13. If $f(l,k) = 3l^3k^6 - 2l^2k^7$, find $f_{klk}(2,1)$.

14. If $f(x,y) = x^3y^2 + x^2y - x^2y^2$, find $f_{xxy}(2,3)$ and $f_{xyx}(2,3)$.

15. If $f(x,y) = y^2e^x + \ln(xy)$, find $f_{xyy}(1,1)$.

16. If $f(x,y) = x^3 - 6xy^2 + x^2 - y^3$, find $f_{xy}(1,-1)$.

17. Cost Function Suppose the cost c of producing q_A units of product A and q_B units of product B is given by

$$c = (3q_A^2 + q_B^3 + 4)^{1/3}$$

and the coupled demand functions for the products are given by

$$q_A = 10 - p_A + p_B^2$$

and

$$q_B = 20 + p_A - 11p_B$$

Find the value of

$$\frac{\partial^2 c}{\partial q_A\, \partial q_B}$$

when $p_A = 25$ and $p_B = 4$.

18. For $f(x,y) = x^4y^4 + 3x^3y^2 - 7x + 4$, show that

$$f_{xyx}(x,y) = f_{xxy}(x,y)$$

19. For $f(x,y) = e^{x^2+xy+y^2}$, show that

$$f_{xy}(x,y) = f_{yx}(x,y)$$

20. For $f(x,y) = e^{xy}$, show that

$$f_{xx}(x,y) + f_{xy}(x,y) + f_{yx}(x,y) + f_{yy}(x,y) = f(x,y)((x+y)^2 + 2)$$

21. For $z = \ln(x^2 + y^2)$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

[17]**22.** If $z^3 - x^3 - x^2y - xy^2 - y^3 = 0$, find $\frac{\partial^2 z}{\partial x^2}$.

[17]**23.** If $z^2 - 3x^2 + y^2 = 0$, find $\frac{\partial^2 z}{\partial y^2}$.

[17]**24.** If $2z^2 = x^2 + 2xy + xz$, find $\frac{\partial^2 z}{\partial x\, \partial y}$.

Objective

To show how to find partial derivatives of composite functions by using the chain rule.

17.5 Chain Rule[18]

Suppose a manufacturer of two related products A and B has a joint-cost function given by

$$c = f(q_A, q_B)$$

where c is the total cost of producing quantities q_A and q_B of A and B, respectively. Furthermore, suppose the demand functions for the products are

$$q_A = g(p_A, p_B) \quad \text{and} \quad q_B = h(p_A, p_B)$$

where p_A and p_B are the prices per unit of A and B, respectively. Since c is a function of q_A and q_B, and since both q_A and q_B are themselves functions of p_A and p_B, c can be viewed as a function of p_A and p_B. (Appropriately, the variables q_A and q_B are called *intermediate variables* of c.) Consequently, we should be able to determine $\partial c/\partial p_A$, the rate of change of total cost with respect to the price of A. One way to do this is to substitute the expressions $g(p_A, p_B)$ and $h(p_A, p_B)$ for q_A and q_B, respectively, into $c = f(q_A, q_B)$. Then c is a function of p_A and p_B, and we can differentiate c with respect to p_A directly. This approach has some drawbacks—especially when f, g, or h is given by a complicated expression. Another way to approach the problem would be to use the chain rule (actually *a* chain rule), which we now state without proof.

Chain Rule

Let $z = f(x,y)$, where both x and y are functions of r and s given by $x = x(r,s)$ and $y = y(r,s)$. If f, x, and y have continuous partial derivatives, then z is a function of r and s, and

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

[17]Omit if Section 17.3 was not covered.

[18]This section can be omitted without loss of continuity.

Note that in the chain rule, the number of intermediate variables of z (two) is the same as the number of terms that compose each of $\partial z/\partial r$ and $\partial z/\partial s$.

Returning to the original situation concerning the manufacturer, we see that if f, q_A, and q_B have continuous partial derivatives, then, by the chain rule,

$$\frac{\partial c}{\partial p_A} = \frac{\partial c}{\partial q_A}\frac{\partial q_A}{\partial p_A} + \frac{\partial c}{\partial q_B}\frac{\partial q_B}{\partial p_A}$$

EXAMPLE 1 Rate of Change of Cost

For a manufacturer of cameras and film, the total cost c of producing q_C cameras and q_F units of film is given by

$$c = 30q_C + 0.015q_Cq_F + q_F + 900$$

The demand functions for the cameras and film are given by

$$q_C = \frac{9000}{p_C\sqrt{p_F}} \quad \text{and} \quad q_F = 2000 - p_C - 400p_F$$

where p_C is the price per camera and p_F is the price per unit of film. Find the rate of change of total cost with respect to the price of the camera when $p_C = 50$ and $p_F = 2$.

