Linear Combinations, Homogeneous Systems

3.57. Write v as a linear combination of u_1, u_2, u_3, where

(a) $v = (4, -9, 2)$, $u_1 = (1, 2, -1)$, $u_2 = (1, 4, 2)$, $u_3 = (1, -3, 2)$;
(b) $v = (1, 3, 2)$, $u_1 = (1, 2, 1)$, $u_2 = (2, 6, 5)$, $u_3 = (1, 7, 8)$;
(c) $v = (1, 4, 6)$, $u_1 = (1, 1, 2)$, $u_2 = (2, 3, 5)$, $u_3 = (3, 5, 8)$.

3.58. Let $u_1 = (1, 1, 2)$, $u_2 = (1, 3, -2)$, $u_3 = (4, -2, -1)$ in $\mathbf{R}^3$. Show that u_1, u_2, u_3 are orthogonal, and write v as a linear combination of u_1, u_2, u_3, where (a) $v = (5, -5, 9)$, (b) $v = (1, -3, 3)$, (c) $v = (1, 1, 1)$. (*Hint:* Use Fourier coefficients.)

3.59. Find the dimension and a basis of the general solution W of each of the following homogeneous systems:

(a) $\begin{aligned} x - y + 2z &= 0 \\ 2x + y + z &= 0 \\ 5x + y + 4z &= 0 \end{aligned}$ (b) $\begin{aligned} x + 2y - 3z &= 0 \\ 2x + 5y + 2z &= 0 \\ 3x - y - 4z &= 0 \end{aligned}$ (c) $\begin{aligned} x + 2y + 3z + t &= 0 \\ 2x + 4y + 7z + 4t &= 0 \\ 3x + 6y + 10z + 5t &= 0 \end{aligned}$

3.60. Find the dimension and a basis of the general solution W of each of the following systems:

(a) $\begin{aligned} x_1 + 3x_2 + 2x_3 - x_4 - x_5 &= 0 \\ 2x_1 + 6x_2 + 5x_3 + x_4 - x_5 &= 0 \\ 5x_1 + 15x_2 + 12x_3 + x_4 - 3x_5 &= 0 \end{aligned}$ (b) $\begin{aligned} 2x_1 - 4x_2 + 3x_3 - x_4 + 2x_5 &= 0 \\ 3x_1 - 6x_2 + 5x_3 - 2x_4 + 4x_5 &= 0 \\ 5x_1 - 10x_2 + 7x_3 - 3x_4 + 18x_5 &= 0 \end{aligned}$

Echelon Matrices, Row Canonical Form

3.61. Reduce each of the following matrices to echelon form and then to row canonical form:

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 9 \\ 1 & 5 & 12 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 5 \\ 3 & 6 & 3 & -7 & 7 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 4 & 2 & -2 & 5 & 1 \\ 3 & 6 & 2 & 2 & 0 & 4 \\ 4 & 8 & 2 & 6 & -5 & 7 \end{bmatrix}$

3.62. Reduce each of the following matrices to echelon form and then to row canonical form:

(a) $\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 & 5 & 7 \\ 3 & 6 & 4 & 9 & 10 & 11 \\ 1 & 2 & 4 & 3 & 6 & 9 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 12 \\ 0 & 0 & 4 & 6 \\ 0 & 2 & 7 & 10 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 8 & 5 & 10 \\ 1 & 7 & 7 & 11 \\ 3 & 11 & 7 & 15 \end{bmatrix}$

3.63. Using only 0's and 1's, list all possible 2×2 matrices in row canonical form.

3.64. Using only 0's and 1's, find the number n of possible 3×3 matrices in row canonical form.

Elementary Matrices, Applications

3.65. Let e_1, e_2, e_3 denote, respectively, the following elementary row operations:

"Interchange R_2 and R_3," "Replace R_2 by $3R_2$," "Replace R_1 by $2R_3 + R_1$"

(a) Find the corresponding elementary matrices E_1, E_2, E_3.
(b) Find the inverse operations e_1^{-1}, e_2^{-1}, e_3^{-1}; their corresponding elementary matrices E_1', E_2', E_3'; and the relationship between them and E_1, E_2, E_3.
(c) Describe the corresponding elementary column operations f_1, f_2, f_3.
(d) Find elementary matrices F_1, F_2, F_3 corresponding to f_1, f_2, f_3, and the relationship between them and E_1, E_2, E_3.

