CHAPTER 8

Determinants

8.1 Introduction

Each n-square matrix $A = [a_{ij}]$ is assigned a special scalar called the *determinant* of A, denoted by $\det(A)$ or $|A|$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

We emphasize that an $n \times n$ array of scalars enclosed by straight lines, called a *determinant of order n*, is not a matrix but denotes the determinant of the enclosed array of scalars (i.e., the enclosed matrix).

The determinant function was first discovered during the investigation of systems of linear equations. We shall see that the determinant is an indispensable tool in investigating and obtaining properties of square matrices.

The definition of the determinant and most of its properties also apply in the case where the entries of a matrix come from a commutative ring.

We begin with a special case of determinants of orders 1, 2, and 3. Then we define a determinant of arbitrary order. This general definition is preceded by a discussion of permutations, which is necessary for our general definition of the determinant.

8.2 Determinants of Orders 1 and 2

Determinants of orders 1 and 2 are defined as follows:

$$|a_{11}| = a_{11} \qquad \text{and} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Thus, the determinant of a 1×1 matrix $A = [a_{11}]$ is the scalar a_{11}; that is, $\det(A) = |a_{11}| = a_{11}$. The determinant of order two may easily be remembered by using the following diagram:

$$+ \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad -$$

That, is, the determinant is equal to the product of the elements along the plus-labeled arrow minus the product of the elements along the minus-labeled arrow. (There is an analogous diagram for determinants of order 3, but not for higher-order determinants.)

EXAMPLE 8.1

(a) Because the determinant of order 1 is the scalar itself, we have:

$$\det(27) = 27, \qquad \det(-7) = -7, \qquad \det(t-3) = t-3$$

(b) $\begin{vmatrix} 5 & 3 \\ 4 & 6 \end{vmatrix} = 5(6) - 3(4) = 30 - 12 = 18, \qquad \begin{vmatrix} 3 & 2 \\ -5 & 7 \end{vmatrix} = 21 + 10 = 31$

Application to Linear Equations

Consider two linear equations in two unknowns, say

$$a_1z + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Let $D = a_1b_2 - a_2b_1$, the determinant of the matrix of coefficients. Then the system has a unique solution if and only if $D \neq 0$. In such a case, the unique solution may be expressed completely in terms of determinants as follows:

$$x = \frac{N_x}{D} = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{N_y}{D} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Here D appears in the denominator of both quotients. The numerators N_x and N_y of the quotients for x and y, respectively, can be obtained by substituting the column of constant terms in place of the column of coefficients of the given unknown in the matrix of coefficients. On the other hand, if $D = 0$, then the system may have no solution or more than one solution.

EXAMPLE 8.2 Solve by determinants the system $\begin{cases} 4x - 3y = 15 \\ 2x + 5y = \ \ 1 \end{cases}$

First find the determinant D of the matrix of coefficients:

$$D = \begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix} = 4(5) - (-3)(2) = 20 + 6 = 26$$

Because $D \neq 0$, the system has a unique solution. To obtain the numerators N_x and N_y, simply replace, in the matrix of coefficients, the coefficients of x and y, respectively, by the constant terms, and then take their determinants:

$$N_x = \begin{vmatrix} 15 & -3 \\ 1 & 5 \end{vmatrix} = 75 + 3 = 78 \qquad N_y = \begin{vmatrix} 4 & 15 \\ 2 & 1 \end{vmatrix} = 4 - 30 = -26$$

Then the unique solution of the system is

$$x = \frac{N_x}{D} = \frac{78}{26} = 3, \qquad y = \frac{N_y}{D} = \frac{-26}{26} = -1$$

8.3 Determinants of Order 3

Consider an arbitrary 3×3 matrix $A = [a_{ij}]$. The determinant of A is defined as follows:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Observe that there are six products, each product consisting of three elements of the original matrix. Three of the products are plus-labeled (keep their sign) and three of the products are minus-labeled (change their sign).

