29. $\displaystyle\int_0^1 \frac{2-2x}{x^2+7x+12}\,dx$

30. $\displaystyle\int_0^1 \frac{x^2+5x+5}{x^2+3x+2}\,dx$

31. Find the area of the region bounded by the graph of

$$y = \frac{6(x^2+1)}{(x+2)^2}$$

and the x-axis from $x = 0$ to $x = 1$.

32. Consumers' Surplus Suppose the demand equation for a manufacturer's product is given by

$$p = \frac{200(q+3)}{q^2+7q+6}$$

where p is the price per unit (in dollars) when q units are demanded. Assume that market equilibrium occurs at the point $(q, p) = (10, 325/22)$. Determine consumers' surplus at market equilibrium.

15.3 Integration by Tables

Objective

To illustrate the use of the table of integrals in Appendix B.

Certain forms of integrals that occur frequently can be found in standard tables of integration formulas.[3] A short table appears in Appendix B, and its use will be illustrated in this section.

A given integral may have to be replaced by an equivalent form before it will fit a formula in the table. The equivalent form must match the formula exactly. Consequently, the steps performed to get the equivalent form should be written carefully rather than performed mentally. Before proceeding with the exercises that use tables, we recommend studying the examples of this section carefully.

In the following examples, the formula numbers refer to the Table of Selected Integrals given in Appendix B.

EXAMPLE 1 Integration by Tables

Find $\displaystyle\int \frac{x\,dx}{(2+3x)^2}$.

Solution: Scanning the table, we identify the integrand with Formula (7):

$$\int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left(\ln|a+bu| + \frac{a}{a+bu}\right) + C$$

Now we see if we can exactly match the given integrand with that in the formula. If we replace x by u, 2 by a, and 3 by b, then $du = dx$, and by substitution we have

$$\int \frac{x\,dx}{(2+3x)^2} = \int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left(\ln|a+bu| + \frac{a}{a+bu}\right) + C$$

Returning to the variable x and replacing a by 2 and b by 3, we obtain

$$\int \frac{x\,dx}{(2+3x)^2} = \frac{1}{9}\left(\ln|2+3x| + \frac{2}{2+3x}\right) + C$$

Note that the answer must be given in terms of x, the *original* variable of integration.

Now Work Problem 5 ◁

EXAMPLE 2 Integration by Tables

Find $\displaystyle\int x^2\sqrt{x^2-1}\,dx$.

Solution: This integral is identified with Formula (24):

$$\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{u}{8}(2u^2 \pm a^2)\sqrt{u^2 \pm a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$$

In the preceding formula, if the bottommost sign in the dual symbol "±" on the left side is used, then the bottommost sign in the dual symbols on the right side must also be

[3] See, for example, W. H. Beyer (ed.), *CRC Standard Mathematical Tables and Formulae*, 30th ed. (Boca Raton, FL: CRC Press, 1996).

used. In the original integral, we let $u = x$ and $a = 1$. Then $du = dx$, and by substitution the integral becomes

$$\int x^2\sqrt{x^2-1}\,dx = \int u^2\sqrt{u^2-a^2}\,du$$

$$= \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$$

Since $u = x$ and $a = 1$,

$$\int x^2\sqrt{x^2-1}\,dx = \frac{x}{8}(2x^2-1)\sqrt{x^2-1} - \frac{1}{8}\ln|x+\sqrt{x^2-1}| + C$$

Now Work Problem 17 ◁

This example, as well as Examples 4, 5, and 7, shows how to adjust an integral so that it conforms to one in the table.

EXAMPLE 3 Integration by Tables

Find $\int \frac{dx}{x\sqrt{16x^2+3}}$.

