result.

THEOREM 4.18: The maximum number of linearly independent rows of any matrix A is equal to the maximum number of linearly independent columns of A. Thus, the dimension of the row space of A is equal to the dimension of the column space of A.

Accordingly, one could restate the above definition of the rank of A using columns instead of rows.

Basis-Finding Problems

This subsection shows how an echelon form of any matrix A gives us the solution to certain problems about A itself. Specifically, let A and B be the following matrices, where the echelon matrix B (whose pivots are circled) is an echelon form of A:

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 5 & 5 & 6 & 4 & 5 \\ 3 & 7 & 6 & 11 & 6 & 9 \\ 1 & 5 & 10 & 8 & 9 & 9 \\ 2 & 6 & 8 & 11 & 9 & 12 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \textcircled{1} & 2 & 1 & 3 & 1 & 2 \\ 0 & \textcircled{1} & 3 & 1 & 2 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We solve the following four problems about the matrix A, where $C_1, C_2, \ldots, C_6$ denote its columns:

(a) Find a basis of the row space of A.

(b) Find each column C_k of A that is a linear combination of preceding columns of A.

(c) Find a basis of the column space of A.

(d) Find the rank of A.

(a) We are given that A and B are row equivalent, so they have the same row space. Moreover, B is in echelon form, so its nonzero rows are linearly independent and hence form a basis of the row space of B. Thus, they also form a basis of the row space of A. That is,

$$\text{basis of rowsp}(A): \qquad (1,2,1,3,1,2), \qquad (0,1,3,1,2,1), \qquad (0,0,0,1,1,2)$$

(b) Let $M_k = [C_1, C_2, \ldots, C_k]$, the submatrix of A consisting of the first k columns of A. Then M_{k-1} and M_k are, respectively, the coefficient matrix and augmented matrix of the vector equation

$$x_1C_1 + x_2C_2 + \cdots + x_{k-1}C_{k-1} = C_k$$

Theorem 3.9 tells us that the system has a solution, or, equivalently, C_k is a linear combination of the preceding columns of A if and only if $\text{rank}(M_k) = \text{rank}(M_{k-1})$, where $\text{rank}(M_k)$ means the number of pivots in an echelon form of M_k. Now the first k column of the echelon matrix B is also an echelon form of M_k. Accordingly,

$$\text{rank}(M_2) = \text{rank}(M_3) = 2 \qquad \text{and} \qquad \text{rank}(M_4) = \text{rank}(M_5) = \text{rank}(M_6) = 3$$

Thus, C_3, C_5, C_6 are each a linear combination of the preceding columns of A.

(c) The fact that the remaining columns C_1, C_2, C_4 are not linear combinations of their respective preceding columns also tells us that they are linearly independent. Thus, they form a basis of the column space of A. That is,

$$\text{basis of colsp}(A): \qquad [1,2,3,1,2]^T, \qquad [2,5,7,5,6]^T, \qquad [3,6,11,8,11]^T$$

Observe that C_1, C_2, C_4 may also be characterized as those columns of A that contain the pivots in any echelon form of A.

(d) Here we see that three possible definitions of the rank of A yield the same value.

(i) There are three pivots in B, which is an echelon form of A.

(ii) The three pivots in B correspond to the nonzero rows of B, which form a basis of the row space of A.

(iii) The three pivots in B correspond to the columns of A, which form a basis of the column space of A.

Thus, $\text{rank}(A) = 3$.