35. Ecology In a forest, natural litter occurs, such as fallen leaves and branches, dead animals, and so on.[7] Let $A = A(t)$ denote the amount of litter present at time t, where $A(t)$ is expressed in grams per square meter and t is in years. Suppose that there is no litter at $t = 0$. Thus, $A(0) = 0$. Assume that

(a) Litter falls to the ground continuously at a constant rate of 200 grams per square meter per year.
(b) The accumulated litter decomposes continuously at the rate of 50% of the amount present per year (which is $0.50A$).

The difference of the two rates is the rate of change of the amount of litter present with respect to time:

$$\begin{pmatrix}\text{rate of change}\\ \text{of litter present}\end{pmatrix} = \begin{pmatrix}\text{rate of falling}\\ \text{to ground}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \text{decomposition}\end{pmatrix}$$

Therefore,

$$\frac{dA}{dt} = 200 - 0.50A$$

Solve for A. To the nearest gram, determine the amount of litter per square meter after one year.

36. Profit and Advertising A certain company determines that the rate of change of monthly net profit P, as a function of monthly advertising expenditure x, is proportional to the difference between a fixed amount, \$150,000, and $2P$; that is, dP/dx is proportional to $\$150{,}000 - 2P$. Furthermore, if no money is spent on monthly advertising, the monthly net profit is \$15,000; if \$1000 is spent on monthly advertising, the monthly net profit is \$70,000. What would the monthly net profit be if \$2000 were spent on advertising each month?

37. Value of a Machine The value of a certain machine depreciates 25% in the first year after the machine is purchased. The rate at which the machine subsequently depreciates is proportional to its value. Suppose that such a machine was purchased new on July 1, 1995, for \$80,000 and was valued at \$38,900 on January 1, 2006.

(a) Determine a formula that expresses the value V of the machine in terms of t, the number of years after July 1, 1996.
(b) Use the formula in part (a) to determine the year and month in which the machine has a value of exactly \$14,000.

Objective

To develop the logistic function as a solution of a differential equation. To model the spread of a rumor. To discuss and apply Newton's law of cooling.

15.6 More Applications of Differential Equations

Logistic Growth

In the previous section, we found that if the number N of individuals in a population at time t follows an exponential law of growth, then $N = N_0e^{kt}$, where $k > 0$ and N_0 is the population when $t = 0$. This law assumes that at time t the rate of growth, dN/dt, of the population is proportional to the number of individuals in the population. That is, $dN/dt = kN$.

Under exponential growth, a population would get infinitely large as time goes on. In reality, however, when the population gets large enough, environmental factors slow down the rate of growth. Examples are food supply, predators, overcrowding, and so on. These factors cause dN/dt to decrease eventually. It is reasonable to assume that the size of a population is limited to some maximum number M, where $0 < N < M$, and as $N \to M$, $dN/dt \to 0$, and the population size tends to be stable.

In summary, we want a population model that has exponential growth initially but that also includes the effects of environmental resistance to large population growth. Such a model is obtained by multiplying the right side of $dN/dt = kN$ by the factor $(M - N)/M$:

$$\frac{dN}{dt} = kN\left(\frac{M-N}{M}\right)$$

Notice that if N is small, then $(M - N)/M$ is close to 1, and we have growth that is approximately exponential. As $N \to M$, then $M - N \to 0$ and $dN/dt \to 0$, as we wanted in our model. Because k/M is a constant, we can replace it by K. Thus,

$$\frac{dN}{dt} = KN(M - N) \tag{1}$$

This states that the rate of growth is proportional to the product of the size of the population and the difference between the maximum size and the actual size of the population. We can solve for N in the differential equation (1) by the method of separation of

[7] R. W. Poole, *An Introduction to Quantitative Ecology* (New York: McGraw-Hill Book Company, 1974).