The diagrams in Fig. 8-1 may help us to remember the above six products in $\det(A)$. That is, the determinant is equal to the sum of the products of the elements along the three plus-labeled arrows in

Fig. 8-1 plus the sum of the negatives of the products of the elements along the three minus-labeled arrows. We emphasize that there are no such diagrammatic devices with which to remember determinants of higher order.

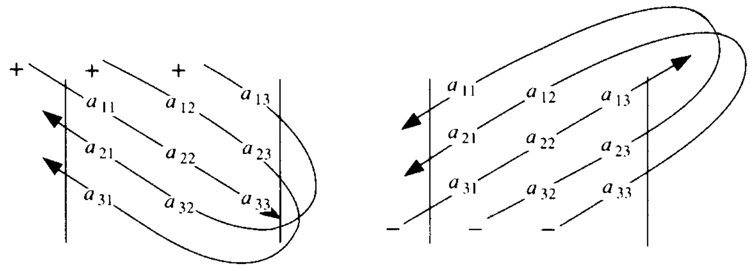

Figure 8-1

EXAMPLE 8.3 Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{bmatrix}$. Find $\det(A)$ and $\det(B)$.

Use the diagrams in Fig. 8-1:

$$\begin{aligned} \det(A) &= 2(5)(4) + 1(-2)(1) + 1(-3)(0) - 1(5)(1) - (-3)(-2)(2) - 4(1)(0) \\ &= 40 - 2 + 0 - 5 - 12 - 0 = 21 \\ \det(B) &= 60 - 4 + 12 - 10 - 9 + 32 = 81 \end{aligned}$$

Alternative Form for a Determinant of Order 3

The determinant of the 3×3 matrix $A = [a_{ij}]$ may be rewritten as follows:

$$\begin{aligned} \det(A) &= a_{11}(a_{22}a_{23} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

which is a linear combination of three determinants of order 2 whose coefficients (with alternating signs) form the first row of the given matrix. This linear combination may be indicated in the form

$$a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Note that each 2×2 matrix can be obtained by deleting, in the original matrix, the row and column containing its coefficient.

EXAMPLE 8.4

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} &= 1\begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} \\ &= 1\begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix} - 2\begin{vmatrix} 4 & 3 \\ 0 & -1 \end{vmatrix} + 3\begin{vmatrix} 4 & -2 \\ 0 & 5 \end{vmatrix} \\ &= 1(2 - 15) - 2(-4 + 0) + 3(20 + 0) = -13 + 8 + 60 = 55 \end{aligned}$$

8.4 Permutations

A permutation σ of the set $\{1, 2, \ldots, n\}$ is a one-to-one mapping of the set onto itself or, equivalently, a rearrangement of the numbers $1, 2, \ldots, n$. Such a permutation σ is denoted by

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix} \qquad \text{or} \qquad \sigma = j_1 j_2 \cdots j_n, \qquad \text{where } j_i = \sigma(i)$$

The set of all such permutations is denoted by S_n, and the number of such permutations is $n!$. If $\sigma \in S_n$, then the inverse mapping $\sigma^{-1} \in S_n$; and if $\sigma, \tau \in S_n$, then the composition mapping $\sigma \circ \tau \in S_n$. Also, the identity mapping $\varepsilon = \sigma \circ \sigma^{-1} \in S_n$. (In fact, $\varepsilon = 123 \ldots n$.)

EXAMPLE 8.5

(a) There are $2! = 2 \cdot 1 = 2$ permutations in S_2; they are 12 and 21.

(b) There are $3! = 3 \cdot 2 \cdot 1 = 6$ permutations in S_3; they are 123, 132, 213, 231, 312, 321.

Sign (Parity) of a Permutation

Consider an arbitrary permutation σ in S_n, say $\sigma = j_1 j_2 \cdots j_n$. We say σ is an even or odd permutation according to whether there is an even or odd number of inversions in σ. By an *inversion* in σ we mean a pair of integers (i, k) such that $i > k$, but i precedes k in σ. We then define the sign or parity of σ, written sgn σ, by

$$\operatorname{sgn} \sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

EXAMPLE 8.6

(a) Find the sign of $\sigma = 35142$ in S_5.

For each element k, we count the number of elements i such that $i > k$ and i precedes k in σ. There are

2 numbers $(3 \text{ and } 5)$ greater than and preceding 1,
3 numbers $(3, 5, \text{and } 4)$ greater than and preceding 2,
1 number (5) greater than and preceding 4.

(There are no numbers greater than and preceding either 3 or 5.) Because there are, in all, six inversions, σ is even and sgn $\sigma = 1$.

(b) The identity permutation $\varepsilon = 123 \ldots n$ is even because there are no inversions in ε.

(c) In S_2, the permutation 12 is even and 21 is odd. In S_3, the permutations 123, 231, 312 are even and the permutations 132, 213, 321 are odd.

(d) Let τ be the permutation that interchanges two numbers i and j and leaves the other numbers fixed. That is,

$$\tau(i) = j, \qquad \tau(j) = i, \qquad \tau(k) = k, \quad \text{where} \quad k \neq i, j$$

We call τ a *transposition*. If $i < j$, then there are $2(j - i) - 1$ inversions in τ, and hence, the transposition τ is odd.

Remark: One can show that, for any n, half of the permutations in S_n are even and half of them are odd. For example, 3 of the 6 permutations in S_3 are even, and 3 are odd.

8.5. Determinants of Arbitrary Order

Let $A = [a_{ij}]$ be a square matrix of order n over a field K.

Consider a product of n elements of A such that one and only one element comes from each row and one and only one element comes from each column. Such a product can be written in the form

$$a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

that is, where the factors come from successive rows, and so the first subscripts are in the natural order $1, 2, \ldots, n$. Now because the factors come from different columns, the sequence of second subscripts forms a permutation $\sigma = j_1 j_2 \cdots j_n$ in S_n. Conversely, each permutation in S_n determines a product of the above form. Thus, the matrix A contains $n!$ such products.

DEFINITION: The determinant of $A = [a_{ij}]$, denoted by $\det(A)$ or $|A|$, is the sum of all the above $n!$ products, where each such product is multiplied by sgn σ. That is,

$$|A| = \sum_{\sigma} (\text{sgn } \sigma) a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

or

$$|A| = \sum_{\sigma \in S_n} (\text{sgn } \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

The determinant of the n-square matrix A is said to be of order n.

The next example shows that the above definition agrees with the previous definition of determinants of orders 1, 2, and 3.

EXAMPLE 8.7

(a) Let $A = [a_{11}]$ be a 1×1 matrix. Because S_1 has only one permutation, which is even, $\det(A) = a_{11}$, the number itself.

(b) Let $A = [a_{ij}]$ be a 2×2 matrix. In S_2, the permutation 12 is even and the permutation 21 is odd. Hence,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

(c) Let $A = [a_{ij}]$ be a 3×3 matrix. In S_3, the permutations 123, 231, 312 are even, and the permutations 321, 213, 132 are odd. Hence,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Remark: As n increases, the number of terms in the determinant becomes astronomical. Accordingly, we use indirect methods to evaluate determinants rather than the definition of the determinant. In fact, we prove a number of properties about determinants that will permit us to shorten the computation considerably. In particular, we show that a determinant of order n is equal to a linear combination of determinants of order $n - 1$, as in the case $n = 3$ above.

8.6 Properties of Determinants

We now list basic properties of the determinant.

THEOREM 8.1: The determinant of a matrix A and its transpose A^T are equal; that is, $|A| = |A^T|$.

By this theorem (proved in Problem 8.22), any theorem about the determinant of a matrix A that concerns the rows of A will have an analogous theorem concerning the columns of A.

The next theorem (proved in Problem 8.24) gives certain cases for which the determinant can be obtained immediately.

THEOREM 8.2: Let A be a square matrix.

(i) If A has a row (column) of zeros, then $|A| = 0$.

(ii) If A has two identical rows (columns), then $|A| = 0$.