Solution: The integrand can be identified with Formula (28):

$$\int \frac{du}{u\sqrt{u^2+a^2}} = \frac{1}{a}\ln\left|\frac{\sqrt{u^2+a^2}-a}{u}\right| + C$$

If we let $u = 4x$ and $a = \sqrt{3}$, then $du = 4\,dx$. Watch closely how, by inserting 4's in the numerator and denominator, we transform the given integral into an equivalent form that matches Formula (28):

$$\int \frac{dx}{x\sqrt{16x^2+3}} = \int \frac{(4\,dx)}{(4x)\sqrt{(4x)^2+(\sqrt{3})^2}} = \int \frac{du}{u\sqrt{u^2+a^2}}$$

$$= \frac{1}{a}\ln\left|\frac{\sqrt{u^2+a^2}-a}{u}\right| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{16x^2+3}-\sqrt{3}}{4x}\right| + C$$

Now Work Problem 7 ◁

EXAMPLE 4 Integration by Tables

Find $\int \frac{dx}{x^2(2-3x^2)^{1/2}}$.

Solution: The integrand is identified with Formula (21):

$$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$$

Letting $u = \sqrt{3}x$ and $a^2 = 2$, we have $du = \sqrt{3}\,dx$. Hence, by inserting two factors of $\sqrt{3}$ in both the numerator and denominator of the original integral, we have

$$\int \frac{dx}{x^2(2-3x^2)^{1/2}} = \sqrt{3}\int \frac{(\sqrt{3}\,dx)}{(\sqrt{3}x)^2[2-(\sqrt{3}x)^2]^{1/2}} = \sqrt{3}\int \frac{du}{u^2(a^2-u^2)^{1/2}}$$

$$= \sqrt{3}\left[-\frac{\sqrt{a^2-u^2}}{a^2u}\right] + C = \sqrt{3}\left[-\frac{\sqrt{2-3x^2}}{2(\sqrt{3}x)}\right] + C$$

$$= -\frac{\sqrt{2-3x^2}}{2x} + C$$

Now Work Problem 35 ◁

EXAMPLE 5 Integration by Tables

Find $\int 7x^2 \ln(4x)\,dx$.

Solution: This is similar to Formula (42) with $n = 2$:

$$\int u^n \ln u\,du = \frac{u^{n+1}\ln u}{n+1} - \frac{u^{n+1}}{(n+1)^2} + C$$

If we let $u = 4x$, then $du = 4\,dx$. Hence,

$$\begin{aligned}
\int 7x^2 \ln(4x)\,dx &= \frac{7}{4^3}\int (4x)^2 \ln(4x)(4\,dx) \\
&= \frac{7}{64}\int u^2 \ln u\,du = \frac{7}{64}\left(\frac{u^3\ln u}{3} - \frac{u^3}{9}\right) + C \\
&= \frac{7}{64}\left(\frac{(4x)^3\ln(4x)}{3} - \frac{(4x)^3}{9}\right) + C \\
&= 7x^3\left(\frac{\ln(4x)}{3} - \frac{1}{9}\right) + C \\
&= \frac{7x^3}{9}(3\ln(4x) - 1) + C
\end{aligned}$$

Now Work Problem 45 ◁

EXAMPLE 6 Integral Table Not Needed

Find $\int \frac{e^{2x}\,dx}{7 + e^{2x}}$.

Solution: At first glance, we do not identify the integrand with any form in the table. Perhaps rewriting the integral will help. Let $u = 7 + e^{2x}$, then $du = 2e^{2x}\,dx$. So

$$\begin{aligned}
\int \frac{e^{2x}\,dx}{7 + e^{2x}} &= \frac{1}{2}\int \frac{(2e^{2x}\,dx)}{7 + e^{2x}} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|7 + e^{2x}| + C = \frac{1}{2}\ln(7 + e^{2x}) + C
\end{aligned}$$

Thus, we had only to use our knowledge of basic integration forms. [Actually, this form appears as Formula (2) in the table, with $a = 0$ and $b = 1$.]

Now Work Problem 39 ◁

EXAMPLE 7 Finding a Definite Integral by Using Tables

Evaluate $\int_1^4 \frac{dx}{(4x^2 + 2)^{3/2}}$.

Solution: We will use Formula (32) to get the indefinite integral first:

$$\int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{\pm u}{a^2\sqrt{u^2 \pm a^2}} + C$$

Letting $u = 2x$ and $a^2 = 2$, we have $du = 2\,dx$. Thus,

$$\begin{aligned}
\int \frac{dx}{(4x^2 + 2)^{3/2}} &= \frac{1}{2}\int \frac{(2\,dx)}{((2x)^2 + 2)^{3/2}} = \frac{1}{2}\int \frac{du}{(u^2 + 2)^{3/2}} \\
&= \frac{1}{2}\left(\frac{u}{2\sqrt{u^2 + 2}}\right) + C
\end{aligned}$$

Here we determine the limits of integration with respect to u.

Instead of substituting back to x and evaluating from $x = 1$ to $x = 4$, we can determine the corresponding limits of integration with respect to u and then evaluate the last expression between those limits. Since $u = 2x$, when $x = 1$, we have $u = 2$; when

CAUTION!

When changing the variable of integration x to the variable of integration u, be sure to change the limits of integration so that they agree with u.

$x = 4$, we have $u = 8$. Hence,

$$\int_1^4 \frac{dx}{(4x^2+2)^{3/2}} = \frac{1}{2}\int_2^8 \frac{du}{(u^2+2)^{3/2}}$$

$$= \frac{1}{2}\left(\frac{u}{2\sqrt{u^2+2}}\right)\Bigg|_2^8 = \frac{2}{\sqrt{66}} - \frac{1}{2\sqrt{6}}$$

Now Work Problem 15 ◁

Integration Applied to Annuities

Tables of integrals are useful when we deal with integrals associated with annuities. Suppose that you must pay out \$100 at the end of each year for the next two years. Recall from Chapter 5 that a series of payments over a period of time, such as this, is called an *annuity*. If you were to pay off the debt now instead, you would pay the present value of the \$100 that is due at the end of the first year, plus the present value of the \$100 that is due at the end of the second year. The sum of these present values is the present value of the annuity. (The present value of an annuity is discussed in Section 5.4.) We will now consider the present value of payments made continuously over the time interval from $t = 0$ to $t = T$, with t in years, when interest is compounded continuously at an annual rate of r.

Suppose a payment is made at time t such that on an annual basis this payment is $f(t)$. If we divide the interval $[0, T]$ into subintervals $[t_{i-1}, t_i]$ of length dt (where dt is small), then the total amount of all payments over such a subinterval is approximately $f(t_i)\,dt$. [For example, if $f(t) = 2000$ and dt were one day, the total amount of the payments would be $2000(\frac{1}{365})$.] The present value of these payments is approximately $e^{-rt_i}f(t_i)\,dt$. (See Section 5.3.) Over the interval $[0, T]$, the total of all such present values is

$$\sum e^{-rt_i}f(t_i)\,dt$$

This sum approximates the present value A of the annuity. The smaller dt is, the better the approximation. That is, as $dt \to 0$, the limit of the sum is the present value. However, this limit is also a definite integral. That is,

$$A = \int_0^T f(t)e^{-rt}\,dt \tag{1}$$

where A is the **present value of a continuous annuity** at an annual rate r (compounded continuously) for T years if a payment at time t is at the rate of $f(t)$ per year.

We say that Equation (1) gives the **present value of a continuous income stream.** Equation (1) can also be used to find the present value of future profits of a business. In this situation, $f(t)$ is the annual rate of profit at time t.

We can also consider the *future* value of an annuity rather than its present value. If a payment is made at time t, then it has a certain value at the *end* of the period of the annuity—that is, $T - t$ years later. This value is

$$\begin{pmatrix}\text{amount of}\\\text{payment}\end{pmatrix} + \begin{pmatrix}\text{interest on this}\\\text{payment for } T-t \text{ years}\end{pmatrix}$$

If S is the total of such values for all payments, then S is called the *accumulated amount of a continuous annuity* and is given by the formula

$$S = \int_0^T f(t)e^{r(T-t)}\,dt$$

where S is the **accumulated amount of a continuous annuity** at the end of T years at an annual rate r (compounded continuously) when a payment at time t is at the rate of $f(t)$ per year.

EXAMPLE 8 Present Value of a Continuous Annuity

Find the present value (to the nearest dollar) of a continuous annuity at an annual rate of 8% for 10 years if the payment at time t is at the rate of t^2 dollars per year.

Solution: The present value is given by

$$A = \int_0^T f(t)e^{-rt}\,dt = \int_0^{10} t^2 e^{-0.08t}\,dt$$

We will use Formula (39),

$$\int u^n e^{au}\,du = \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}\,du$$

This is called a *reduction formula,* since it reduces one integral to an expression that involves another integral that is easier to determine. If $u = t, n = 2$, and $a = -0.08$, then $du = dt$, and we have

$$A = \frac{t^2 e^{-0.08t}}{-0.08}\Bigg|_0^{10} - \frac{2}{-0.08}\int_0^{10} te^{-0.08t}\,dt$$

In the new integral, the exponent of t has been reduced to 1. We can match this integral with Formula (38),

$$\int ue^{au}\,du = \frac{e^{au}}{a^2}(au - 1) + C$$

by letting $u = t$ and $a = -0.08$. Then $du = dt$, and

$$A = \int_0^{10} t^2 e^{-0.08t}\,dt = \frac{t^2 e^{-0.08t}}{-0.08}\Bigg|_0^{10} - \frac{2}{-0.08}\left(\frac{e^{-0.08t}}{(-0.08)^2}(-0.08t - 1)\right)\Bigg|_0^{10}$$

$$= \frac{100e^{-0.8}}{-0.08} - \frac{2}{-0.08}\left(\frac{e^{-0.8}}{(-0.08)^2}(-0.8 - 1) - \frac{1}{(-0.08)^2}(-1)\right)$$

$$\approx 185$$

The present value is \$185.

Now Work Problem 59 ◁

PROBLEMS 15.3

In Problems 1 and 2, use Formula (19) in Appendix B to determine the integrals.

1. $\int \frac{dx}{(6 - x^2)^{3/2}}$

2. $\int \frac{dx}{(25 - 4x^2)^{3/2}}$

In Problems 3 and 4, use Formula (30) in Appendix B to determine the integrals.

3. $\int \frac{dx}{x^2\sqrt{16x^2 + 3}}$

4. $\int \frac{3\,dx}{x^3\sqrt{x^4 - 9}}$

In Problems 5–38, find the integrals by using the table in Appendix B.

5. $\int \frac{dx}{x(6 + 7x)}$

6. $\int \frac{5x^2\,dx}{(2 + 3x)^2}$

7. $\int \frac{dx}{x\sqrt{x^2 + 9}}$

8. $\int \frac{dx}{(x^2 + 7)^{3/2}}$

9. $\int \frac{x\,dx}{(2 + 3x)(4 + 5x)}$

10. $\int 2^{5x}\,dx$

11. $\int \frac{dx}{1 + 2e^{3x}}$

12. $\int x^2\sqrt{1 + x}\,dx$

13. $\int \frac{7\,dx}{x(5 + 2x)^2}$

14. $\int \frac{dx}{x\sqrt{5 - 11x^2}}$

15. $\int_0^1 \frac{x\,dx}{2 + x}$

16. $\int \frac{-3x^2\,dx}{2 - 5x}$

17. $\int \sqrt{x^2 - 3}\,dx$

18. $\int \frac{dx}{(1 + 5x)(2x + 3)}$

19. $\int_0^{1/12} xe^{12x}\,dx$

20. $\int \sqrt{\frac{2 + 3x}{5 + 3x}}\,dx$

21. $\int x^3 e^x\,dx$

22. $\int_1^2 \frac{4\,dx}{x^2(1 + x)}$

23. $\int \frac{\sqrt{5x^2 + 1}}{2x^2}\,dx$

24. $\int \frac{dx}{x\sqrt{2 - x}}$

25. $\int \frac{x\,dx}{(1 + 3x)^2}$

26. $\int \frac{2\,dx}{\sqrt{(1 + 2x)(3 + 2x)}}$

27. $\int \frac{dx}{7 - 5x^2}$

28. $\int 7x^2\sqrt{3x^2 - 6}\,dx$

29. $\int 36x^5 \ln(3x)\,dx$

30. $\int \frac{5\,dx}{x^2(3 + 2x)^2}$