3.66. Express each of the following matrices as a product of elementary matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 3 & 8 & 7 \end{bmatrix}$$

3.67. Find the inverse of each of the following matrices (if it exists):

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -3 & 1 \\ 3 & -4 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 1 \\ 3 & 10 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 8 & -3 \\ 1 & 7 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 1 & -1 \\ 5 & 2 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

3.68. Find the inverse of each of the following $n \times n$ matrices:

(a) A has 1's on the diagonal and *superdiagonal* (entries directly above the diagonal) and 0's elsewhere.
(b) B has 1's on and above the diagonal, and 0's below the diagonal.

Lu Factorization

3.69. Find the LU factorization of each of the following matrices:

$$\text{(a)} \begin{bmatrix} 1 & -1 & -1 \\ 3 & -4 & -2 \\ 2 & -3 & -2 \end{bmatrix}, \text{(b)} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \\ 3 & 4 & 2 \end{bmatrix}, \text{(c)} \begin{bmatrix} 2 & 3 & 6 \\ 4 & 7 & 9 \\ 3 & 5 & 4 \end{bmatrix}, \text{(d)} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 10 \end{bmatrix}$$

3.70. Let A be the matrix in Problem 3.69(a). Find X_1, X_2, X_3, X_4, where

(a) X_1 is the solution of $AX = B_1$, where $B_1 = (1, 1, 1)^T$.
(b) For $k > 1$, X_k is the solution of $AX = B_k$, where $B_k = B_{k-1} + X_{k-1}$.

3.71. Let B be the matrix in Problem 3.69(b). Find the LDU factorization of B.

Miscellaneous Problems

3.72. Consider the following systems in unknowns x and y:

$$\text{(a)} \quad \begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned} \qquad\qquad \text{(b)} \quad \begin{aligned} ax + by &= 0 \\ cx + dy &= 1 \end{aligned}$$

Suppose $D = ad - bc \neq 0$. Show that each system has the unique solution:

(a) $x = d/D, y = -c/D$, (b) $x = -b/D, y = a/D$.

3.73. Find the inverse of the row operation "Replace R_i by $kR_j + k'R_i$ ($k' \neq 0$)."

3.74. Prove that deleting the last column of an echelon form (respectively, the row canonical form) of an augmented matrix $M = [A, B]$ yields an echelon form (respectively, the row canonical form) of A.

3.75. Let e be an elementary row operation and E its elementary matrix, and let f be the corresponding elementary column operation and F its elementary matrix. Prove

(a) $f(A) = (e(A^T))^T$, (b) $F = E^T$, (c) $f(A) = AF$.

3.76. Matrix A is *equivalent* to matrix B, written $A \approx B$, if there exist nonsingular matrices P and Q such that $B = PAQ$. Prove that $\approx$ is an *equivalence* relation; that is,

(a) $A \approx A$, (b) If $A \approx B$, then $B \approx A$, (c) If $A \approx B$ and $B \approx C$, then $A \approx C$.

ANSWERS TO SUPPLEMENTARY PROBLEMS

Notation: $A = [R_1; \quad R_2; \quad \ldots]$ denotes the matrix A with rows $R_1, R_2, \ldots$. The elements in each row are separated by commas (which may be omitted with single digits), the rows are separated by semicolons, and 0 denotes a zero row. For example,

$$A = [1,2,3,4; \quad 5,-6,7,-8; \quad 0] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & -6 & 7 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.49. (a) no, (b) yes, (c) linear in x, y, z, not linear in x, y, z, k

3.50. (a) $x = 2/\pi$, (b) no solution, (c) every scalar k is a solution

3.51. (a) $(2, -1)$, (b) no solution, (c) $(5, 2)$, (d) $(5 - 2a, \ a)$

3.52. (a) $a \neq \pm 2$, $(2, 2)$, $(-2, -2)$, (b) $a \neq \pm 6$, $(6, 4)$, $(-6, -4)$, (c) $a \neq \frac{5}{2}$, $(\frac{5}{2}, 6)$

3.53. (a) $(2, 1, \frac{1}{2})$, (b) no solution, (c) $u = (-7a - 1, \ 2a + 2, \ a)$.