variables:

$$\frac{dN}{N(M-N)} = K\,dt$$

$$\int \frac{1}{N(M-N)}\,dN = \int K\,dt \qquad (2)$$

The integral on the left side can be found by using Formula (5) in the table of integrals in Appendix B. Thus, Equation (2) becomes

$$\frac{1}{M}\ln\left|\frac{N}{M-N}\right| = Kt + C$$

so

$$\ln\left|\frac{N}{M-N}\right| = MKt + MC$$

Since $N > 0$ and $M - N > 0$, we can write

$$\ln\frac{N}{M-N} = MKt + MC$$

In exponential form, we have

$$\frac{N}{M-N} = e^{MKt+MC} = e^{MKt}e^{MC}$$

Replacing the positive constant e^{MC} by A and solving for N gives

$$\begin{aligned}
\frac{N}{M-N} &= Ae^{MKt}\\
N &= (M-N)Ae^{MKt}\\
N &= MAe^{MKt} - NAe^{MKt}\\
NAe^{MKt} + N &= MAe^{MKt}\\
N(Ae^{MKt}+1) &= MAe^{MKt},\\
N &= \frac{MAe^{MKt}}{Ae^{MKt}+1}
\end{aligned}$$

Dividing numerator and denominator by Ae^{MKt}, we have

$$N = \frac{M}{1 + \frac{1}{Ae^{MKt}}} = \frac{M}{1 + \frac{1}{A}e^{-MKt}}$$

Replacing $1/A$ by b and MK by c yields the so-called *logistic function:*

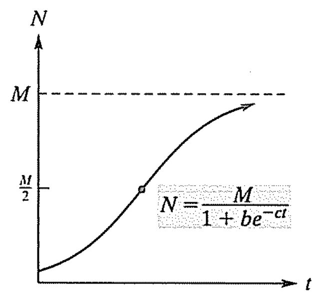


FIGURE 15.3 Logistic curve.

Logistic Function

The function defined by

$$N = \frac{M}{1+be^{-ct}} \qquad (3)$$

is called the **logistic function** or the **Verhulst–Pearl logistic function.**

The graph of Equation (3), called a *logistic curve,* is S-shaped and appears in Figure 15.3. Notice that the line $N = M$ is a horizontal asymptote; that is,

$$\lim_{t\to\infty}\frac{M}{1+be^{-ct}} = \frac{M}{1+b(0)} = M$$

Moreover, from Equation (1), the rate of growth is

$$KN(M-N)$$

which can be considered a function of N. To find when the maximum rate of growth occurs, we solve $\frac{d}{dN}[KN(M-N)] = 0$ for N:

$$\begin{aligned}\frac{d}{dN}[KN(M-N)] &= \frac{d}{dN}[K(MN-N^2)]\\ &= K[M-2N] = 0\end{aligned}$$

Thus, $N = M/2$. In other words, the rate of growth increases until the population size is $M/2$ and decreases thereafter. The maximum rate of growth occurs when $N = M/2$ and corresponds to a point of inflection in the graph of N. To find the value of t for which this occurs, we substitute $M/2$ for N in Equation (3) and solve for t:

$$\begin{aligned}\frac{M}{2} &= \frac{M}{1+be^{-ct}}\\ 1+be^{-ct} &= 2\\ e^{-ct} &= \frac{1}{b}\\ e^{ct} &= b\\ ct &= \ln b \qquad \text{logarithmic form}\\ t &= \frac{\ln b}{c}\end{aligned}$$

Therefore, the maximum rate of growth occurs at the point $((\ln b)/c, M/2)$.

We remark that in Equation (3) we can replace e^{-c} by C, and then the logistic function has the following form:

Alternative Form of Logistic Function

$$N = \frac{M}{1+bC^t}$$

EXAMPLE 1 Logistic Growth of Club Membership

Suppose the membership in a new country club is to be a maximum of 800 persons, due to limitations of the physical plant. One year ago the initial membership was 50 persons, and now there are 200. Provided that enrollment follows a logistic function, how many members will there be three years from now?

Solution: Let N be the number of members enrolled t years after the formation of the club. Then, from Equation (3),

$$N = \frac{M}{1+be^{-ct}}$$

Here $M = 800$, and when $t = 0$, we have $N = 50$. So

$$\begin{aligned}50 &= \frac{800}{1+b}\\ 1+b &= \frac{800}{50} = 16\\ b &= 15\end{aligned}$$

Thus,

$$N = \frac{800}{1+15e^{-ct}} \tag{4}$$

When $t = 1$, then $N = 200$, so we have

$$200 = \frac{800}{1 + 15e^{-c}}$$
$$1 + 15e^{-c} = \frac{800}{200} = 4$$
$$e^{-c} = \frac{3}{15} = \frac{1}{5}$$

Hence, $c = -\ln\frac{1}{5} = \ln 5$. Rather than substituting this value of c into Equation (4), it is more convenient to substitute the value of e^{-c} there:

$$N = \frac{800}{1 + 15\left(\frac{1}{5}\right)^t}$$

Three years from now, t will be 4. Therefore,

$$N = \frac{800}{1 + 15\left(\frac{1}{5}\right)^4} \approx 781$$

Now Work Problem 5 ◁

Modeling the Spread of a Rumor

Let us now consider a simplified model[8] of how a rumor spreads in a population of size M. A similar situation would be the spread of an epidemic or new fad.

Let $N = N(t)$ be the number of persons who know the rumor at time t. We will assume that those who know the rumor spread it randomly in the population and that those who are told the rumor become spreaders of the rumor. Furthermore, we will assume that each knower tells the rumor to k individuals per unit of time. (Some of these k individuals may already know the rumor.) We want an expression for the rate of increase of the knowers of the rumor. Over a unit of time, each of approximately N persons will tell the rumor to k persons. Thus, the total number of persons who are told the rumor over the unit of time is (approximately) Nk. However, we are interested only in *new* knowers. The proportion of the population that does not know the rumor is $(M - N)/M$. Hence, the total number of new knowers of the rumor is

$$Nk\left(\frac{M-N}{M}\right)$$

which can be written $(k/M)N(M - N)$. Therefore,

$$\frac{dN}{dt} = \frac{k}{M}N(M-N)$$
$$= KN(M-N), \qquad \text{where } K = \frac{k}{M}$$

This differential equation has the form of Equation (1), so its solution, from Equation (3), is a *logistic function:*

$$N = \frac{M}{1 + be^{-ct}}$$

EXAMPLE 2 **Campus Rumor**

In a large university of 45,000 students, a sociology major is researching the spread of a new campus rumor. When she begins her research, she determines that 300 students know the rumor. After one week, she finds that 900 know it. Estimate the number of students who know it four weeks after the research begins by assuming logistic growth. Give the answer to the nearest thousand.

Solution: Let N be the number of students who know the rumor t weeks after the research begins. Then

$$N = \frac{M}{1 + be^{-ct}}$$

[8]More simplified, that is, than the model described in the Explore and Extend for Chapter 8.

Here M, the size of the population, is 45,000, and when $t = 0, N = 300$. So we have

$$300 = \frac{45{,}000}{1+b}$$
$$1+b = \frac{45{,}000}{300} = 150$$
$$b = 149$$

Thus,

$$N = \frac{45{,}000}{1+149e^{-ct}}$$

When $t = 1$, then $N = 900$. Hence,

$$900 = \frac{45{,}000}{1+149e^{-c}}$$
$$1 + 149e^{-c} = \frac{45{,}000}{900} = 50$$

Therefore, $e^{-c} = \frac{49}{149}$, so

$$N = \frac{45{,}000}{1+149\left(\frac{49}{149}\right)^t}$$

When $t = 4$,

$$N = \frac{45{,}000}{1+149\left(\frac{49}{149}\right)^4} \approx 16{,}000$$

After four weeks, approximately 16,000 students know the rumor.

Now Work Problem 3 ◁

Newton's Law of Cooling

We conclude this section with an interesting application of a differential equation. If a homicide is committed, the temperature of the victim's body will gradually decrease from 37°C (normal body temperature) to the temperature of the surroundings (ambient temperature). In general, the temperature of the cooling body changes at a rate proportional to the difference between the temperature of the body and the ambient temperature. This statement is known as **Newton's law of cooling.** Thus, if $T(t)$ is the temperature of the body at time t and the ambient temperature is a, then

$$\frac{dT}{dt} = k(T-a)$$

where k is the constant of proportionality. Therefore, Newton's law of cooling is a differential equation. It can be applied to determine the time at which a homicide was committed, as the next example illustrates.

EXAMPLE 3 Time of Murder

A wealthy industrialist was found murdered in his home. Police arrived on the scene at 11:00 P.M. The temperature of the body at that time was 31°C, and one hour later it was 30°C. The temperature of the room in which the body was found was 22°C. Estimate the time at which the murder occurred.

Solution: Let t be the number of hours after the body was discovered and $T(t)$ be the temperature (in degrees Celsius) of the body at time t. We want to find the value of t for which $T = 37$ (normal body temperature). This value of t will, of course, be negative. By Newton's law of cooling,

$$\frac{dT}{dt} = k(T-a)$$

where k is a constant and a (the ambient temperature) is 22. Thus,

$$\frac{dT}{dt} = k(T - 22)$$

Separating variables, we have

$$\frac{dT}{T - 22} = k\,dt$$

$$\int \frac{dT}{T - 22} = \int k\,dt$$

$$\ln|T - 22| = kt + C$$

Because $T - 22 > 0$,

$$\ln(T - 22) = kt + C$$

When $t = 0$, then $T = 31$. Therefore,

$$\ln(31 - 22) = k \cdot 0 + C$$

$$C = \ln 9$$

Hence,

$$\ln(T - 22) = kt + \ln 9$$

$$\ln(T - 22) - \ln 9 = kt$$

$$\ln\frac{T - 22}{9} = kt \qquad \ln a - \ln b = \ln\frac{a}{b}$$

When $t = 1$, then $T = 30$, so

$$\ln\frac{30 - 22}{9} = k \cdot 1$$

$$k = \ln\frac{8}{9}$$

Thus,

$$\ln\frac{T - 22}{9} = t\ln\frac{8}{9}$$

Now we find t when $T = 37$:

$$\ln\frac{37 - 22}{9} = t\ln\frac{8}{9}$$

$$t = \frac{\ln(15/9)}{\ln(8/9)} \approx -4.34$$

Accordingly, the murder occurred about 4.34 hours *before* the time of discovery of the body (11:00 P.M.). Since 4.34 hours is (approximately) 4 hours and 20 minutes, the industrialist was murdered about 6:40 P.M.

Now Work Problem 9 ◁

PROBLEMS 15.6

1. Population The population of a city follows logistic growth and is limited to 100,000. If the population in 1995 was 50,000 and in 2000 was 60,000, what will the population be in 2005? Give your answer to the nearest hundred.

2. Production A company believes that the production of its product in present facilities will follow logistic growth. Presently, 200 units per day are produced, and production will increase to 300 units per day in one year. If production is limited to 500 units per day, what is the anticipated daily production in two years? Give your answer to the nearest unit.

3. Spread of Rumor In a university of 40,000 students, the administration holds meetings to discuss the idea of bringing in a major rock band for homecoming weekend. Before the plans are officially announced, students representatives on the administrative council spread information about the event as a rumor. At the end of one week, 100 people know the rumor. Assuming logistic growth, how many people know the rumor after two weeks? Give your answer to the nearest hundred.

4. Spread of a Fad At a university with 50,000 students, it is believed that the number of students with a particular ring tone on their mobile phones is following a logistic growth pattern. The student newspaper investigates when a survey reveals that 500 students have the ring tone. One week later, a similar survey reveals that 1500 students have it. The newspaper writes a story about it and includes a formula predicting the number $N = N(t)$ of students who will have the ring tone t weeks after the first survey. What is the formula that the newspaper publishes?

5. Flu Outbreak In a city whose population is 100,000, an outbreak of flu occurs. When the city health department begins its recordkeeping, there are 500 infected persons. One week later, there are 1000 infected persons. Assuming logistic growth, estimate the number of infected persons two weeks after recordkeeping begins.

6. Sigmoid Function A very special case of the logistic function defined by Equation (3) is the *sigmoid function,* obtained by taking $M = b = c = 1$ so that we have

$$N(t) = \frac{1}{1+e^{-t}}$$

(a) Show directly that the sigmoid function is the solution of the differential equation

$$\frac{dN}{dt} = N(1-N)$$

and the initial condition $N(0) = 1/2$.
(b) Show that $(0, 1/2)$ is an inflection point on the graph of the sigmoid function.
(c) Show that the function

$$f(t) = \frac{1}{1+e^{-t}} - \frac{1}{2}$$

is symmetric about the origin.
(d) Explain how (c) above shows that the sigmoid function is *symmetric about the point* $(0, 1/2)$, explaining at the same time what this means.
(e) Sketch the graph of the sigmoid function.

7. Biology In an experiment,[9] five *Paramecia* were placed in a test tube containing a nutritive medium. The number N of *Paramecia* in the tube at the end of t days is given approximately by

$$N = \frac{375}{1+e^{5.2-2.3t}}$$

(a) Show that this equation can be written as

$$N = \frac{375}{1+181.27e^{-2.3t}}$$

and hence is a logistic function.
(b) Find $\lim_{t\to\infty} N$.

8. Biology In a study of the growth of a colony of unicellular organisms,[10] the equation

$$N = \frac{0.2524}{e^{-2.128x}+0.005125} \qquad 0 \leq x \leq 5$$

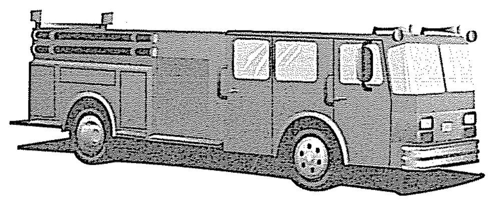
was obtained, where N is the estimated area of the growth in square centimeters and x is the age of the colony in days after being first observed.

(a) Put this equation in the form of a logistic function.
(b) Find the area when the age of the colony is 0.

9. Time of a Murder A murder was committed in an abandoned warehouse, and the victim's body was discovered at 3:17 A.M. by the police. At that time, the temperature of the body was 27°C and the temperature in the warehouse was −5°C. One hour later, the body temperature was 19°C and the warehouse temperature was unchanged. The police forensic mathematician calculates using Newton's law of cooling. What is the time she reports as the time of the murder?

10. Enzyme Formation An enzyme is a protein that acts as a catalyst for increasing the rate of a chemical reaction that occurs in cells. In a certain reaction, an enzyme A is converted to another enzyme B. Enzyme B acts as a catalyst for its own formation. Let p be the amount of enzyme B at time t and I be the total amount of both enzymes when $t = 0$. Suppose the rate of formation of B is proportional to $p(I - p)$. Without directly using calculus, find the value of p for which the rate of formation will be a maximum.

11. Fund-Raising A small town decides to conduct a fund-raising drive for a fire engine whose cost is \$200,000. The initial amount in the fund is \$50,000. On the basis of past drives, it is determined that t months after the beginning of the drive, the rate dx/dt at which money is contributed to such a fund is proportional to the difference between the desired goal of \$200,000 and the total amount x in the fund at that time. After one month, a total of \$100,000 is in the fund. How much will be in the fund after three months?



12. Birthrate In a discussion of unexpected properties of mathematical models of population, Bailey[11] considers the case in which the birthrate per *individual* is proportional to the population size N at time t. Since the growth rate per individual is $\frac{1}{N}\frac{dN}{dt}$, this means that

$$\frac{1}{N}\frac{dN}{dt} = kN$$

so that

$$\frac{dN}{dt} = kN^2 \qquad \text{subject to } N = N_0 \text{ at } t = 0$$

where $k > 0$. Show that

$$N = \frac{N_0}{1 - kN_0 t}$$

Use this result to show that

$$\lim N = \infty \quad \text{as} \quad t \to \left(\frac{1}{kN_0}\right)^-$$

This means that over a finite interval of time, there is an infinite amount of growth. Such a model might be useful only for rapid growth over a short interval of time.

13. Population Suppose that the rate of growth of a population is proportional to the difference between some maximum size M and the number N of individuals in the population at time t. Suppose that when $t = 0$, the size of the population is N_0. Find a formula for N.

[9] G. F. Gause, *The Struggle for Existence* (New York: Hafner Publishing Co., 1964).

[10] A. J. Lotka, *Elements of Mathematical Biology* (New York: Dover Publications, Inc., 1956).

[11] N. T. J. Bailey, *The Mathematical Approach to Biology and Medicine* (New York: John Wiley & Sons, Inc., 1967).