Vector Space Isomorphism

The notion of two vector spaces being isomorphic was defined in Chapter 4 when we investigated the coordinates of a vector relative to a basis. We now redefine this concept.

DEFINITION: Two vector spaces V and U over K are *isomorphic*, written $V \cong U$, if there exists a bijective (one-to-one and onto) linear mapping $F: V \to U$. The mapping F is then called an *isomorphism* between V and U.

Consider any vector space V of dimension n and let S be any basis of V. Then the mapping

$$v \mapsto [v]_S$$

which maps each vector $v \in V$ into its coordinate vector $[v]_S$, is an isomorphism between V and K^n.

5.4 Kernel and Image of a Linear Mapping

We begin by defining two concepts.

DEFINITION: Let $F: V \to U$ be a linear mapping. The *kernel* of F, written $\operatorname{Ker} F$, is the set of elements in V that map into the zero vector 0 in U; that is,

$$\operatorname{Ker} F = \{v \in V : F(v) = 0\}$$

The *image* (or *range*) of F, written $\operatorname{Im} F$, is the set of image points in U; that is,

$$\operatorname{Im} F = \{u \in U : \text{there exists } v \in V \text{ for which } F(v) = u\}$$

The following theorem is easily proved (Problem 5.22).

THEOREM 5.3: Let $F: V \to U$ be a linear mapping. Then the kernel of F is a subspace of V and the image of F is a subspace of U.

Now suppose that $v_1, v_2, \ldots, v_m$ span a vector space V and that $F: V \to U$ is linear. We show that $F(v_1), F(v_2), \ldots, F(v_m)$ span $\operatorname{Im} F$. Let $u \in \operatorname{Im} F$. Then there exists $v \in V$ such that $F(v) = u$. Because the v_i's span V and $v \in V$, there exist scalars $a_1, a_2, \ldots, a_m$ for which

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m$$

Therefore,

$$u = F(v) = F(a_1 v_1 + a_2 v_2 + \cdots + a_m v_m) = a_1 F(v_1) + a_2 F(v_2) + \cdots + a_m F(v_m)$$

Thus, the vectors $F(v_1), F(v_2), \ldots, F(v_m)$ span $\operatorname{Im} F$.

We formally state the above result.

PROPOSITION 5.4: Suppose $v_1, v_2, \ldots, v_m$ span a vector space V, and suppose $F: V \to U$ is linear. Then $F(v_1), F(v_2), \ldots, F(v_m)$ span $\operatorname{Im} F$.

EXAMPLE 5.7

(a) Let $F: \mathbf{R}^3 \to \mathbf{R}^3$ be the projection of a vector v into the xy-plane [as pictured in Fig. 5-2(a)]; that is,

$$F(x, y, z) = (x, y, 0)$$

Clearly the image of F is the entire xy-plane—that is, points of the form $(x, y, 0)$. Moreover, the kernel of F is the z-axis—that is, points of the form $(0, 0, c)$. That is,

$$\operatorname{Im} F = \{(a, b, c) : c = 0\} = xy\text{-plane} \qquad \text{and} \qquad \operatorname{Ker} F = \{(a, b, c) : a = 0, b = 0\} = z\text{-axis}$$

(b) Let $G: \mathbf{R}^3 \to \mathbf{R}^3$ be the linear mapping that rotates a vector v about the z-axis through an angle θ [as pictured in Fig. 5-2(b)]; that is,

$$G(x, y, z) = (x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta,\ z)$$

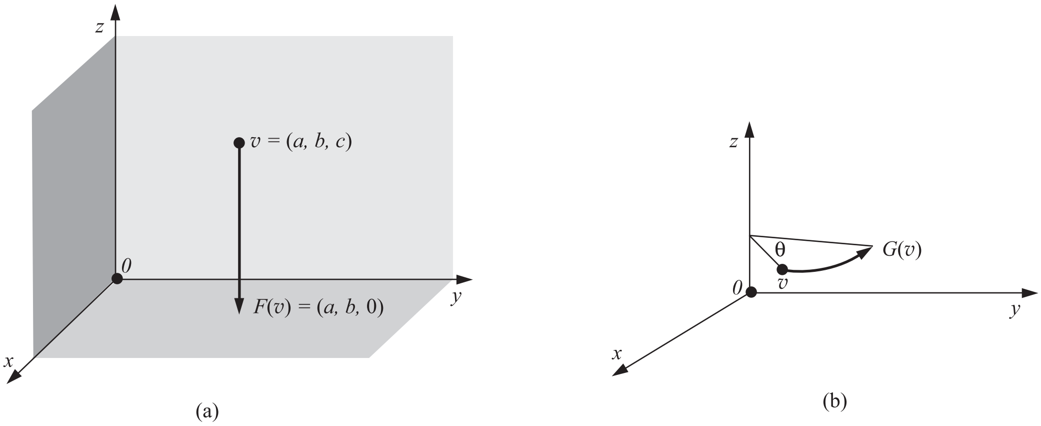

Figure 5-2

Observe that the distance of a vector v from the origin O does not change under the rotation, and so only the zero vector 0 is mapped into the zero vector 0. Thus, Ker $G = \{0\}$. On the other hand, every vector u in $\mathbf{R}^3$ is the image of a vector v in $\mathbf{R}^3$ that can be obtained by rotating u back by an angle of θ. Thus, Im $G = \mathbf{R}^3$, the entire space.

EXAMPLE 5.8 Consider the vector space $V = \mathbf{P}(t)$ of polynomials over the real field $\mathbf{R}$, and let $H : V \to V$ be the third-derivative operator; that is, $H[f(t)] = d^3f/dt^3$. [Sometimes the notation $\mathbf{D}^3$ is used for H, where $\mathbf{D}$ is the derivative operator.] We claim that

$$\text{Ker } H = \{\text{polynomials of degree} \le 2\} = \mathbf{P}_2(t) \qquad \text{and} \qquad \text{Im } H = V$$

The first comes from the fact that $H(at^2 + bt + c) = 0$ but $H(t^n) \ne 0$ for $n \ge 3$. The second comes from that fact that every polynomial $g(t)$ in V is the third derivative of some polynomial $f(t)$ (which can be obtained by taking the antiderivative of $g(t)$ three times).

Kernel and Image of Matrix Mappings

Consider, say, a 3×4 matrix A and the usual basis $\{e_1, e_2, e_3, e_4\}$ of K^4 (written as columns):

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad e_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Recall that A may be viewed as a linear mapping $A : K^4 \to K^3$, where the vectors in K^4 and K^3 are viewed as column vectors. Now the usual basis vectors span K^4, so their images Ae_1, Ae_2, Ae_3, Ae_4 span the image of A. But the vectors Ae_1, Ae_2, Ae_3, Ae_4 are precisely the columns of A:

$$Ae_1 = [a_1, b_1, c_1]^T, \qquad Ae_2 = [a_2, b_2, c_2]^T, \qquad Ae_3 = [a_3, b_3, c_3]^T, \qquad Ae_4 = [a_4, b_4, c_4]^T$$

Thus, the image of A is precisely the column space of A.

On the other hand, the kernel of A consists of all vectors v for which $Av = 0$. This means that the kernel of A is the solution space of the homogeneous system $AX = 0$, called the *null space* of A.

We state the above results formally.

PROPOSITION 5.5: Let A be any $m \times n$ matrix over a field K viewed as a linear map $A : K^n \to K^m$. Then

$$\text{Ker } A = \text{nullsp}(A) \qquad \text{and} \qquad \text{Im } A = \text{colsp}(A)$$

Here colsp(A) denotes the column space of A, and nullsp(A) denotes the null space of A.

Rank and Nullity of a Linear Mapping

Let $F: V \to U$ be a linear mapping. The *rank* of F is defined to be the dimension of its image, and the *nullity* of F is defined to be the dimension of its kernel; namely,

$$\text{rank}(F) = \dim(\text{Im } F) \qquad \text{and} \qquad \text{nullity}(F) = \dim(\text{Ker } F)$$

The following important theorem (proved in Problem 5.23) holds.

THEOREM 5.6 Let V be of finite dimension, and let $F: V \to U$ be linear. Then

$$\dim V = \dim(\text{Ker } F) + \dim(\text{Im } F) = \text{nullity}(F) + \text{rank}(F)$$

Recall that the rank of a matrix A was also defined to be the dimension of its column space and row space. If we now view A as a linear mapping, then both definitions correspond, because the image of A is precisely its column space.

EXAMPLE 5.9 Let $F: \mathbf{R}^4 \to \mathbf{R}^3$ be the linear mapping defined by

$$F(x, y, z, t) = (x - y + z + t, \quad 2x - 2y + 3z + 4t, \quad 3x - 3y + 4z + 5t)$$

(a) Find a basis and the dimension of the image of F.

First find the image of the usual basis vectors of $\mathbf{R}^4$,

$$F(1,0,0,0) = (1,2,3), \qquad F(0,0,1,0) = (1,3,4)$$
$$F(0,1,0,0) = (-1,-2,-3), \qquad F(0,0,0,1) = (1,4,5)$$

By Proposition 5.4, the image vectors span Im F. Hence, form the matrix M whose rows are these image vectors and row reduce to echelon form:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $(1,2,3)$ and $(0,1,1)$ form a basis of Im F. Hence, $\dim(\text{Im } F) = 2$ and $\text{rank}(F) = 2$.

(b) Find a basis and the dimension of the kernel of the map F.

Set $F(v) = 0$, where $v = (x, y, z, t)$,

$$F(x, y, z, t) = (x - y + z + t, \quad 2x - 2y + 3z + 4t, \quad 3x - 3y + 4z + 5t) = (0, 0, 0)$$

Set corresponding components equal to each other to form the following homogeneous system whose solution space is Ker F:

$$\begin{aligned} x - y + z + t &= 0 \\ 2x - 2y + 3z + 4t &= 0 \\ 3x - 3y + 4z + 5t &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y + z + t &= 0 \\ z + 2t &= 0 \\ z + 2t &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y + z + t &= 0 \\ z + 2t &= 0 \end{aligned}$$

The free variables are y and t. Hence, $\dim(\text{Ker } F) = 2$ or $\text{nullity}(F) = 2$.

(i) Set $y = 1$, $t = 0$ to obtain the solution $(-1, 1, 0, 0)$,

(ii) Set $y = 0$, $t = 1$ to obtain the solution $(1, 0, -2, 1)$.

Thus, $(-1, 1, 0, 0)$ and $(1, 0, -2, 1)$ form a basis for Ker F.

As expected from Theorem 5.6, $\dim(\text{Im } F) + \dim(\text{Ker } F) = 4 = \dim \mathbf{R}^4$.

Application to Systems of Linear Equations

Let $AX = B$ denote the matrix form of a system of m linear equations in n unknowns. Now the matrix A may be viewed as a linear mapping

$$A: K^n \to K^m$$