3.54. (a) $(3, -1)$, (b) $u = (-a + 2b, \ 1 + 2a - 2b, \ a, \ b)$, (c) no solution

3.55. (a) $u = (\frac{1}{2}a + 2, \quad a, \quad \frac{1}{2})$, (b) $u = (\frac{1}{2}(7 - 5b - 4a), \quad a, \ \frac{1}{2}(1 + b), \ b)$

3.56. (a) $a \neq \pm 3$, $(3, 3)$, $(-3, -3)$, (b) $a \neq 5$ and $a \neq -1$, $(5, 7)$, $(-1, -5)$,
(c) $a \neq 1$ and $a \neq -2$, $(-2, 5)$

3.57. (a) $2, -1, 3$, (b) $6, -3, 1$, (c) not possible

3.58. (a) $3, -2, 1$, (b) $\frac{2}{3}, -1, \frac{1}{3}$, (c) $\frac{2}{3}, \frac{1}{7}, \frac{1}{21}$

3.59. (a) $\dim W = 1$, $u_1 = (-1, 1, 1)$, (b) $\dim W = 0$, no basis,
(c) $\dim W = 2$, $u_1 = (-2, 1, 0, 0)$, $u_2 = (5, 0, -2, 1)$

3.60. (a) $\dim W = 3$, $u_1 = (-3, 1, 0, 0, 0)$, $u_2 = (7, 0, -3, 1, 0)$, $u_3 = (3, 0, -1, 0, 1)$,
(b) $\dim W = 2$, $u_1 = (2, 1, 0, 0, 0)$, $u_2 = (5, 0, -5, -3, 1)$

3.61. (a) $[1, 0, -\frac{1}{2}; \quad 0, 1, \frac{5}{2}; \quad 0]$, (b) $[1, 2, 0, 0, 2; \quad 0, 0, 1, 0, 5; \quad 0, 0, 0, 1, 2]$,
(c) $[1, 2, 0, 4, -5, 3; \quad 0, 0, 1, -5, \frac{15}{2}, -\frac{5}{2}; \quad 0]$

3.62. (a) $[1, 2, 0, 0, -4, -2; \quad 0, 0, 1, 0, 1, 2; \quad 0, 0, 0, 1, 2, 1; \quad 0]$,
(b) $[0, 1, 0, 0; \quad 0, 0, 1, 0; \quad 0, 0, 0, 1; \quad 0]$, (c) $[1, 0, 0, 4; \quad 0, 1, 0, -1; \quad 0, 0, 1, 2; \quad 0]$

3.63. 5: $[1, 0; \quad 0, 1]$, $[1, 1; \quad 0, 0]$, $[1, 0; \quad 0, 0]$, $[0, 1; \quad 0, 0]$, 0

3.64. 16

3.65. (a) $[1, 0, 0; \quad 0, 0, 1; \quad 0, 1, 0]$, $[1, 0, 0; \quad 0, 3, 0; \quad 0, 0, 1]$, $[1, 0, 2; \quad 0, 1, 0; \quad 0, 0, 1]$,
(b) $R_2 \leftrightarrow R_3$; $\frac{1}{3}R_2 \to R_2$; $-2R_3 + R_1 \to R_1$; each $E_i' = E_i^{-1}$,
(c) $C_2 \leftrightarrow C_3, 3C_2 \to C_2, 2C_3 + C_1 \to C_1$, (d) each $F_i = E_i^T$.

