Another method is to notice that the values being added are $2k + 12$, for $k = 1$ to 44. The sum can thus also be written as

$$\sum_{k=1}^{44} (2k + 12)$$

Now Work Problem 9 ◁

Since summation notation is used to express the addition of terms, we can use the properties of addition when performing operations on sums written in summation notation. By applying these properties, we can create a list of properties, and formulas for summation notation.

By the distributive property of addition,

$$ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n)$$

So, in summation notation,

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i \tag{6}$$

Note that c must be constant with respect to i for Equation (6) to be used.

By the commutative property of addition,

$$a_1 + b_1 + a_2 + b_2 + \cdots + a_n + b_n = a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n$$

So we have

$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \tag{7}$$

Sometimes we want to change the bounds of summation.

$$\sum_{i=m}^{n} a_i = \sum_{i=p}^{p+n-m} a_{i+m-p} \tag{8}$$

A sum of 37 terms can be regarded as the sum of the first 17 terms plus the sum of the next 20 terms. The next rule generalzes this observation.

$$\sum_{i=m}^{p-1} a_i + \sum_{i=p}^{n} a_i = \sum_{i=m}^{n} a_i \tag{9}$$

In addition to these four basic rules, there are some other rules worth noting.

$$\sum_{i=1}^{n} 1 = n \tag{10}$$

This is because $\sum_{i=1}^{n} 1$ is a sum of n terms, each of which is 1. The next follows from combining Equation (6) and Equation (10).

$$\sum_{i=1}^{n} c = cn \tag{11}$$

Similarly, from Equations (6) and (7) we have

$$\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i \tag{12}$$

Establishing the next three formulas is best done by a proof method called mathematical induction, which we will not demonstrate here.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{13}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{14}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{15}$$

However, we can deduce Equation (13). If we add the following equations, "vertically," term by term,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$$\sum_{i=1}^{n} i = n + (n-1) + (n-2) + \cdots + 1$$

we get

$$2\sum_{i=1}^{n} i = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$

and since there are n terms on the right, we conclude

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Observe that if a teacher assigns the task of finding

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots + 104 + 105$$

as a *punishment* and if he or she knows the formula given by Equation (13), then a student's work can be checked quickly by

$$\sum_{i=1}^{105} i = \frac{105(106)}{2} = 105 \cdot 53 = 5300 + 265 = 5565$$

EXAMPLE 3 Applying the Properties of Summation Notation

Evaluate the given sums.

a. $\sum_{j=30}^{100} 4$ **b.** $\sum_{k=1}^{100} (5k+3)$ **c.** $\sum_{k=1}^{200} 9k^2$

Solutions:

a.

$$\begin{aligned} \sum_{j=30}^{100} 4 &= \sum_{j=1}^{71} 4 && \text{by Equation (8)} \\ &= 4 \cdot 71 && \text{by Equation (11)} \\ &= 284 \end{aligned}$$

b.

$$\begin{aligned} \sum_{k=1}^{100} (5k+3) &= \sum_{k=1}^{100} 5k + \sum_{k=1}^{100} 3 && \text{by Equation (7)} \\ &= 5\left(\sum_{k=1}^{100} k\right) + 3\left(\sum_{k=1}^{100} 1\right) && \text{by Equation (6)} \end{aligned}$$

$$= 5\left(\frac{100 \cdot 101}{2}\right) + 3(100) \quad \text{by Equations (13) and (10)}$$
$$= 25{,}250 + 300$$
$$= 25{,}550$$

c.
$$\sum_{k=1}^{200} 9k^2 = 9\sum_{k=1}^{200} k^2 \quad \text{by Equation (6)}$$
$$= 9\left(\frac{200 \cdot 201 \cdot 401}{6}\right) \quad \text{by Equation (14)}$$
$$= 24{,}180{,}300$$

Now Work Problem 19 ◁

PROBLEMS 1.5

In Problems 1 and 2, give the bounds of summation and the index of summation for each expression.

1. $\sum_{t=12}^{17}(8t^2 - 5t + 3)$ **2.** $\sum_{m=3}^{450}(8m - 4)$

In Problems 3–6, evaluate the given sums.

3. $\sum_{i=1}^{5} 3i$ **4.** $\sum_{p=0}^{4} 10p$

5. $\sum_{k=3}^{9}(10k + 16)$ **6.** $\sum_{n=7}^{11}(2n - 3)$

In Problems 7–12, express the given sums in summation notation.

7. $36 + 37 + 38 + 39 + \cdots + 60$

8. $1 + 8 + 27 + 64 + 125$

9. $5^3 + 5^4 + 5^5 + 5^6 + 5^7 + 5^8$

10. $11 + 15 + 19 + 23 + \cdots + 71$

11. $2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$

12. $10 + 100 + 1000 + \cdots + 100{,}000{,}000$

In Problems 13–26, evaluate the given sums.

13. $\sum_{k=1}^{875} 10$ **14.** $\sum_{k=35}^{135} 2$

15. $\sum_{k=1}^{n}\left(5 \cdot \frac{1}{n}\right)$ **16.** $\sum_{k=1}^{200}(k - 100)$

17. $\sum_{k=51}^{100} 10k$ **18.** $\sum_{k=1}^{n}\frac{n}{n+1}k^3$

19. $\sum_{k=1}^{20}(5k^2 + 3k)$ **20.** $\sum_{k=1}^{100}\frac{3k^2 - 200k}{101}$

21. $\sum_{k=51}^{100} k^2$ **22.** $\sum_{k=1}^{50}(k + 50)^2$

23. $\sum_{k=1}^{9}\left\{\left[3 - \left(\frac{k}{10}\right)^2\right]\left(\frac{1}{10}\right)\right\}$

24. $\sum_{k=1}^{100}\left\{\left[4 - \left(\frac{2}{100}k\right)^2\right]\left(\frac{2}{100}\right)\right\}$

25. $\sum_{k=1}^{n}\left\{\left[5 - \left(\frac{3}{n} \cdot k\right)^2\right]\frac{3}{n}\right\}$

26. $\sum_{k=1}^{n}\frac{k^2}{(n+1)(2n+1)}$

Objective

To introduce sequences, particularly arithmetic and geometric sequences, and their sums.

1.6 Sequences

Introduction

Consider the following list of five numbers:

$$2,\ 2 + \sqrt{3},\ 2 + 2\sqrt{3},\ 2 + 3\sqrt{3},\ 2 + 4\sqrt{3} \quad (1)$$

If it is understood that the ordering of the numbers is to be taken into account, then such a list is called a **sequence of length** 5 and it is considered to be different from

$$2,\ 2 + 3\sqrt{3},\ 2 + \sqrt{3},\ 2 + 4\sqrt{3},\ 2 + 2\sqrt{3} \quad (2)$$

which is also a sequence of length 5. Both of these sequences are different again from

$$2, \ 2, \ 2+\sqrt{3}, \ 2+2\sqrt{3}, \ 2+3\sqrt{3}, \ 2+4\sqrt{3} \tag{3}$$

which is a sequence of length 6. However, each of the sequences (1), (2), and (3) takes on all the numbers in the 5-element *set*

$$\{2, \ 2+\sqrt{3}, \ 2+2\sqrt{3}, \ 2+3\sqrt{3}, \ 2+4\sqrt{3}\}$$

CAUTION!

Both rearrangements and repetitions *do* affect a sequence.

In Section 0.1 we emphasized that "*a set is determined by its elements*, and neither repetitions nor rearrangements in a listing affect the set". Since both repetitions and rearrangements do affect a sequence, it follows that sequences are not the same as sets.

We will also consider listings such as

$$2, \ 2+\sqrt{3}, \ 2+2\sqrt{3}, \ 2+3\sqrt{3}, \ 2+4\sqrt{3}, \ \cdots, 2+k\sqrt{3}, \ \cdots \tag{4}$$

and

$$1, \ -1, \ 1, \ -1, \ 1, \cdots, \ (-1)^{k+1}, \ \cdots \tag{5}$$

Both are examples of what is called an **infinite sequence**. However, note that the infinite sequence (4) involves the infinitely many different numbers in the set

$$\{2 + k\sqrt{3} | k \text{ a nonnegative integer}\}$$

while the infinite sequence (5) involves only the numbers in the finite set

$$\{-1, 1\}$$

For n a positive integer, taking the first n numbers of an infinite sequence results in a sequence of length n. For example, taking the first five numbers of the infinite sequence (4) gives the sequence (1). The following more formal definitions are helpful in understanding the somewhat subtle idea of a sequence.

Definition

For n a positive integer, a ***sequence of length*** n is a rule which assigns to each element of the set $\{1, 2, 3, \cdots, n\}$ exactly one real number. The set $\{1, 2, 3, \cdots, n\}$ is called the ***domain*** of the sequence of length n. A ***finite sequence*** is a sequence of length n for some positive integer n.

Definition

An ***infinite sequence*** is a rule which assigns to each element of the set of all positive integers $\{1, 2, 3, \cdots\}$ exactly one real number. The set $\{1, 2, 3, \cdots\}$ is called the ***domain*** of the infinite sequence.

The word *rule* in both definitions may appear vague but the point is that for any sequence there must be a definite way of specifying exactly one number for each of the elements in its domain. For a finite sequence the rule can be given by simply listing the numbers in the sequence. There is no requirement that there be a discernible pattern (although in practice there often is). For example,

$$99, \ -\pi, \ \frac{3}{5}, \ 102.7$$

is a perfectly good sequence of length 4. For an infinite sequence there should be some sort of procedure for generating its numbers, one after the other. However, the procedure may fail to be given by a simple formula. The infinite sequence

$$2, \ 3, \ 5, \ 7, \ 11, \ 13, \ 17, \ 19, \ 23, \ \cdots$$

is very important in number theory but its rule is not given by a mere formula. (What is *apparently* the rule which gives rise to this sequence? In that event, what is the next number in this sequence after those displayed?)

We often use letters like a, b, c, and so on, for the names of sequences. If the sequence is called a, we write a_1 for the number assigned to 1, a_2 for the number assigned to 2, a_3 for the number assigned to 3, and so on. In general, for k in the domain of the sequence, we write a_k for the number assigned to k and call it the kth ***term*** of the sequence. (If you have studied Section 1.5 on summation, you will already be familiar with this notation.) In fact, rather than listing all the numbers of a sequence by

$$a_1, a_2, a_3, \ldots, a_n$$

or an indication of all the numbers such as

$$a_1, a_2, a_3, \ldots, a_k, \ldots$$

a sequence is often denoted by (a_k). Sometimes $(a_k)_{k=1}^{n}$ is used to indicate that the sequence is finite, of length n, or $(a_k)_{k=1}^{\infty}$ is used to emphasize that the sequence is infinite. The ***range*** of a sequence (a_k) is the *set*

$$\{a_k | k \text{ is in the domain of } a\}$$

Notice that

$$\{(-1)^{k+1} | k \text{ is a positive integer}\} = \{-1, 1\}$$

so an infinite sequence may have a finite range. If a and b are sequences, then, by definition, $a = b$ if and only if a and b have the same domain and, for all k in the common domain, $a_k = b_k$.

APPLY IT ▷

4. A chain of hip coffee shops has 183 outlets in 2009. Starting in 2010 it plans to expand its number of outlets by 18 each year for five years. Writing c_k for the number of coffee shops in year k, measured from 2008, list the terms in the sequence $(c_k)_{k=1}^{6}$.

EXAMPLE 1 Listing the Terms in a Sequence

a. List the first four terms of the infinite sequence $(a_k)_{k=1}^{\infty}$ whose kth term is given by $a_k = 2k^2 + 3k + 1$.

Solution: We have $a_1 = 2(1^2) + 3(1) + 1 = 6$, $a_2 = 2(2^2) + 3(2) + 1 = 15$, $a_3 = 2(3^2) + 3(3) + 1 = 28$, and $a_4 = 2(4^2) + 3(4) + 1 = 45$. So the first four terms are

$$6, \quad 15, \quad 28, \quad 45$$

b. List the first four terms of the infinite sequence (e_k), where $e_k = \left(\dfrac{k+1}{k}\right)^k$.

Solution: We have $e_1 = \left(\dfrac{1+1}{1}\right)^1 = \left(\dfrac{2}{1}\right)^1 = 2$, $e_2 = \left(\dfrac{2+1}{2}\right)^2 = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$, $e_3 = \left(\dfrac{3+1}{3}\right)^3 = \left(\dfrac{4}{3}\right)^3 = \dfrac{64}{27}$, $e_4 = \left(\dfrac{4+1}{4}\right)^4 = \left(\dfrac{5}{4}\right)^4 = \dfrac{625}{256}$.

c. Display the sequence $\left(\dfrac{3}{2^{k-1}}\right)_{k=1}^{6}$.

Solution: Noting that $2^0 = 1$, we have

$$3, \ \frac{3}{2}, \ \frac{3}{4}, \ \frac{3}{8}, \ \frac{3}{16}, \ \frac{3}{32}$$

Now Work Problem 3 ◁

APPLY IT ▷

5. A certain inactive bank account which bears interest at the rate of 6% compounded yearly shows year-end balances, for four consecutive years, of \$9.57, \$10.14, \$10.75, \$11.40. Write the sequence of amounts in the form $(a_k)_{k=1}^{4}$.

EXAMPLE 2 Giving a Formula for a Sequence

a. Write 41, 44, 47, 50, 53 in the form $(a_k)_{k=1}^{5}$.

Solution: Each term of the sequence is obtained by adding three to the previous term. Since the first term is 41, we can write the sequence as $(41 + (k-1)3)_{k=1}^{5}$. Observe that this formula is not unique. The sequence is also described by $(38 + 3k)_{k=1}^{5}$ and by $(32 + (k+2)3)_{k=1}^{5}$, to give just two more possibilities.

b. Write the sequence 1, 4, 9, 16, ... in the form (a_k).

Solution: The sequence is *apparently* the sequence of squares of positive integers so that (k^2) or $(k^2)_{k=1}^{\infty}$ would be regarded as the correct answer by most people. But the sequence described by $(k^4 - 10k^3 + 36k^2 - 50k + 24)$ also has its first four terms given by 1, 4, 9, 16 and yet its fifth term is 49. The sixth and seventh terms are 156 and 409, respectively. The point we are making is that an infinite sequence cannot be determined by finitely many values alone.

On the other hand, it is correct to write

$$1, 4, 9, 16, \ldots, k^2, \ldots = (k^2)$$

because the display on the left side of the equation makes it clear that the ***general term*** is k^2.

Now Work Problem 9 ◁

EXAMPLE 3 Demonstrating Equality of Sequences

Show that the sequences $((i+3)^2)_{i=1}^{\infty}$ and $(j^2+6j+9)_{j=1}^{\infty}$ are equal.

Solution: Both $((i+3)^2)_{i=1}^{\infty}$ and $(j^2+6j+9)_{j=1}^{\infty}$ are explicitly given to have the same domain, namely $\{1, 2, 3, \ldots\}$, the infinite set of all positive integers The names i and j being used to name a typical element of the domain are unimportant. The first sequence is the same as $((k+3)^2)_{k=1}^{\infty}$ and the second sequence is the same as $(k^2+6k+9)_{k=1}^{\infty}$. The first rule assigns, to any positive integer k, the number $(k+3)^2$ and the second assigns, to any positive integer k, the number k^2+6k+9. However, for all k, $(k+3)^2 = k^2+6k+9$, so by the definition of equality of sequences the sequences are equal.

Now Work Problem 13 ◁

Recursively Defined Sequences

Suppose that a is a sequence with

$$a_1 = 1 \text{ and, for each positive integer } k,\ a_{k+1} = (k+1)a_k \qquad (6)$$

Taking $k = 1$, we see that $a_2 = (2)a_1 = (2)1 = 2$, while with $k = 2$ we have $a_3 = (3)a_2 = (3)2 = 6$. A sequence whose rule is defined in terms of itself evaluated at smaller values, and some explicitly given small values, is said to be ***recursively defined***. Thus we can say that there is a sequence a recursively defined by (6) above.

Another famous example of a recursively defined sequence is the Fibonacci sequence:

$$F_1 = 1 \text{ and } F_2 = 1 \text{ and, for each positive integer } k,\ F_{k+2} = F_{k+1} + F_k \qquad (7)$$

Taking $k = 1$, we see that $F_3 = F_2 + F_1 = 1 + 1 = 2$, $F_4 = F_3 + F_2 = 2 + 1 = 3$, $F_5 = F_4 + F_3 = 3 + 2 = 5$. In fact, the first ten terms of (F_k) are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

EXAMPLE 4 Applying a Recursive Definition

a. Use the recursive definition (6) to determine a_5 (without referring to the earlier calculations).

Solution: We have

$$\begin{aligned} a_5 &= (5)a_4 \\ &= (5)(4)a_3 \\ &= (5)(4)(3)a_2 \\ &= (5)(4)(3)(2)a_1 \\ &= (5)(4)(3)(2)(1) \\ &= 120 \end{aligned}$$

The standard notation for a_k as defined by (6) is $k!$ and it is read "k factorial". We also define $0! = 1$.

b. Use the recursive definition (7) to determine F_6.

Solution:

$$\begin{aligned} F_6 &= F_5 + F_4 \\ &= (F_4 + F_3) + (F_3 + F_2) \\ &= F_4 + 2F_3 + F_2 \\ &= (F_3 + F_2) + 2(F_2 + F_1) + F_2 \\ &= F_3 + 4F_2 + 2F_1 \\ &= (F_2 + F_1) + 4F_2 + 2F_1 \\ &= 5F_2 + 3F_1 \\ &= 5(1) + 3(1) \\ &= 8 \end{aligned}$$

Now Work Problem 17 ◁

In Example 4 we deliberately avoided making any numerical evaluations until *all* terms had been expressed using only those terms whose values were given explicitly in the recursive definition. This helps to illustrate the structure of the recursive definition in each case.

While recursive definitions are very useful in applications, the computations above underscore that, for large values of k, the computation of the kth term may be time-consuming. It is desirable to have a simple formula for a_k that does not refer to a_l, for $l < k$. Sometimes it is possible to find such a *closed* formula. In the case of (6) it is easy to see that $a_k = k \cdot (k-1) \cdot (k-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. On the other hand, in the case of (7), it is not so easy to derive

$$F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

Arithmetic Sequences and Geometric Sequences

Definition

An ***arithmetic sequence*** is a sequence (b_k) defined recursively by

$$b_1 = a \text{ and, for each positive integer } k,\ b_{k+1} = d + b_k \tag{8}$$

for fixed real numbers a and d.

In words, the definition tells us to start the sequence at a and get the *next* term by adding d (no matter which term is currently under consideration). The number a is simply the first term of the arithmetic sequence. Since the recursive definition gives $b_{k+1} - b_k = d$, for every positive integer k, we see that the number d is the difference between any pair of successive terms. It is, accordingly, called the ***common difference*** of the arithmetic sequence. Any pair of real numbers a and d determines an infinite arithmetic sequence. By restricting to a finite number of terms, we can speak of finite arithmetic sequences.

APPLY IT ▷

6. In 2009 the enrollment at Springfield High was 1237, and demographic studies suggest that it will decline by 12 students a year for the next seven years. List the projected enrollments of Springfield High.

EXAMPLE 5 **Listing an Arithmetic Sequence**

Write explicitly the terms of an arithmetic sequence of length 6 with first term $a = 1.5$ and common difference $d = 0.7$.

Solution: Let us write (b_k) for the arithmetic sequence. Then

$$\begin{aligned} b_1 &= 1.5 \\ b_2 &= 0.7 + b_1 = 0.7 + 1.5 = 2.2 \\ b_3 &= 0.7 + b_2 = 0.7 + 2.2 = 2.9 \\ b_4 &= 0.7 + b_3 = 0.7 + 2.9 = 3.6 \\ b_5 &= 0.7 + b_4 = 0.7 + 3.6 = 4.3 \\ b_6 &= 0.7 + b_5 = 0.7 + 4.3 = 5.0 \end{aligned}$$

Thus the required sequence is

$$1.5, \ 2.2, \ 2.9, \ 3.6, \ 4.3, \ 5.0$$

Now Work Problem 21 ◁

Definition

A ***geometric sequence*** is a sequence (c_k) defined recursively by

$$c_1 = a \text{ and, for each positive integer } k, \ c_{k+1} = c_k \cdot r \qquad (9)$$

for fixed real numbers a and r.

In words, the definition tells us to start the sequence at a and get the *next* term by multiplying by r (no matter which term is currently under consideration). The number a is simply the first term of the geometric sequence. Since the recursive definition gives $c_{k+1}/c_k = r$, for every positive integer k with $c_k \neq 0$, we see that the number r is the ratio between any pair of successive terms, with the first of these not 0. It is, accordingly, called the ***common ratio*** of the geometric sequence. Any pair of real numbers a and r determines an infinite geometric sequence. By restricting to a finite number of terms, we can speak of finite geometric sequences.

APPLY IT ▷

7. The population of the rural area surrounding Springfield is declining as a result of movement to the urban core. In 2009 it was 23,500 and each year, for the next four years, it is expected to be only 92% of the previous year's population. List the anticipated annual population numbers for the rural area.

EXAMPLE 6 Listing a Geometric Sequence

Write explicitly the terms of a geometric sequence of length 5 with first term $a = \sqrt{2}$ and common ratio $r = 1/2$.

Solution: Let us write (c_k) for the geometric sequence. Then

$$\begin{aligned}
c_1 &&&= \sqrt{2}\\
c_2 &= (c_1) \cdot 1/2 = &(\sqrt{2})1/2 &= \sqrt{2}/2\\
c_3 &= (c_2) \cdot 1/2 = &(\sqrt{2}/2)1/2 &= \sqrt{2}/4\\
c_4 &= (c_3) \cdot 1/2 = &(\sqrt{2}/4)1/2 &= \sqrt{2}/8\\
c_5 &= (c_4) \cdot 1/2 = &(\sqrt{2}/8)1/2 &= \sqrt{2}/16
\end{aligned}$$

Thus the required sequence is

$$\sqrt{2}, \ \sqrt{2}/2, \ \sqrt{2}/4, \ \sqrt{2}/8, \ \sqrt{2}/16$$

Now Work Problem 25 ◁

We have remarked that sometimes it is possible to determine an explicit formula for the kth term of a recursively-defined sequence. This is certainly the case for arithmetic and geometric sequences.

EXAMPLE 7 Finding the kth term of an Arithmetic Sequence

Find an explicit formula for the kth term of an arithmetic sequence (b_k) with first term a and common difference d.

Solution: We have

$$\begin{aligned}
b_1 &= &a &= 0d + a\\
b_2 &= d + (b_1) = &d + a &= 1d + a\\
b_3 &= d + (b_2) = &d + (1d + a) &= 2d + a\\
b_4 &= d + (b_3) = &d + (2d + a) &= 3d + a\\
b_5 &= d + (b_4) = &d + (3d + a) &= 4d + a
\end{aligned}$$

It *appears* that, for each positive integer k, the kth term of an arithmetic sequence (b_k) with first term a and common difference d is given by

$$b_k = (k - 1)d + a \qquad (10)$$

This is true and follows easily via the proof method called mathematical induction, which we will not demonstrate here.

Now Work Problem 29 ◁

EXAMPLE 8 **Finding the kth term of a Geometric Sequence**

Find an explicit formula for the kth term of a geometric sequence (c_k) with first term a and common ratio r.

Solution: We have

$$\begin{aligned} c_1 &= & a &= ar^0 \\ c_2 &= (c_1)\cdot r = & ar &= ar^1 \\ c_3 &= (c_2)\cdot r = & ar^1 r &= ar^2 \\ c_4 &= (c_3)\cdot r = & ar^2 r &= ar^3 \\ c_5 &= (c_4)\cdot r = & ar^3 r &= ar^4 \end{aligned}$$

It *appears* that, for each positive integer k, the kth term of a geometric sequence (c_k) with first term a and common difference r is given by

$$c_k = ar^{k-1} \tag{11}$$

This is true and also follows easily via mathematical induction.

Now Work Problem 31 ◁

It is clear that any arithmetic sequence has a unique first term a and a unique common difference d. For a geometric sequence we have to be a little more careful. From (11) we see that if any term c_k is 0, then either $a = 0$ or $r = 0$. If $a = 0$, then every term in the geometric sequence is 0. In this event, there is not a uniquely determined r because $r \cdot 0 = 0$, for *any* r. If $a \neq 0$ but $r = 0$, then every term except the first is 0.

Sums of Sequences

For any sequence (c_k) we can speak of the sum of the first k terms. Let us call this sum s_k. Using the summation notation introduced in Section 1.5, we can write

$$s_k = \sum_{i=1}^{k} c_i = c_1 + c_2 + \cdots + c_k \tag{12}$$

We can regard the s_k as terms of a new sequence (s_k), of sums, associated to the original sequence (s_k). If a sequence (c_k) is finite, of length n then s_n can be regarded as ***the sum of the sequence***.

APPLY IT ▷

8. If a company has an annual revenue of 27M$ in 2009 and revenue grows by 1.5M$ each year, find the total revenue through 2009–2015 inclusive.

EXAMPLE 9 **Finding the Sum of an Arithmetic Sequence**

Find a formula for the sum s_n of the first n terms of an arithmetic sequence (b_k) with first term a and common difference d.

Solution: Since the arithmetic sequence (b_k) in question has, by Example 7, $b_k = (k-1)d + a$, the required sum is given by

$$s_n = \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}((k-1)d + a) = \sum_{k=1}^{n}(dk - (d-a)) = \sum_{k=1}^{n} dk - \sum_{k=1}^{n}(d-a)$$

$$= d\sum_{k=1}^{n} k - (d-a)\sum_{k=1}^{n} 1 \stackrel{\star}{=} d\frac{n(n+1)}{2} - (d-a)n = \frac{n}{2}((n-1)d + 2a)$$

Notice that the equality labeled $\star$ uses both (13) and (10) of Section 1.5. We remark that the last term under consideration in the sum is $b_n = (n-1)d + a$ so that in our formula for s_n the factor $((n-1)d + 2a)$ is the first term a plus the last term $(n-1)d + a$. If we write $z = (n-1)d + a$ for the last term, then we can summarize with

$$s_n = \frac{n}{2}((n-1)d + 2a) = \frac{n}{2}(a+z) \tag{13}$$

Note that we could also have found (13) by the same technique used to find (13) of Section 1.5. We preferred to calculate using summation notation here. Finally, we should remark that the sum (13) in Section 1.5 is the sum of the first n terms of the special arithmetic sequence with $a = 1$ and $d = 1$.

Now Work Problem 33 ◁

APPLY IT ▷

9. Mrs. Simpson put \$1000 in a special account for Bart on each of his first 21 birthdays. The account earned interest at the rate of 7% compounded annually. It follows (see Chapter 5) that the amount deposited on Bart's $(22-k)$th birthday is worth $\$1000(1.07)^{k-1}$ on Bart's 21st birthday. Find the total amount in the special account on Bart's 21st birthday.

EXAMPLE 10 Finding the Sum of a Geometric Sequence

Find a formula for the sum s_n of the first n terms of a geometric sequence (c_k) with first term a and common ratio r.

Solution: Since the geometric sequence (c_k) in question has, by Example 8, $c_k = ar^{k-1}$, the required sum is given by

$$s_n = \sum_{k=1}^{n} c_k = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1} \tag{14}$$

It follows that if we multiply (14) by r we have

$$rs_n = r\sum_{k=1}^{n} c_k = r\sum_{k=1}^{n} ar^{k-1} = \sum_{k=1}^{n} ar^k = ar + ar^2 + \cdots + ar^{n-1} + ar^n \tag{15}$$

If we subtract (15) from (14) we get

$$s_n - rs_n = a - ar^n \text{ so that } (1-r)s_n = a(1-r^n)$$

Thus we have

$$s_n = \frac{a(1-r^n)}{1-r} \quad \text{for } r \neq 1 \tag{16}$$

(Note that if $r = 1$, then each term in the sum is a and, since there are n terms, the answer in this easy case is $s_n = na$)

Now Work Problem 37 ◁

For *some* infinite sequences $(c_k)_{k=1}^{\infty}$ the sequence of sums $(s_k)_{k=1}^{\infty} = \left(\sum_{i=1}^{k} c_i\right)_{k=1}^{\infty}$ appears to approach a definite number. When this is indeed the case we write the number as $\sum_{i=1}^{\infty} c_i$. Here we consider only the case of a geometric sequence. As we see from (16), if $c_k = ar^{k-1}$ then, for $r \neq 1$, $s_k = \dfrac{a(1-r^k)}{1-r}$. Observe that only the factor $1 - r^k$ depends on k. If $|r| > 1$, then for large values of k, $|r^k|$ will become large, as will $|1 - r^k|$. In fact, for $|r| > 1$ we can make the values $|1 - r^k|$ as large as we like by taking k to be sufficiently large. It follows that, for $|r| > 1$, the sums $\dfrac{a(1-r^k)}{1-r}$ *do not* approach a definite number. If $r = 1$, then $s_k = ka$ and, again, the sums do not approach a definite *number*.

However, for $|r| < 1$ (that is for $-1 < r < 1$), we can make the values r^k as close to 0 as we like, by taking k to be sufficiently large. (Be sure to convince yourself that this is true before reading further because the rest of the argument hinges on this point.) Thus, for $|r| < 1$, we can make the values $1 - r^k$ as close to 1 as we like by taking k to be sufficiently large. Finally, for $|r| < 1$, we can make the values $\dfrac{a(1-r^k)}{1-r}$ as close to $\dfrac{a}{1-r}$ as we like by taking k to be sufficiently large. In precisely this sense, an infinite geometric sequence with $|r| < 1$ has a sum and we have

$$\text{for } |r| < 1, \quad \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r} \tag{17}$$

EXAMPLE 11 **Finding the Sum of an Infinite Geometric Sequence**

A rich woman would like to leave \$100,000 a year, starting now, to be divided equally among all her direct descendants. She puts no time limit on this bequeathment and is able to invest for this long-term outlay of funds at 2% compounded annually. How much must she invest now to meet such a long-term commitment?

Solution: Let us write $R = 100,000$, set the clock to 0 now, and measure time in years from now. With these conventions we are to account for payments of R at times $0, 1, 2, 3, \dots, k, \dots$ by making a single investment now. (Such a sequence of payments is called a *perpetuity*.) The payment now simply costs her R. The payment at time 1 has a *present value* of $R(1.02)^{-1}$. (See Chapter 5.) The payment at time 2 has a present value of $R(1.02)^{-2}$. The payment at time 3 has a present value of $R(1.02)^{-3}$, and, quite generally, the payment at time k has a present value of $R(1.02)^{-k}$. Her investment *now* must exactly cover the present value of *all* these future payments. In other words, the investment must equal the sum

$$R + R(1.02)^{-1} + R(1.02)^{-2} + R(1.02)^{-3} + \cdots + R(1.02)^{-k} + \cdots$$

We recognize the infinite sum as that of a geometric series, with first term $a = R = 100,000$ and common ratio $r = (1.02)^{-1}$. Since $|r| = (1.02)^{-1} < 1$, we can evaluate the required investment as

$$\frac{a}{1-r} = \frac{100,000}{1 - \dfrac{1}{1.02}} = \frac{100,000}{\dfrac{0.02}{1.02}} = \frac{100,000(1.02)}{0.02} = 5,100,000$$

In other words, an investment of a mere \$5,100,000 now will allow her to leave \$100,000 per year to her descendants *forever*!

Now Work Problem 57 ◁

PROBLEMS 1.6

In Problems 1–8, write the indicated term of the given sequence.

1. $a = \sqrt{2},\ -\frac{3}{7},\ 2.3,\ 57;\ a_3$

2. $b = 1,\ 13,\ -0.9,\ \frac{5}{2},\ 100,\ 39;\ b_6$

3. $(a_k)_{k=1}^{7} = (3^k);\ a_4$

4. $(c_k)_{k=1}^{9} = (3^k + k);\ c_4$

5. $(a_k) = (2 + (k-1)3);\ a_{24}$

6. $(b_k) = (5 \cdot 2^{k-1});\ b_6$

7. $(a_k) = (k^4 - 2k^2 + 1);\ a_2$

8. $(a_k) = (k^3 + k^2 - 2k + 7);\ a_3$

In Problems 9–12, find a general term, (a_k), description that fits the displayed terms of the given sequence.

9. $-1,\ 2,\ 5,\ 8$

10. $5,\ 3,\ 1,\ -1,\ \dots$

11. $2,\ -4,\ 8,\ -16$

12. $5,\ \frac{5}{3},\ \frac{5}{9},\ \frac{5}{27},\ \dots$

In Problems 13–16, determine whether the given sequences are equal to each other.

13. $((i+3)^3)$ and $(j^3 - 9j^2 + 9j - 27)$

14. $(k^2 - 4)$ and $((k+2)(k-2))$

15. $\left(\pi\frac{1}{2^{k-1}}\right)_{k=1}^{\infty}$ and $\left(\frac{\pi}{2^k}\right)_{k=1}^{\infty}$

16. $(j^3 - 9j^2 + 27j - 27)_{j=1}^{\infty}$ and $((k-3)^3)_{k=1}^{\infty}$

In Problems 17–20, determine the indicated term of the given recursively defined sequence.

17. $a_1 = 1,\ a_2 = 2,\ a_{k+2} = a_{k+1} \cdot a_k;\ a_7$

18. $a_1 = 1,\ a_{k+1} = a_{a_k};\ a_{17}$

19. $b_1 = 1,\ b_{k+1} = \frac{b_k}{k};\ b_6$

20. $a_1 = 1,\ a_{k+1} = (k+1) + a_k;\ a_8$

In Problems 21–24, write the first five terms of the arithmetic sequence with the given first term a and common difference d.

21. $a = 22.5, d = 0.9$

22. $a = 0, d = 1$

23. $a = 96, d = -1.5$

24. $a = A, d = D$

In Problems 25–28, write the first five terms of the geometric sequence with the given first term a and common ratio r.

25. $a = -2, r = -0.5$

26. $a = 50, r = (1.06)^{-1}$

27. $a = 100, r = 1.05$

28. $a = 3, r = \frac{1}{3}$

In Problems 29–32, write the indicated term of the arithmetic sequence with given parameters a and d or of the geometric sequence with given parameters a and r.

29. 27th term, $a = 3, d = 2$

30. 9th term, $a = 2.7, d = -0.3$

31. 11th term, $a = 1, r = 2$

32. 7th term, $a = 2, r = 10$

In Problems 33–40, find the required sums.