Solution: We must first determine $\partial c/\partial p_C$. By the chain rule,

$$\begin{aligned}\frac{\partial c}{\partial p_C} &= \frac{\partial c}{\partial q_C}\frac{\partial q_C}{\partial p_C} + \frac{\partial c}{\partial q_F}\frac{\partial q_F}{\partial p_C}\\ &= (30 + 0.015q_F)\left[\frac{-9000}{p_C^2\sqrt{p_F}}\right] + (0.015q_C + 1)(-1)\end{aligned}$$

When $p_C = 50$ and $p_F = 2$, then $q_C = 90\sqrt{2}$ and $q_F = 1150$. Substituting these values into $\partial c/\partial p_C$ and simplifying, we have

$$\left.\frac{\partial c}{\partial p_C}\right|_{\substack{p_C=50\\p_F=2}} \approx -123.2$$

Now Work Problem 1 ◁

The chain rule can be extended. For example, suppose $z = f(v, w, x, y)$ and v, w, x, and y are all functions of r, s, and t. Then, if certain conditions of continuity are assumed, z is a function of r, s, and t, and we have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Observe that the number of intermediate variables of z (four) is the same as the number of terms that form each of $\partial z/\partial r$, $\partial z/\partial s$, and $\partial z/\partial t$.

Now consider the situation where $z = f(x, y)$ such that $x = x(t)$ and $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Use the partial derivative symbols and the ordinary derivative symbols appropriately.

Here we use the symbol dz/dt rather than $\partial z/\partial t$, since z can be considered a function of *one* variable t. Likewise, the symbols dx/dt and dy/dt are used rather than $\partial x/\partial t$ and $\partial y/\partial t$. As is typical, the number of terms that compose dz/dt equals the number of intermediate variables of z. Other situations would be treated in a similar way.

EXAMPLE 2 **Chain Rule**

a. If $w = f(x, y, z) = 3x^2y + xyz - 4y^2z^3$, where

$$x = 2r - 3s \quad y = 6r + s \quad z = r - s$$

determine $\partial w/\partial r$ *and* $\partial w/\partial s$.

Solution: Since x, y, and z are functions of r and s, then, by the chain rule,

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ &= (6xy + yz)(2) + (3x^2 + xz - 8yz^3)(6) + (xy - 12y^2z^2)(1) \\ &= x(18x + 13y + 6z) + 2yz(1 - 24z^2 - 6yz)\end{aligned}$$

Also,

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (6xy + yz)(-3) + (3x^2 + xz - 8yz^3)(1) + (xy - 12y^2z^2)(-1) \\ &= x(3x - 19y + z) - yz(3 + 8z^2 - 12yz)\end{aligned}$$

b. If $z = \dfrac{x + e^y}{y}$, where $x = rs + se^{rt}$ and $y = 9 + rt$, evaluate $\partial z/\partial s$ when $r = -2$, $s = 5$, and $t = 4$.

Solution: Since x and y are functions of r, s, and t (note that we can write $y = 9 + rt + 0 \cdot s$), by the chain rule,

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= \left(\frac{1}{y}\right)(r + e^{rt}) + \frac{\partial z}{\partial y} \cdot (0) = \frac{r + e^{rt}}{y}\end{aligned}$$

If $r = -2, s = 5$, and $t = 4$, then $y = 1$. Thus,

$$\left.\frac{\partial z}{\partial s}\right|_{\substack{r=-2\\ s=5\\ t=4}} = \frac{-2 + e^{-8}}{1} = -2 + e^{-8}$$

Now Work Problem 13 ◁

EXAMPLE 3 **Chain Rule**

a. Determine $\partial y/\partial r$ if $y = x^2 \ln(x^4 + 6)$ and $x = (r + 3s)^6$.

Solution: By the chain rule,

$$\begin{aligned}\frac{\partial y}{\partial r} &= \frac{dy}{dx}\frac{\partial x}{\partial r} \\ &= \left[x^2 \cdot \frac{4x^3}{x^4 + 6} + 2x \cdot \ln(x^4 + 6)\right][6(r + 3s)^5] \\ &= 12x(r + 3s)^5 \left[\frac{2x^4}{x^4 + 6} + \ln(x^4 + 6)\right]\end{aligned}$$

b. Given that $z = e^{xy}, x = r - 4s$, and $y = r - s$, find $\partial z/\partial r$ in terms of r and s.

Solution:

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= (ye^{xy})(1) + (xe^{xy})(1) \\ &= (x+y)e^{xy}\end{aligned}$$

Since $x = r - 4s$ and $y = r - s$,

$$\begin{aligned}\frac{\partial z}{\partial r} &= [(r-4s) + (r-s)]e^{(r-4s)(r-s)} \\ &= (2r - 5s)e^{r^2 - 5rs + 4s^2}\end{aligned}$$

Now Work Problem 15 ◁

PROBLEMS 17.5

In Problems 1–12, find the indicated derivatives by using the chain rule.

1. $z = 5x + 3y, x = 2r + 3s, y = r - 2s;\quad \partial z/\partial r, \partial z/\partial s$

2. $z = 2x^2 + 3xy + 2y^2, x = r^2 - s^2, y = r^2 + s^2;\quad \partial z/\partial r, \partial z/\partial s$

3. $z = e^{x+y}, x = t^2 + 3, y = \sqrt{t^3};\quad dz/dt$

4. $z = \sqrt{8x + y}, x = t^2 + 3t + 4,\ y = t^3 + 4;\quad dz/dt$

5. $w = x^2yz + xy^2z + xyz^2, x = e^t, y = te^t, z = t^2e^t;\quad dw/dt$

6. $w = \ln(x^2 + y^2 + z^2),\ x = 2 - 3t, y = t^2 + 3, z = 4 - t; dw/dt$

7. $z = (x^2 + xy^2)^3, x = r + s + t,\ y = 2r - 3s + 8t;\quad \partial z/\partial t$

8. $z = \sqrt{x^2 + y^2}, x = r^2 + s - t,\ y = r - s + t;\quad \partial z/\partial r$

9. $w = x^2 + xyz + z^2, x = r^2 - s^2,\ y = rs, z = r^2 + s^2;\quad \partial w/\partial s$

10. $w = \ln(xyz), x = r^2s, y = rs, z = rs^2;\quad \partial w/\partial r$

11. $y = x^2 - 7x + 5, x = 19rs + 2s^2t^2;\quad \partial y/\partial r$

12. $y = 4 - x^2, x = 2r + 3s - 4t;\quad \partial y/\partial t$

13. If $z = (4x + 3y)^3$, where $x = r^2s$ and $y = r - 2s$, evaluate $\partial z/\partial r$ when $r = 0$ and $s = 1$.

14. If $z = \sqrt{2x + 3y}$, where $x = 3t + 5$ and $y = t^2 + 2t + 1$, evaluate dz/dt when $t = 1$.

15. If $w = e^{x+y+z}(x^2 + y^2 + z^2)$, where $x = (r - s)^2, y = (r + s)^2$, and $z = (s - r)^2$, evaluate $\partial w/\partial s$ when $r = 1$ and $s = 1$.

16. If $y = x/(x - 5)$, where $x = 2t^2 - 3rs - r^2t$, evaluate $\partial y/\partial t$ when $r = 0, s = 2$, and $t = -1$.

17. Cost Function Suppose the cost c of producing q_A units of product A and q_B units of product B is given by

$$c = (3q_A^2 + q_B^3 + 4)^{1/3}$$

and the coupled demand functions for the products are given by

$$q_A = 10 - p_A + p_B^2$$

and

$$q_B = 20 + p_A - 11p_B$$

Use a chain rule to evaluate $\dfrac{\partial c}{\partial p_A}$ and $\dfrac{\partial c}{\partial p_B}$ when $p_A = 25$ and $p_B = 4$.

18. Suppose $w = f(x, y)$, where $x = g(t)$ and $y = h(t)$.
(a) State a chain rule that gives dw/dt.
(b) Suppose $h(t) = t$, so that $w = f(x, t)$, where $x = g(t)$. Use part (a) to find dw/dt and simplify your answer.

19. (a) Suppose w is a function of x and y, where both x and y are functions of s and t. State a chain rule that expresses $\partial w/\partial t$ in terms of derivatives of these functions.
(b) Let $w = 2x^2 \ln|3x - 5y|$, where $x = s\sqrt{t^2 + 2}$ and $y = t - 3e^{2-s}$. Use part (a) to evaluate $\partial w/\partial t$ when $s = 1$ and $t = 0$.

20. Production Function In considering a production function $P = f(l, k)$, where l is labor input and k is capital input, Fon, Boulier, and Goldfarb[19] assume that $l = Lg(h)$, where L is the number of workers, h is the number of hours per day per worker, and $g(h)$ is a labor effectiveness function. In maximizing profit p given by

$$p = aP - whL$$

where a is the price per unit of output and w is the hourly wage per worker, Fon, Boulier, and Goldfarb determine $\partial p/\partial L$ and $\partial p/\partial h$. Assume that k is independent of L and h, and determine these partial derivatives.

17.6 Maxima and Minima for Functions of Two Variables

Objective

To discuss relative maxima and relative minima, to find critical points, and to apply the second-derivative test for a function of two variables.

We now extend the notion of relative maxima and minima (or relative extrema) to functions of two variables.

[19] V. Fon, B. L. Boulier, and R. S. Goldfarb, "The Firm's Demand for Daily Hours of Work: Some Implications," *Atlantic Economic Journal*, XIII, no. 1 (1985), 36–42.

Definition

A function $z = f(x, y)$ is said to have a ***relative maximum*** at the point (a, b) if, for all points (x, y) in the plane that are sufficiently close to (a, b), we have

$$f(a, b) \geq f(x, y) \tag{1}$$

For a ***relative minimum,*** we replace $\geq$ by $\leq$ in Equation (1).

To say that $z = f(x, y)$ has a relative maximum at (a, b) means, geometrically, that the point $(a, b, f(a, b))$ on the graph of f is higher than (or is as high as) all other points on the surface that are "near" $(a, b, f(a, b))$. In Figure 17.4(a), f has a relative maximum at (a, b). Similarly, the function f in Figure 17.4(b) has a relative minimum when $x = y = 0$, which corresponds to a *low* point on the surface.

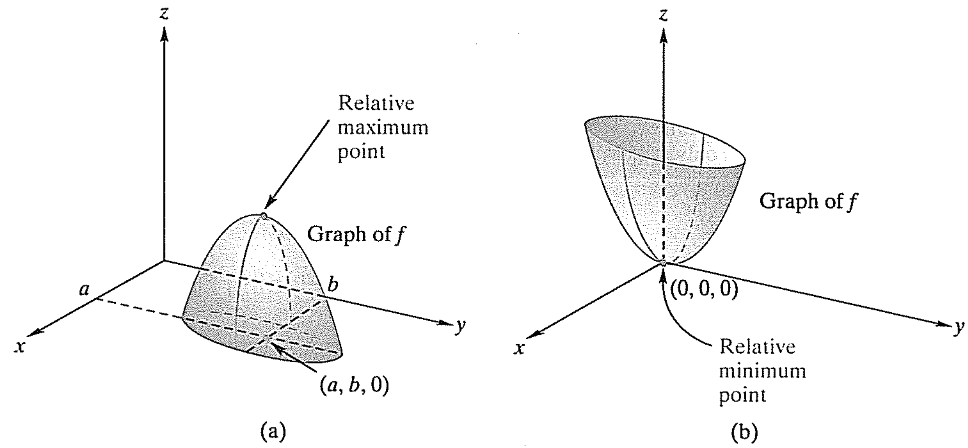

FIGURE 17.4 Relative extrema.

Recall that in locating extrema for a function $y = f(x)$ of one variable, we examine those values of x in the domain of f for which $f'(x) = 0$ or $f'(x)$ does not exist. For functions of two (or more) variables, a similar procedure is followed. However, for the functions that concern us, extrema will not occur where a derivative does not exist, and such situations will be excluded from our consideration.

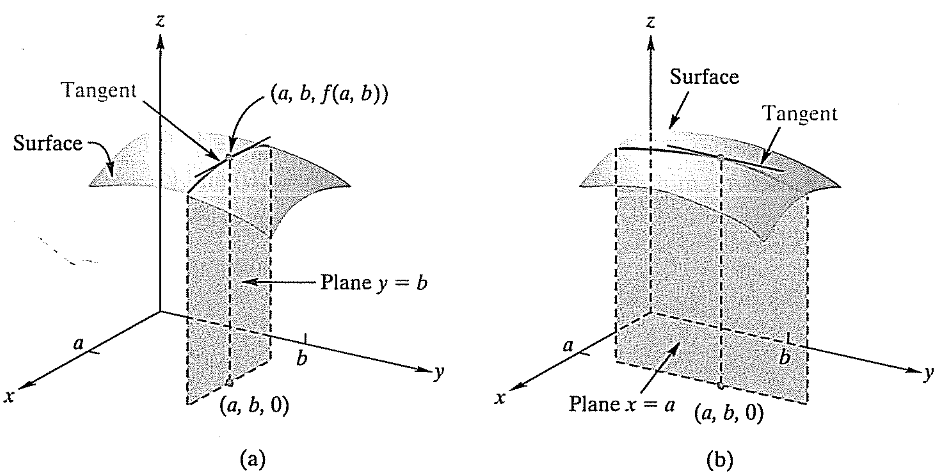

FIGURE 17.5 At relative extremum, $f_x(x, y) = 0$ and $f_y(x, y) = 0$.

Suppose $z = f(x, y)$ has a relative maximum at (a, b), as indicated in Figure 17.5(a). Then the curve where the plane $y = b$ intersects the surface must have a relative maximum when $x = a$. Hence, the slope of the tangent line to the surface in the x-direction must be 0 at (a, b). Equivalently, $f_x(x, y) = 0$ at (a, b). Similarly, on the curve where the plane $x = a$ intersects the surface [Figure 17.5(b)], there must be a relative maximum when $y = b$. Thus, in the y-direction, the slope of the tangent to the surface must be 0 at (a, b). Equivalently, $f_y(x, y) = 0$ at (a, b). Since a similar discussion applies to a relative minimum, we can combine these results as follows:

Rule 1

If $z = f(x, y)$ has a relative maximum or minimum at (a, b), and if both f_x and f_y are defined for all points close to (a, b), it is necessary that (a, b) be a solution of the system

$$\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases}$$

CAUTION!

Rule 1 does not imply that there must be an extremum at a critical point. Just as in the case of functions of one variable, a critical point can give rise to a relative maximum, a relative minimum, or neither. A critical point is only a *candidate* for a relative extremum.

A point (a, b) for which $f_x(a, b) = f_y(a, b) = 0$ is called a **critical point** of f. Thus, from Rule 1, we infer that, to locate relative extrema for a function, we should examine its critical points.

Two additional comments are in order: First, Rule 1, as well as the notion of a critical point, can be extended to functions of more than two variables. For example, to locate possible extrema for $w = f(x, y, z)$, we would examine those points for which $w_x = w_y = w_z = 0$. Second, for a function whose domain is restricted, a thorough examination for absolute extrema would include a consideration of boundary points.

EXAMPLE 1 Finding Critical Points

Find the critical points of the following functions.

a. $f(x, y) = 2x^2 + y^2 - 2xy + 5x - 3y + 1$.

Solution: Since $f_x(x, y) = 4x - 2y + 5$ and $f_y(x, y) = 2y - 2x - 3$, we solve the system

$$\begin{cases} 4x - 2y + 5 = 0 \\ -2x + 2y - 3 = 0 \end{cases}$$

This gives $x = -1$ and $y = \frac{1}{2}$. Thus, $\left(-1, \frac{1}{2}\right)$ is the only critical point.

b. $f(l, k) = l^3 + k^3 - lk$.

Solution:

$$\begin{cases} f_l(l, k) = 3l^2 - k = 0 & (2) \\ f_k(l, k) = 3k^2 - l = 0 & (3) \end{cases}$$

From Equation (2), $k = 3l^2$. Substituting for k in Equation (3) gives

$$0 = 27l^4 - l = l(27l^3 - 1)$$

Hence, either $l = 0$ or $l = \frac{1}{3}$. If $l = 0$, then $k = 0$; if $l = \frac{1}{3}$, then $k = \frac{1}{3}$. The critical points are therefore $(0, 0)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$.

c. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 100)$.

Solution: Solving the system

$$\begin{cases} f_x(x, y, z) = 4x + y - z = 0 \\ f_y(x, y, z) = x + 2y - z = 0 \\ f_z(x, y, z) = -x - y + 100 = 0 \end{cases}$$

gives the critical point $(25, 75, 175)$, as the reader should verify.

Now Work Problem 1 ◁

EXAMPLE 2 Finding Critical Points

Find the critical points of

$$f(x, y) = x^2 - 4x + 2y^2 + 4y + 7$$

Solution: We have $f_x(x,y) = 2x - 4$ and $f_y(x,y) = 4y + 4$. The system

$$\begin{cases} 2x - 4 = 0 \\ 4y + 4 = 0 \end{cases}$$

gives the critical point $(2, -1)$. Observe that we can write the given function as

$$\begin{aligned} f(x,y) &= x^2 - 4x + 4 + 2(y^2 + 2y + 1) + 1 \\ &= (x-2)^2 + 2(y+1)^2 + 1 \end{aligned}$$

and $f(2,-1) = 1$. Clearly, if $(x,y) \neq (2,-1)$, then $f(x,y) > 1$. Hence, a relative minimum occurs at $(2,-1)$. Moreover, there is an *absolute minimum* at $(2,-1)$, since $f(x,y) > f(2,-1)$ for *all* $(x,y) \neq (2,-1)$.

Now Work Problem 3 ◁

Although in Example 2 we were able to show that the critical point gave rise to a relative extremum, in many cases this is not so easy to do. There is, however, a second-derivative test that gives conditions under which a critical point will be a relative maximum or minimum. We state it now, omitting the proof.

Rule 2 Second-Derivative Test for Functions of Two Variables

Suppose $z = f(x,y)$ has continuous partial derivatives f_{xx}, f_{yy}, and f_{xy} at all points (x,y) near a critical point (a,b). Let D be the function defined by

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$$

Then

1. if $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a relative maximum at (a,b);
2. if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a relative minimum at (a,b);
3. if $D(a,b) < 0$, then f has a *saddle point* at (a,b) (see Example 4);
4. if $D(a,b) = 0$, then no conclusion about an extremum at (a,b) can be drawn, and further analysis is required.

We remark that when $D(a,b) > 0$, the sign of $f_{xx}(a,b)$ is necessarily the same as the sign of $f_{yy}(a,b)$. Thus, when $D(a,b) > 0$ we can test either $f_{xx}(a,b)$ or $f_{yy}(a,b)$, whichever is easiest, to make the determination required in parts 1 and 2 of the second derivative test.

EXAMPLE 3 Applying the Second-Derivative Test

Examine $f(x,y) = x^3 + y^3 - xy$ for relative maxima or minima by using the second-derivative test.

Solution: First we find critical points:

$$f_x(x,y) = 3x^2 - y \quad f_y(x,y) = 3y^2 - x$$

In the same manner as in Example 1(b), solving $f_x(x,y) = f_y(x,y) = 0$ gives the critical points $(0,0)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$. Now,

$$f_{xx}(x,y) = 6x \qquad f_{yy}(x,y) = 6y \qquad f_{xy}(x,y) = -1$$

Thus,

$$D(x,y) = (6x)(6y) - (-1)^2 = 36xy - 1$$

Since $D(0,0) = 36(0)(0) - 1 = -1 < 0$, there is no relative extremum at $(0,0)$. Also, since $D\left(\frac{1}{3}, \frac{1}{3}\right) = 36\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - 1 = 3 > 0$ and $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = 6\left(\frac{1}{3}\right) = 2 > 0$, there is a relative minimum at $\left(\frac{1}{3}, \frac{1}{3}\right)$. At this point, the value of the function is

$$f\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{27}$$

Now Work Problem 7 ◁

EXAMPLE 4 A Saddle Point

The surface in Figure 17.6 is called a hyperbolic paraboloid.

Examine $f(x, y) = y^2 - x^2$ for relative extrema.

Solution: Solving

$$f_x(x, y) = -2x = 0 \quad \text{and} \quad f_y(x, y) = 2y = 0$$

we get the critical point (0, 0). Now we apply the second-derivative test. At (0, 0), and indeed at any point,

$$f_{xx}(x, y) = -2 \quad f_{yy}(x, y) = 2 \quad f_{xy}(x, y) = 0$$

Because $D(0, 0) = (-2)(2) - (0)^2 = -4 < 0$, no relative extremum exists at (0, 0). A sketch of $z = f(x, y) = y^2 - x^2$ appears in Figure 17.6. Note that, for the surface curve cut by the plane $y = 0$, there is a *maximum* at (0, 0); but for the surface curve cut by the plane $x = 0$, there is a *minimum* at (0, 0). Thus, on the *surface,* no relative extremum can exist at the origin, although (0, 0) is a critical point. Around the origin the curve is saddle shaped, and (0, 0) is called a *saddle point* of f.

Now Work Problem 11 ◁

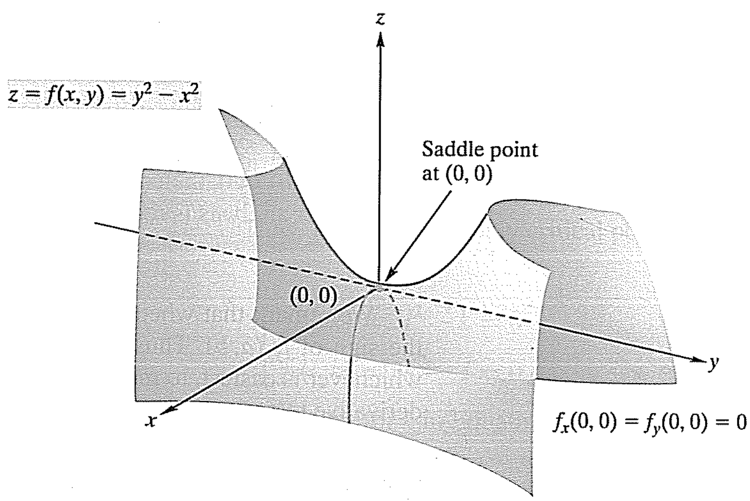

FIGURE 17.6 Saddle point.

EXAMPLE 5 Finding Relative Extrema

Examine $f(x, y) = x^4 + (x - y)^4$ for relative extrema.

Solution: If we set

$$f_x(x, y) = 4x^3 + 4(x - y)^3 = 0 \tag{4}$$

and

$$f_y(x, y) = -4(x - y)^3 = 0 \tag{5}$$

then, from Equation (5), we have $x - y = 0$, or $x = y$. Substituting into Equation (4) gives $4x^3 = 0$, or $x = 0$. Thus, $x = y = 0$, and (0, 0) is the only critical point. At (0, 0),

$$f_{xx}(x, y) = 12x^2 + 12(x - y)^2 = 0$$
$$f_{yy}(x, y) = 12(x - y)^2 = 0$$

and

$$f_{xy}(x, y) = -12(x - y)^2 = 0$$

Hence, $D(0, 0) = 0$, and the second-derivative test gives no information. However, for all $(x, y) \neq (0, 0)$, we have $f(x, y) > 0$, whereas $f(0, 0) = 0$. Therefore, at (0, 0) the graph of f has a low point, and we conclude that f has a relative (and absolute) minimum at (0, 0).

Now Work Problem 13 ◁

Applications

In many situations involving functions of two variables, and especially in their applications, the nature of the given problem is an indicator of whether a critical point is in fact a relative (or absolute) maximum or a relative (or absolute) minimum. In such cases, the second-derivative test is not needed. Often, in mathematical studies of applied problems, the appropriate second-order conditions are assumed to hold.

EXAMPLE 6 **Maximizing Output**

Let P be a production function given by

$$P = f(l, k) = 0.54l^2 - 0.02l^3 + 1.89k^2 - 0.09k^3$$

where l and k are the amounts of labor and capital, respectively, and P is the quantity of output produced. Find the values of l and k that maximize P.

Solution: To find the critical points, we solve the system $P_l = 0$ and $P_k = 0$:

$$\begin{aligned} P_l &= 1.08l - 0.06l^2 & P_k &= 3.78k - 0.27k^2 \\ &= 0.06l(18 - l) = 0 & &= 0.27k(14 - k) = 0 \\ l &= 0, l = 18 & k &= 0, k = 14 \end{aligned}$$

There are four critical points: (0, 0), (0, 14), (18, 0), and (18, 14).

Now we apply the second-derivative test to each critical point. We have

$$P_{ll} = 1.08 - 0.12l \quad P_{kk} = 3.78 - 0.54k \quad P_{lk} = 0$$

Thus,

$$\begin{aligned} D(l, k) &= P_{ll}P_{kk} - [P_{lk}]^2 \\ &= (1.08 - 0.12l)(3.78 - 0.54k) \end{aligned}$$

At (0, 0),

$$D(0, 0) = 1.08(3.78) > 0$$

Since $D(0, 0) > 0$ and $P_{ll} = 1.08 > 0$, there is a relative minimum at (0, 0). At (0, 14),

$$D(0, 14) = 1.08(-3.78) < 0$$

Because $D(0, 14) < 0$, there is no relative extremum at (0, 14). At (18, 0),

$$D(18, 0) = (-1.08)(3.78) < 0$$

Since $D(18, 0) < 0$, there is no relative extremum at (18, 0). At (18, 14),

$$D(18, 14) = (-1.08)(-3.78) > 0$$

Because $D(18, 14) > 0$ and $P_{ll} = -1.08 < 0$, there is a relative maximum at (18, 14). Hence, the maximum output is obtained when $l = 18$ and $k = 14$.

Now Work Problem 21 ◁

EXAMPLE 7 Profit Maximization

A candy company produces two types of candy, A and B, for which the average costs of production are constant at \$2 and \$3 per pound, respectively. The quantities q_A, q_B (in pounds) of A and B that can be sold each week are given by the joint-demand functions

$$q_A = 400(p_B - p_A)$$

and

$$q_B = 400(9 + p_A - 2p_B)$$

where p_A and p_B are the selling prices (in dollars per pound) of A and B, respectively. Determine the selling prices that will maximize the company's profit P.

Solution: The total profit is given by

$$P = \begin{pmatrix}\text{profit}\\ \text{per pound}\\ \text{of A}\end{pmatrix}\begin{pmatrix}\text{pounds}\\ \text{of A}\\ \text{sold}\end{pmatrix} + \begin{pmatrix}\text{profit}\\ \text{per pound}\\ \text{of B}\end{pmatrix}\begin{pmatrix}\text{pounds}\\ \text{of B}\\ \text{sold}\end{pmatrix}$$

For A and B, the profits per pound are $p_A - 2$ and $p_B - 3$, respectively. Thus,

$$\begin{aligned} P &= (p_A - 2)q_A + (p_B - 3)q_B \\ &= (p_A - 2)[400(p_B - p_A)] + (p_B - 3)[400(9 + p_A - 2p_B)] \end{aligned}$$

Notice that P is expressed as a function of two variables, p_A and p_B. To maximize P, we set its partial derivatives equal to 0:

$$\begin{aligned} \frac{\partial P}{\partial p_A} &= (p_A - 2)[400(-1)] + [400(p_B - p_A)](1) + (p_B - 3)[400(1)] \\ &= 0 \\ \frac{\partial P}{\partial p_B} &= (p_A - 2)[400(1)] + (p_B - 3)[400(-2)] + 400(9 + p_A - 2p_B)](1) \\ &= 0 \end{aligned}$$

Simplifying the preceding two equations gives

$$\begin{cases} -2p_A + 2p_B - 1 = 0 \\ 2p_A - 4p_B + 13 = 0 \end{cases}$$

whose solution is $p_A = 5.5$ and $p_B = 6$. Moreover, we find that

$$\frac{\partial^2 P}{\partial p_A^2} = -800 \quad \frac{\partial^2 P}{\partial p_B^2} = -1600 \quad \frac{\partial^2 P}{\partial p_B \partial p_A} = 800$$

Therefore,

$$D(5.5, 6) = (-800)(-1600) - (800)^2 > 0$$

Since $\partial^2 P/\partial p_A^2 < 0$, we indeed have a maximum, and the company should sell candy A at \$5.50 per pound and B at \$6.00 per pound.

Now Work Problem 23 ◁

EXAMPLE 8 Profit Maximization for a Monopolist[20]

Suppose a monopolist is practicing price discrimination by selling the same product in two separate markets at different prices. Let q_A be the number of units sold in market

[20]Omit if Section 17.5 was not covered.

A, where the demand function is $p_A = f(q_A)$, and let q_B be the number of units sold in market B, where the demand function is $p_B = g(q_B)$. Then the revenue functions for the two markets are

$$r_A = q_A f(q_A) \quad \text{and} \quad r_B = q_B g(q_B)$$

Assume that all units are produced at one plant, and let the cost function for producing $q = q_A + q_B$ units be $c = c(q)$. Keep in mind that r_A is a function of q_A and r_B is a function of q_B. The monopolist's profit P is

$$P = r_A + r_B - c$$

To maximize P with respect to outputs q_A and q_B, we set its partial derivatives equal to zero. To begin with,

$$\begin{aligned}\frac{\partial P}{\partial q_A} &= \frac{dr_A}{dq_A} + 0 - \frac{\partial c}{\partial q_A}\\ &= \frac{dr_A}{dq_A} - \frac{dc}{dq}\frac{\partial q}{\partial q_A} = 0 \qquad \text{chain rule}\end{aligned}$$

Because

$$\frac{\partial q}{\partial q_A} = \frac{\partial}{\partial q_A}(q_A + q_B) = 1$$

we have

$$\frac{\partial P}{\partial q_A} = \frac{dr_A}{dq_A} - \frac{dc}{dq} = 0 \tag{6}$$

Similarly,

$$\frac{\partial P}{\partial q_B} = \frac{dr_B}{dq_B} - \frac{dc}{dq} = 0 \tag{7}$$

From Equations (6) and (7), we get

$$\frac{dr_A}{dq_A} = \frac{dc}{dq} = \frac{dr_B}{dq_B}$$

But dr_A/dq_A and dr_B/dq_B are marginal revenues, and dc/dq is marginal cost. Hence, to maximize profit, it is necessary to charge prices (and distribute output) so that the marginal revenues in both markets will be the same and, loosely speaking, will also be equal to the cost of the last unit produced in the plant.

Now Work Problem 25 ◁

PROBLEMS 17.6

In Problems 1–6, find the critical points of the functions.

1. $f(x, y) = x^2 - 3y^2 - 8x + 9y + 3xy$

2. $f(x, y) = x^2 + 4y^2 - 6x + 16y$

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

4. $f(x, y) = xy - x + y$

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

In Problems 7–20, find the critical points of the functions. For each critical point, determine, by the second-derivative test, whether it corresponds to a relative maximum, to a relative minimum, or to neither, or whether the test gives no information.

7. $f(x, y) = x^2 + 3y^2 + 4x - 9y + 3$

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

9. $f(x, y) = y - y^2 - 3x - 6x^2$

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

11. $f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3$

12. $f(x, y) = \frac{x^3}{3} + y^2 - 2x + 2y - 2xy$

13. $f(x, y) = \frac{1}{3}(x^3 + 8y^3) - 2(x^2 + y^2) + 1$

14. $f(x, y) = x^2 + y^2 - xy + x^3$

15. $f(l, k) = \frac{l^2}{2} + 2lk + 3k^2 - 69l - 164k + 17$

16. $f(l, k) = l^2 + 4k^2 - 4lk$

17. $f(p, q) = pq - \frac{1}{p} - \frac{1}{q}$

18. $f(x, y) = (x - 3)(y - 3)(x + y - 3)$

19. $f(x, y) = (y^2 - 4)(e^x - 1)$

20. $f(x, y) = \ln(xy) + 2x^2 - xy - 6x$

21. Maximizing Output Suppose

$$P = f(l, k) = 2.18l^2 - 0.02l^3 + 1.97k^2 - 0.03k^3$$

is a production function for a firm. Find the quantities of inputs l and k that maximize output P.