3.66. $A = [1, 0; \quad 3, 1][1, 0; \quad 0, -2][1, 2; \quad 0, 1]$, B is not invertible,
$C = [1, 0; \quad -\frac{3}{2}, 1][1, 0; \quad 0, 2][1, 6; \quad 0, 1][2, 0; \quad 0, 1]$,
$D = [100; \quad 010; \quad 301][100; \quad 010; \quad 021][100; \quad 013; \quad 001][120; \quad 010; \quad 001]$

3.67. $A^{-1} = [-8, 12, -5; \quad -5, 7, -3; \quad 1, -2, 1]$, B has no inverse,
$C^{-1} = [\frac{29}{2}, -\frac{17}{2}, \frac{7}{2}; \quad -\frac{5}{2}, \frac{3}{2}, -\frac{1}{2}; \quad 3, -2, 1]$, $D^{-1} = [8, -3, -1; \quad -5, 2, 1; \quad 10, -4, -1]$

3.68. $A^{-1} = [1, -1, 1, -1, \ldots; \quad 0, 1, -1, 1, -1, \ldots; \quad 0, 0, 1, -1, 1, -1, 1, \ldots; \quad \ldots; \quad \ldots; \quad 0, \ldots 0, 1]$
B^{-1} has 1's on diagonal, -1's on superdiagonal, and 0's elsewhere.

3.69. (a) $[100; \quad 310; \quad 211][1, -1, -1; \quad 0, -1, 1; \quad 0, 0, -1]$,
(b) $[100; \quad 210; \quad 351][1, 3, -1; \quad 0, -1, 3; \quad 0, 0, -10]$,
(c) $[100; \quad 210; \quad \frac{3}{2}, \frac{1}{2}, 1][2, 3, 6; \quad 0, 1, -3; \quad 0, 0, -\frac{7}{2}]$,
(d) There is no LU decomposition.

3.70. $X_1 = [1, 1, -1]^T, \quad B_2 = [2, 2, 0]^T, \quad X_2 = [6, 4, 0]^T, \quad B_3 = [8, 6, 0]^T, \quad X_3 = [22, 16, -2]^T,$
$B_4 = [30, 22, -2]^T, \quad X_4 = [86, 62, -6]^T$

3.71. $B = [100; \quad 210; \quad 351] \operatorname{diag}(1, -1, -10) [1, 3, -1; \quad 0, 1, 3; \quad 0, 0, 1]$

3.73. Replace R_i by $-kR_j + (1/k')R_i$.

3.75. (c) $f(A) = (e(A^T))^T = (EA^T)^T = (A^T)^T E^T = AF$

3.76. (a) $A = IAI$. (b) If $A = PBQ$, then $B = P^{-1}AQ^{-1}$.
(c) If $A = PBQ$ and $B = P'CQ'$, then $A = (PP')C(Q'Q)$.

CHAPTER 4

Vector Spaces

4.1 Introduction

This chapter introduces the underlying structure of linear algebra, that of a finite-dimensional vector space. The definition of a vector space V, whose elements are called *vectors*, involves an arbitrary field K, whose elements are called *scalars*. The following notation will be used (unless otherwise stated or implied):

V	the given vector space
u, v, w	vectors in V
K	the given number field
$a, b, c,$ or k	scalars in K

Almost nothing essential is lost if the reader assumes that K is the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.

The reader might suspect that the real line $\mathbf{R}$ has ''dimension'' one, the cartesian plane $\mathbf{R}^2$ has ''dimension'' two, and the space $\mathbf{R}^3$ has ''dimension'' three. This chapter formalizes the notion of ''dimension,'' and this definition will agree with the reader's intuition.

Throughout this text, we will use the following set notation:

$a \in A$	Element a belongs to set A
$a, b \in A$	Elements a and b belong to A
$\forall x \in A$	For every x in A
$\exists x \in A$	There exists an x in A
$A \subseteq B$	A is a subset of B
$A \cap B$	Intersection of A and B
$A \cup B$	Union of A and B
$\emptyset$	Empty set

4.2 Vector Spaces

The following defines the notion of a vector space V where K is the field of scalars.

DEFINITION: Let V be a nonempty set with two operations:

(i) ***Vector Addition:*** This assigns to any $u, v \in V$ a *sum* $u + v$ in V.

(ii) ***Scalar Multiplication:*** This assigns to any $u \in V$, $k \in K$ a *product* $ku \in V$.

Then V is called a *vector space* (over the field K) if the following axioms hold for any vectors $u, v, w \in V$: