where F_1 and α (a Greek letter read "alpha") are constants.[6] Find the average flow $\overline{F}$ on the interval $[0, T]$.

Solution:

$$\overline{F} = \frac{1}{T-0}\int_0^T F(t)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{F_1}{(1+\alpha t)^2}\,dt = \frac{F_1}{\alpha T}\int_0^T (1+\alpha t)^{-2}(\alpha\,dt)$$

$$= \frac{F_1}{\alpha T}\left(\frac{(1+\alpha t)^{-1}}{-1}\right)\Bigg|_0^T = \frac{F_1}{\alpha T}\left(-\frac{1}{1+\alpha T}+1\right)$$

$$= \frac{F_1}{\alpha T}\left(\frac{-1+1+\alpha T}{1+\alpha T}\right) = \frac{F_1}{\alpha T}\left(\frac{\alpha T}{1+\alpha T}\right) = \frac{F_1}{1+\alpha T}$$

Now Work Problem 11 ◁

PROBLEMS 15.4

In Problems 1–8, find the average value of the function over the given interval.

1. $f(x) = x^2$; $[-1, 3]$

2. $f(x) = 2x + 1$; $[0, 1]$

3. $f(x) = 2 - 3x^2$; $[-1, 2]$

4. $f(x) = x^2 + x + 1$; $[1, 3]$

5. $f(t) = 2t^5$; $[-3, 3]$

6. $f(t) = t\sqrt{t^2+9}$; $[0, 4]$

7. $f(x) = \sqrt{x}$; $[0, 1]$

8. $f(x) = 5/x^2$; $[1, 3]$

9. Profit The profit (in dollars) of a business is given by

$$P = P(q) = 369q - 2.1q^2 - 400$$

where q is the number of units of the product sold. Find the average profit on the interval from $q = 0$ to $q = 100$.

10. Cost Suppose the cost (in dollars) of producing q units of a product is given by

$$c = 4000 + 10q + 0.1q^2$$

Find the average cost on the interval from $q = 100$ to $q = 500$.

11. Investment An investment of \$3000 earns interest at an annual rate of 5% compounded continuously. After t years, its value S (in dollars) is given by $S = 3000e^{0.05t}$. Find the average value of a two-year investment.

12. Medicine Suppose that colored dye is injected into the bloodstream at a constant rate R. At time t, let

$$C(t) = \frac{R}{F(t)}$$

be the concentration of dye at a location distant (distal) from the point of injection, where $F(t)$ is as given in Example 2. Show that the average concentration on $[0, T]$ is

$$\overline{C} = \frac{R\left(1 + \alpha T + \frac{1}{3}\alpha^2 T^2\right)}{F_1}$$

13. Revenue Suppose a manufacturer receives revenue r from the sale of q units of a product. Show that the average value of the marginal-revenue function over the interval $[0, q_0]$ is the price per unit when q_0 units are sold.

14. Find the average value of $f(x) = \dfrac{1}{x^2 - 4x + 5}$ over the interval $[0, 1]$ using an approximate integration technique. Round your answer to two decimal places.

Objective

To solve a differential equation by using the method of separation of variables. To discuss particular solutions and general solutions. To develop interest compounded continuously in terms of a differential equation. To discuss exponential growth and decay.

15.5 Differential Equations

Occasionally, you may have to solve an equation that involves the derivative of an unknown function. Such an equation is called a **differential equation.** An example is

$$y' = xy^2 \tag{1}$$

More precisely, Equation (1) is called a **first-order differential equation,** since it involves a derivative of the first order and none of higher order. A solution of Equation (1) is any function $y = f(x)$ that is defined on an interval and satisfies the equation for all x in the interval.

[6] W. Simon, *Mathematical Techniques for Physiology and Medicine* (New York: Academic Press, Inc., 1972).

To solve $y' = xy^2$, equivalently,

$$\frac{dy}{dx} = xy^2 \tag{2}$$

we think of dy/dx as a quotient of differentials and algebraically "separate variables" by rewriting the equation so that each side contains only one variable and a differential is not in a denominator:

$$\frac{dy}{y^2} = x\,dx$$

Integrating both sides and combining the constants of integration, we obtain

$$\int \frac{1}{y^2}\,dy = \int x\,dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C_1$$

$$-\frac{1}{y} = \frac{x^2 + 2C_1}{2}$$

Since $2C_1$ is an arbitrary constant, we can replace it by C.

$$-\frac{1}{y} = \frac{x^2 + C}{2} \tag{3}$$

Solving Equation (3) for y, we have

$$y = -\frac{2}{x^2 + C} \tag{4}$$

We can verify that y is a solution to the differential equation (2):

For if y is given by Equation (4), then

$$\frac{dy}{dx} = \frac{4x}{(x^2 + C)^2}$$

while also

$$xy^2 = x\left[-\frac{2}{x^2 + C}\right]^2 = \frac{4x}{(x^2 + C)^2}$$

showing that our y satisfies (2). Note in Equation (4) that, for *each* value of C, a different solution is obtained. We call Equation (4) the **general solution** of the differential equation. The method that we used to find it is called **separation of variables.**

In the foregoing example, suppose we are given the condition that $y = -\frac{2}{3}$ when $x = 1$; that is, $y(1) = -\frac{2}{3}$. Then the *particular* function that satisfies both Equation (2) and this condition can be found by substituting the values $x = 1$ and $y = -\frac{2}{3}$ into Equation (4) and solving for C:

$$-\frac{2}{3} = -\frac{2}{1^2 + C}$$

$$C = 2$$

Therefore, the solution of $dy/dx = xy^2$ such that $y(1) = -\frac{2}{3}$ is

$$y = -\frac{2}{x^2 + 2} \tag{5}$$

We call Equation (5) a **particular solution** of the differential equation.

APPLY IT ▷

5. For a clear liquid, light intensity diminishes at a rate of $\frac{dI}{dx} = -kI$, where I is the intensity of the light and x is the number of feet below the surface of the liquid. If $k = 0.0085$ and $I = I_0$ when $x = 0$, find I as a function of x.

EXAMPLE 1 Separation of Variables

Solve $y' = -\frac{y}{x}$ if $x, y > 0$.

Solution: Writing y' as dy/dx, separating variables, and integrating, we have

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$$

$$\ln|y| = C_1 - \ln|x|$$

Since $x, y > 0$, we can omit the absolute-value bars:

$$\ln y = C_1 - \ln x \tag{6}$$

To solve for y, we convert Equation (6) to exponential form:

$$y = e^{C_1 - \ln x}$$

So

$$y = e^{C_1} e^{-\ln x} = \frac{e^{C_1}}{e^{\ln x}}$$

Replacing e^{C_1} by C, where $C > 0$, and rewriting $e^{\ln x}$ as x gives

$$y = \frac{C}{x} \qquad C, x > 0$$

Now Work Problem 1 ◁

In Example 1, note that Equation (6) expresses the solution implicitly, whereas the final equation ($y = C/x$) states the solution y explicitly in terms of x. Solutions of certain differential equations are often expressed in implicit form for convenience (or necessity because of the difficulty involved in obtaining an explicit form).

Exponential Growth and Decay

In Section 5.3, the notion of interest compounded continuously was developed. Let us now take a different approach to this topic that involves a differential equation. Suppose P dollars are invested at an annual rate r compounded n times a year. Let the function $S = S(t)$ give the compound amount S (or total amount present) after t years from the date of the initial investment. Then the initial principal is $S(0) = P$. Furthermore, since there are n interest periods per year, each period has length $1/n$ years, which we will denote by dt. At the end of the first period, the accrued interest for that period is added to the principal, and the sum acts as the principal for the second period, and so on. Hence, if the beginning of an interest period occurs at time t, then the increase in the amount present at the end of a period dt is $S(t + dt) - S(t)$, which we write as ΔS. This increase, ΔS, is also the interest earned for the period. Equivalently, the interest earned is principal times rate times time:

$$\Delta S = S \cdot r \cdot dt$$

Dividing both sides by dt, we obtain

$$\frac{\Delta S}{dt} = rS \tag{7}$$

As $dt \to 0$, then $n = \frac{1}{dt} \to \infty$, and consequently interest is being *compounded continuously;* that is, the principal is subject to continuous growth at every instant. However, as $dt \to 0$, then $\Delta S/dt \to dS/dt$, and Equation (7) takes the form

$$\frac{dS}{dt} = rS \tag{8}$$

This differential equation means that *when interest is compounded continuously, the rate of change of the amount of money present at time t is proportional to the amount present at time t.* The constant of proportionality is r.

To determine the actual function S, we solve the differential equation (8) by the method of separation of variables:

$$\frac{dS}{dt} = rS$$

$$\frac{dS}{S} = r\,dt$$

$$\int \frac{1}{S}\,dS = \int r\,dt$$

$$\ln|S| = rt + C_1$$

We assume that $S > 0$, so $\ln|S| = \ln S$. Thus,

$$\ln S = rt + C_1$$

To get an explicit form, we can solve for S by converting to exponential form.

$$S = e^{rt+C_1} = e^{C_1}e^{rt}$$

For simplicity, e^{C_1} can be replaced by C (and then necessarily $C > 0$) to obtain the general solution

$$S = Ce^{rt}$$

The condition $S(0) = P$ allows us to find the value of C:

$$P = Ce^{r(0)} = C \cdot 1$$

Hence $C = P$, so

$$S = Pe^{rt} \tag{9}$$

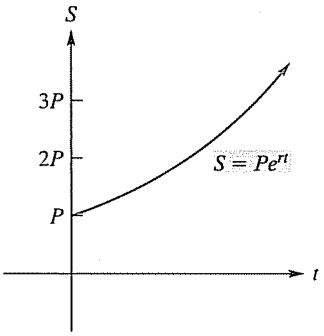

FIGURE 15.2 Compounding continuously.

Equation (9) gives the total value after t years of an initial investment of P dollars compounded continuously at an annual rate r. (See Figure 15.2.)

In our discussion of compound interest, we saw from Equation (8) that the rate of change in the amount present was proportional to the amount present. There are many natural quantities, such as population, whose rate of growth or decay at any time is considered proportional to the amount of that quantity present. If N denotes the amount of such a quantity at time t, then this rate of growth means that

$$\frac{dN}{dt} = kN$$

where k is a constant. If we separate variables and solve for N as we did for Equation (8), we get

$$N = N_0e^{kt} \tag{10}$$

where N_0 is a constant. In particular, if $t = 0$, then $N = N_0e^0 = N_0 \cdot 1 = N_0$. Thus, the constant N_0 is simply $N(0)$. Due to the form of Equation (10), we say that the quantity follows an **exponential law of growth** if k is positive and an **exponential law of decay** if k is negative.

EXAMPLE 2 Population Growth

In a certain city, the rate at which the population grows at any time is proportional to the size of the population. If the population was 125,000 in 1970 and 140,000 in 1990, what is the expected population in 2010?

Solution: Let N be the size of the population at time t. Since the exponential law of growth applies,

$$N = N_0e^{kt}$$

To find the population in 2010, we must first find the particular law of growth involved by determining the values of N_0 and k. Let the year 1970 correspond to $t = 0$. Then

$t = 20$ in 1990 and $t = 40$ in 2010. We have

$$N_0 = N(0) = 125{,}000$$

Thus,

$$N = 125{,}000e^{kt}$$

To find k, we use the fact that $N = 140{,}000$ when $t = 20$:

$$140{,}000 = 125{,}000e^{20k}$$

Hence,

$$e^{20k} = \frac{140{,}000}{125{,}000} = 1.12$$

Therefore, the law of growth is

$$\begin{aligned} N &= 125{,}000e^{kt} \\ &= 125{,}000(e^{20k})^{t/20} \\ &= 125{,}000(1.12)^{t/20} \end{aligned} \tag{11}$$

Setting $t = 40$ gives the expected population in 2010:

$$N = N(40) = 125{,}000(1.12)^2 = 156{,}800$$

We remark that from $e^{20k} = 1.12$ we have $20k = \ln(1.12)$ and hence $k = \frac{\ln(1.12)}{20} \approx 0.0057$, which can be placed in $N = 125{,}000e^{kt}$ to give

$$N \approx 125{,}000e^{0.0057t} \tag{12}$$

Now Work Problem 23 ◁

In Chapter 4, radioactive decay was discussed. Here we will consider this topic from the perspective of a differential equation. The rate at which a radioactive element decays at any time is found to be proportional to the amount of that element present. If N is the amount of a radioactive substance at time t, then the rate of decay is given by

$$\frac{dN}{dt} = -\lambda N. \tag{13}$$

The positive constant λ (a Greek letter read "lambda") is called the **decay constant,** and the minus sign indicates that N is decreasing as t increases. Thus, we have exponential decay. From Equation (10), the solution of this differential equation is

$$N = N_0e^{-\lambda t} \tag{14}$$

If $t = 0$, then $N = N_0 \cdot 1 = N_0$, so N_0 represents the amount of the radioactive substance present when $t = 0$.

The time for one-half of the substance to decay is called the **half-life** of the substance. In Section 4.2, it was shown that the half-life is given by

$$\text{half-life} = \frac{\ln 2}{\lambda} \approx \frac{0.69315}{\lambda} \tag{15}$$

Note that the half-life depends on λ. In Chapter 4, Figure 4.13 shows the graph of radioactive decay.

EXAMPLE 3 Finding the Decay Constant and Half-Life

If 60% of a radioactive substance remains after 50 days, find the decay constant and the half-life of the element.

Solution: From Equation (14),

$$N = N_0e^{-\lambda t}$$

where N_0 is the amount of the element present at $t = 0$. When $t = 50$, then $N = 0.6N_0$, and we have

$$\begin{aligned} 0.6N_0 &= N_0 e^{-50\lambda} \\ 0.6 &= e^{-50\lambda} \\ -50\lambda &= \ln(0.6) && \text{logarithmic form} \\ \lambda &= -\frac{\ln(0.6)}{50} \approx 0.01022 \end{aligned}$$

Thus, $N \approx N_0 e^{-0.01022t}$. The half-life, from Equation (15), is

$$\frac{\ln 2}{\lambda} \approx 67.82 \text{ days}$$

Now Work Problem 27 ◁

Radioactivity is useful in dating such things as fossil plant remains and archaeological remains made from organic material. Plants and other living organisms contain a small amount of radioactive carbon 14 (^{14}C) in addition to ordinary carbon (^{12}C). The ^{12}C atoms are stable, but the ^{14}C atoms are decaying exponentially. However, ^{14}C is formed in the atmosphere due to the effect of cosmic rays. This ^{14}C is taken up by plants during photosynthesis and replaces what has decayed. As a result, the ratio of ^{14}C atoms to ^{12}C atoms is considered constant in living tissues over a long period of time. When a plant dies, it stops absorbing ^{14}C, and the remaining ^{14}C atoms decay. By comparing the proportion of ^{14}C to ^{12}C in a fossil plant to that of plants found today, we can estimate the age of the fossil. The half-life of ^{14}C is approximately 5730 years. Thus, if a fossil is found to have a ^{14}C-to-^{12}C ratio that is half that of a similar substance found today, we would estimate the fossil to be 5730 years old.

EXAMPLE 4 Estimating the Age of an Ancient Tool

A wood tool found in a Middle East excavation site is found to have a ^{14}C-to-^{12}C ratio that is 0.6 of the corresponding ratio in a present-day tree. Estimate the age of the tool to the nearest hundred years.

Solution: Let N be the amount of ^{14}C present in the wood t years after the tool was made. Then $N = N_0 e^{-\lambda t}$, where N_0 is the amount of ^{14}C when $t = 0$. Since the ratio of ^{14}C to ^{12}C is 0.6 of the corresponding ratio in a present-day tree, this means that we want to find the value of t for which $N = 0.6N_0$. Thus, we have

$$\begin{aligned} 0.6N_0 &= N_0 e^{-\lambda t} \\ 0.6 &= e^{-\lambda t} \\ -\lambda t &= \ln(0.6) && \text{logarithmic form} \\ t &= -\frac{1}{\lambda}\ln(0.6) \end{aligned}$$

From Equation (15), the half-life is $(\ln 2)/\lambda$, which equals 5730, so $\lambda = (\ln 2)/5730$. Consequently,

$$\begin{aligned} t &= -\frac{1}{(\ln 2)/5730}\ln(0.6) \\ &= -\frac{5730\ln(0.6)}{\ln 2} \\ &\approx 4200 \text{ years} \end{aligned}$$

Now Work Problem 29 ◁

PROBLEMS 15.5

In Problems 1–8, solve the differential equations.

1. $y' = 2xy^2$
2. $y' = x^2y^2$
3. $\dfrac{dy}{dx} - 2x\ln(x^2+1) = 0$
4. $\dfrac{dy}{dx} = \dfrac{x}{y}$
5. $\dfrac{dy}{dx} = y, y > 0$
6. $y' = e^x y^3$
7. $y' = \dfrac{y}{x}, x, y > 0$
8. $\dfrac{dy}{dx} - x\ln x = 0$

In Problems 9–18, solve each of the differential equations, subject to the given conditions.

9. $y' = \dfrac{1}{y^2}$; $y(1) = 1$

10. $y' = e^{x-y}$; $y(0) = 0$ (*Hint:* $e^{x-y} = e^x/e^y$.)

11. $e^y y' - x^2 = 0$; $y = 0$ when $x = 0$

12. $x^2y' + \dfrac{1}{y^2} = 0$; $y(1) = 2$

13. $(3x^2+2)^3 y' - xy^2 = 0$; $y(0) = 2$

14. $y' + x^3y = 0$; $y = e$ when $x = 0$

15. $\dfrac{dy}{dx} = \dfrac{3x\sqrt{1+y^2}}{y}$; $y > 0, y(1) = \sqrt{8}$

16. $2y(x^3+2x+1)\dfrac{dy}{dx} = \dfrac{3x^2+2}{\sqrt{y^2+9}}$; $y(0) = 0$

17. $2\dfrac{dy}{dx} = \dfrac{xe^{-y}}{\sqrt{x^2+3}}$; $y(1) = 0$

18. $dy = 2xye^{x^2}dx$, $y > 0$; $y(0) = e$

19. **Cost** Find the manufacturer's cost function $c = f(q)$ given that

$$(q+1)^2\frac{dc}{dq} = cq$$

and fixed cost is e.

20. Find $f(2)$, given that $f(1) = 0$ and that $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = xe^{x-y}$$

21. **Circulation of Money** A country has 1.00 billion dollars of paper money in circulation. Each week 25 million dollars is brought into the banks for deposit, and the same amount is paid out. The government decides to issue new paper money; whenever the old money comes into the banks, it is destroyed and replaced by new money. Let y be the amount of old money (in millions of dollars) in circulation at time t (in weeks). Then y satisfies the differential equation

$$\frac{dy}{dt} = -0.025y$$

How long will it take for 95% of the paper money in circulation to be new? Round your answer to the nearest week. (*Hint:* If money is 95% new, then y is 5% of 1000.)

22. **Marginal Revenue and Demand** Suppose that a monopolist's marginal-revenue function is given by the differential equation

$$\frac{dr}{dq} = (50-4q)e^{-r/5}$$

Find the demand equation for the monopolist's product.

23. **Population Growth** In a certain town, the population at any time changes at a rate proportional to the population. If the population in 1990 was 60,000 and in 2000 was 64,000, find an equation for the population at time t, where t is the number of years past 1990. What is the expected population in 2010?

24. **Population Growth** The population of a town increases by natural growth at a rate proportional to the number N of persons present. If the population at time $t = 0$ is 50,000, find two expressions for the population N, t years later, if the population doubles in 50 years. Assume that $\ln 2 = 0.69$. Also, find N for $t = 100$.

25. **Population Growth** Suppose that the population of the world in 1930 was 2 billion and in 1960 was 3 billion. If the exponential law of growth is assumed, what is the expected population in 2015? Give your answer in terms of e.

26. **Population Growth** If exponential growth is assumed, in approximately how many years will a population double if it triples in 100 years? (*Hint:* Let the population at $t = 0$ be N_0.)

27. **Radioactivity** If 30% of the initial amount of a radioactive sample remains after 100 seconds, find the decay constant and the half-life of the element.

28. **Radioactivity** If 20% of the initial amount of a radioactive sample has *decayed* after 100 seconds, find the decay constant and the half-life of the element.

29. **Carbon Dating** An Egyptian scroll was found to have a ^{14}C-to-^{12}C ratio 0.7 of the corresponding ratio in similar present-day material. Estimate the age of the scroll, to the nearest hundred years.

30. **Carbon Dating** A recently discovered archaeological specimen has a ^{14}C-to-^{12}C ratio 0.1 of the corresponding ratio found in present-day organic material. Estimate the age of the specimen, to the nearest hundred years.

31. **Population Growth** Suppose that a population follows exponential growth given by $dN/dt = kN$ for $t \geq t_0$. Suppose also that $N = N_0$ when $t = t_0$. Find N, the population size at time t.

32. **Radioactivity** Polonium-210 has a half-life of about 140 days. (a) Find the decay constant in terms of ln 2. (b) What fraction of the original amount of a sample of polonium-210 remains after one year?

33. **Radioactivity** Radioactive isotopes are used in medical diagnoses as tracers to determine abnormalities that may exist in an organ. For example, if radioactive iodine is swallowed, after some time it is taken up by the thyroid gland. With the use of a detector, the rate at which it is taken up can be measured, and a determination can be made as to whether the uptake is normal. Suppose radioactive technetium-99m, which has a half-life of six hours, is to be used in a brain scan two hours from now. What should be its activity now if the activity when it is used is to be 12 units? Give your answer to one decimal place. [*Hint:* In Equation (14), let N = activity t hours from now and N_0 = activity now.]

34. **Radioactivity** A radioactive substance that has a half-life of eight days is to be temporarily implanted in a hospital patient until three-fifths of the amount originally present remains. How long should the implant remain in the patient?