33. $\sum_{k=1}^{7}((k-1)3 + 5)$

34. $\sum_{k=1}^{9}(k \cdot 2 + 9)$

35. $\sum_{k=1}^{6}((k-1)0.5+2.3)$ **36.** $\sum_{k=1}^{34}((k-1)10+5)$

37. $\sum_{k=1}^{10}100(1/2)^{k-1}$ **38.** $\sum_{k=1}^{10}50(1.07)^{k-1}$

39. $\sum_{k=1}^{10}50(1.07)^{1-k}$ **40.** $\sum_{k=1}^{7}5\cdot 2^{k}$

In Problems 41–46, find the infinite sums, if possible, or state why this cannot be done.

41. $\sum_{k=1}^{\infty}3\left(\frac{1}{2}\right)^{k-1}$ **42.** $\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^{i}$

43. $\sum_{k=1}^{\infty}\frac{1}{2}(17)^{k-1}$ **44.** $\sum_{k=1}^{\infty}\frac{2}{3}(1.5)^{k-1}$

45. $\sum_{k=1}^{\infty}50(1.05)^{1-k}$ **46.** $\sum_{j=1}^{\infty}75(1.09)^{1-j}$

47. Inventory Every thirty days a grocery store stocks 90 cans of elephant noodle soup and, rather surprisingly, sells 3 cans each day. Describe the inventory levels of elephant noodle soup at the end of each day, as a sequence, and determine the inventory level 19 days after restocking.

48. Inventory If a corner store has 95 previously viewed DVD movies for sale today and manages to sell 6 each day, write the first seven terms of the store's daily inventory sequence for the DVDs. How many DVDs will the store have on hand after 10 days?

49. Checking Account A checking account, which earns no interest, contains \$125.00 and is forgotten. It is nevertheless subject to a \$5.00 per month service charge. The account is remembered after 9 months. How much does it then contain?

50. Savings Account A savings account, which earns interest at a rate of 5% compounded annually, contains \$125.00 and is forgotten. It is remembered 9 years later. How much does it then contain?

51. Population Change A town with a population of 50,000 in 2009 is growing at the rate of 8% per year. In other words, at the end of each year the population is 1.08 times the population at the end of the preceding year. Describe the population sequence and determine what the population will be at the end of 2020, if this rate of growth is maintained.

52. Population Change Each year 5% of the inhabitants of a rural area move to the city. If the current population is 24,000, and this rate of decrease continues, give a formula for the population k years from now.

53. Revenue Current daily revenue at a campus burger restaurant is \$12,000. Over the next seven days revenue is expected to increase by \$1000 each day as students return for the fall semester. What is the projected total revenue for the eight days for which we have projected data?

54. Revenue A car dealership's finance department is going to receive payments of \$300 per month for the next 60 months to pay for Bart's car. The kth such payment has a present value of $\$300(1.01)^{-k}$. The sum of the present values of all 60 payments must equal the selling price of the car. Write an expression for the selling price of the car and evaluate it using your calculator.

55. Future Value Six years from now, Nicole will need a new tractor for her farm. Starting next month, she is going to put \$100 in the bank each month to save for the inevitable purchase. Six years from now the kth bank deposit will be worth $\$100(1.005)^{72-k}$ (due to compounded interest). Write a formula for the accumulated amount of money from her 72 bank deposits. Use your calculator to determine how much Nicole will have available towards her tractor purchase.

56. Future Value Lisa has just turned seven years old. She would like to save some money each month, starting next month, so that on her 21st birthday she will have \$1000 in her bank account. Marge told her that with current interest rates her kth deposit will be worth, on her 21st birthday, $(1.004)^{168-k}$ times the deposited amount. Lisa wants to deposit the same amount each month. Write a formula for the amount Lisa needs to deposit each month to meet her goal. Use your calculator to evaluate the required amount.

57. Perpetuity Brad's will includes an endowment to Dalhousie University that is to provide each year after his death, forever, a \$500 prize for the top student in the business mathematics class, MATH 1115. Brad's estate can make an investment at 5% compounded annually to pay for this endowment. Adapt the solution of Example 11 to determine how much this endowment will cost Brad's estate.

58. Perpetuity Rework Problem 57 under the assumption that Brad's estate can make an investment at 10% compounded annually.

59. The Fibonacci sequence given in (7) is defined recursively using addition. Is it an arithmetic sequence? Explain.

60. The factorial sequence given in (6) is defined recursively using multiplication. Is it a geometric sequence? Explain.

61. The recursive definition for an arithmetic sequence (b_k) called for starting with a number a and adding a fixed number d to each term to get the next term. Similarly, the recursive definition for a geometric sequence (c_k) called for starting with a number a and multiplying each term by a fixed number r to get the next term. If instead of addition or multiplication we use *exponentiation*, we get two other classes of recursively defined sequences:

$$d_1 = a \text{ and, for each positive integer } k,\ d_{k+1} = (d_k)^p$$

for fixed real numbers a and p and

$$e_1 = a \text{ and, for each positive integer } k,\ e_{k+1} = b^{e_k}$$

for fixed real numbers a and b. To get an idea of how sequences can grow in size, take each of the parameters a, d, r, p, and b that have appeared in these definitions to be the number 2 and write the first five terms of each of the arithmetic sequence (b_k), the geometric sequence (c_k), and the sequences (d_k) and (e_k) defined above.