22. Maximizing Output In a certain office, computers C and D are utilized for c and d hours, respectively. If daily output Q is a function of c and d, namely,

$$Q = 18c + 20d - 2c^2 - 4d^2 - cd$$

find the values of c and d that maximize Q.

In Problems 23–35, unless otherwise indicated, the variables p_A and p_B denote selling prices of products A and B, respectively. Similarly, q_A and q_B denote quantities of A and B that are produced and sold during some time period. In all cases, the variables employed will be assumed to be units of output, input, money, and so on.

23. Profit A candy company produces two varieties of candy, A and B, for which the constant average costs of production are 60 and 70 (cents per lb), respectively. The demand functions for A and B are given by

$$q_A = 5(p_B - p_A) \quad \text{and} \quad q_B = 500 + 5(p_A - 2p_B)$$

Find the selling prices p_A and p_B that maximize the company's profit.

24. Profit Repeat Problem 23 if the constant costs of production of A and B are a and b (cents per lb), respectively.

25. Price Discrimination Suppose a monopolist is practicing price discrimination in the sale of a product by charging different prices in two separate markets. In market A the demand function is

$$p_A = 100 - q_A$$

and in B it is

$$p_B = 84 - q_B$$

where q_A and q_B are the quantities sold per week in A and B, and p_A and p_B are the respective prices per unit. If the monopolist's cost function is

$$c = 600 + 4(q_A + q_B)$$

how much should be sold in each market to maximize profit? What selling prices give this maximum profit? Find the maximum profit.

26. Profit A monopolist sells two competitive products, A and B, for which the demand functions are

$$q_A = 16 - p_A + p_B \quad \text{and} \quad q_B = 24 + 2p_A - 4p_B$$

If the constant average cost of producing a unit of A is 2 and a unit of B is 4, how many units of A and B should be sold to maximize the monopolist's profit?

27. Profit For products A and B, the joint-cost function for a manufacturer is

$$c = \frac{3}{2}q_A^2 + 3q_B^2$$

and the demand functions are $p_A = 60 - q_A^2$ and $p_B = 72 - 2q_B^2$. Find the level of production that maximizes profit.

28. Profit For a monopolist's products A and B, the joint-cost function is $c = 2(q_A + q_B + q_A q_B)$, and the demand functions are $q_A = 20 - 2p_A$ and $q_B = 10 - p_B$. Find the values of p_A and p_B that maximize profit. What are the quantities of A and B that correspond to these prices? What is the total profit?

29. Cost An open-top rectangular box is to have a volume of 6 ft^3. The cost per square foot of materials is \$3 for the bottom, \$1 for the front and back, and \$0.50 for the other two sides. Find the dimensions of the box so that the cost of materials is minimized. (See Figure 17.7.)

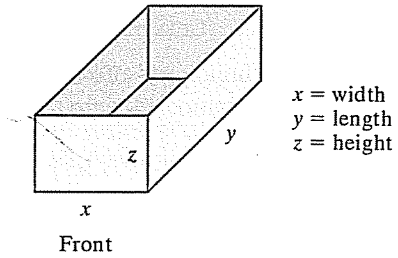

FIGURE 17.7

30. Collusion Suppose A and B are the only two firms in the market selling the same product. (We say that they are *duopolists*.) The industry demand function for the product is

$$p = 92 - q_A - q_B$$

where q_A and q_B denote the output produced and sold by A and B, respectively. For A, the cost function is $c_A = 10q_A$; for B, it is $c_B = 0.5q_B^2$. Suppose the firms decide to enter into an agreement on output and price control by jointly acting as a monopoly. In this case, we say they enter into *collusion*. Show that the profit function for the monopoly is given by

$$P = pq_A - c_A + pq_B - c_B$$

Express P as a function of q_A and q_B, and determine how output should be allocated so as to maximize the profit of the monopoly.

31. Suppose $f(x, y) = x^2 + 3y^2 + 9$, where x and y must satisfy the equation $x + y = 2$. Find the relative extrema of f, subject to the given condition on x and y, by first solving the second equation for y (or x). Substitute the result in the first equation. Thus, f is expressed as a function of one variable. Now find where relative extrema for f occur.

32. Repeat Problem 31 if $f(x, y) = x^2 + 4y^2 + 6$, subject to the condition that $2x - 8y = 20$.

33. Suppose the joint-cost function

$$c = q_A^2 + 3q_B^2 + 2q_A q_B + aq_A + bq_B + d$$

has a relative minimum value of 15 when $q_A = 3$ and $q_B = 1$. Determine the values of the constants a, b, and d.

34. Suppose that the function $f(x, y)$ has continuous partial derivatives f_{xx}, f_{yy}, and f_{xy} at all points (x, y) near a critical point (a, b). Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$ and suppose that $D(a, b) > 0$.

(a) Show that $f_{xx}(a, b) < 0$ if and only if $f_{yy}(a, b) < 0$.
(b) Show that $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

35. Profit from Competitive Products A monopolist sells two competitive products, A and B, for which the demand equations are

$$p_A = 35 - 2q_A^2 + q_B$$