Objective

To develop and apply the formula for integration by parts.

15.1 Integration by Parts[1]

Many integrals cannot be found by our previous methods. However, there are ways of changing certain integrals to forms that are easier to integrate. Of these methods, we will discuss two: *integration by parts* and (in Section 15.2) *integration using partial fractions.*

If u and v are differentiable functions of x, we have, by the product rule,

$$(uv)' = uv' + vu'$$

Rearranging gives

$$uv' = (uv)' - vu'$$

Integrating both sides with respect to x, we get

$$\int uv'\,dx = \int (uv)'\,dx - \int vu'\,dx \tag{1}$$

For $\int (uv)'\,dx$, we must find a function whose derivative with respect to x is $(uv)'$. Clearly, uv is such a function. Hence $\int (uv)'\,dx = uv + C_1$, and Equation (1) becomes

$$\int uv'\,dx = uv + C_1 - \int vu'\,dx$$

Absorbing C_1 into the constant of integration for $\int vu'\,dx$ and replacing $v'\,dx$ by dv and $u'\,dx$ by du, we have the *formula for integration by parts:*

Formula for Integration by Parts

$$\int u\,dv = uv - \int v\,du \tag{2}$$

This formula expresses an integral, $\int u\,dv$, in terms of another integral, $\int v\,du$, that may be easier to find.

To apply the formula to a given integral $\int f(x)\,dx$, we must write $f(x)\,dx$ as the product of two factors (or *parts*) by choosing a function u and a differential dv such that $f(x)\,dx = u\,dv$. However, for the formula to be useful, we must be able to integrate the part chosen for dv. To illustrate, consider

$$\int xe^x\,dx$$

This integral cannot be determined by previous integration formulas. One way to write $xe^x\,dx$ in the form $u\,dv$ is by letting

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

To apply the formula for integration by parts, we must find du and v:

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x + C_1$$

Thus,

$$\begin{aligned}
\int xe^x\,dx &= \int u\,dv \\
&= uv - \int v\,du \\
&= x(e^x + C_1) - \int (e^x + C_1)\,dx \\
&= xe^x + C_1x - e^x - C_1x + C \\
&= xe^x - e^x + C \\
&= e^x(x-1) + C
\end{aligned}$$

[1]This section can be omitted without loss of continuity.

The first constant, C_1, does not appear in the final answer. It is easy to prove that the constant involved in finding v from dv will always drop out, so from now on we will not write it when we find v.

When using the formula for integration by parts, sometimes the *best choice* for u and dv is not obvious. In some cases, one choice may be as good as another; in other cases, only one choice may be suitable. Insight into making a good choice (if any exists) will come only with practice and, of course, trial and error.

APPLY IT ▷

1. The monthly sales of a computer keyboard are estimated to decline at the rate of $S'(t) = -4te^{0.1t}$ keyboards per month, where t is time in months and $S(t)$ is the number of keyboards sold each month. If 5000 keyboards are sold now ($S(0) = 5000$), find $S(t)$.

EXAMPLE 1 Integration by Parts

Find $\int \frac{\ln x}{\sqrt{x}}\,dx$ by integration by parts.

Solution: We try

$$u = \ln x \quad \text{and} \quad dv = \frac{1}{\sqrt{x}}\,dx$$

Then

$$du = \frac{1}{x}\,dx \quad \text{and} \quad v = \int x^{-1/2}\,dx = 2x^{1/2}$$

Thus,

$$\begin{aligned}\int \ln x\left(\frac{1}{\sqrt{x}}\,dx\right) &= \int u\,dv = uv - \int v\,du\\ &= (\ln x)(2\sqrt{x}) - \int (2x^{1/2})\left(\frac{1}{x}\,dx\right)\\ &= 2\sqrt{x}\ln x - 2\int x^{-1/2}\,dx\\ &= 2\sqrt{x}\ln x - 2(2\sqrt{x}) + C \qquad x^{1/2} = \sqrt{x}\\ &= 2\sqrt{x}[\ln(x) - 2] + C\end{aligned}$$

Now Work Problem 3 ◁

Example 2 shows how a poor choice for u and dv can be made. If a choice does not work, there may be another that does.

EXAMPLE 2 Integration by Parts

Evaluate $\int_1^2 x\ln x\,dx$.

Solution: Since the integral does not fit a familiar form, we will try integration by parts. Let $u = x$ and $dv = \ln x\,dx$. Then $du = dx$, but $v = \int \ln x\,dx$ is not apparent by inspection. So we will make a different choice for u and dv. Let

$$u = \ln x \quad \text{and} \quad dv = x\,dx$$

Then

$$du = \frac{1}{x}\,dx \quad \text{and} \quad v = \int x\,dx = \frac{x^2}{2}$$

Therefore,

$$\begin{aligned}\int_1^2 x\ln x\,dx &= (\ln x)\left(\frac{x^2}{2}\right)\Big|_1^2 - \int_1^2 \left(\frac{x^2}{2}\right)\frac{1}{x}\,dx\\ &= (\ln x)\left(\frac{x^2}{2}\right)\Big|_1^2 - \frac{1}{2}\int_1^2 x\,dx\\ &= \frac{x^2\ln x}{2}\Big|_1^2 - \frac{1}{2}\left(\frac{x^2}{2}\right)\Big|_1^2\\ &= (2\ln 2 - 0) - \left(1 - \tfrac{1}{4}\right) = 2\ln 2 - \frac{3}{4}\end{aligned}$$

Now Work Problem 5 ◁

EXAMPLE 3 Integration by Parts where u Is the Entire Integrand

Determine $\int \ln y\, dy$.

Solution: We cannot integrate $\ln y$ by previous methods, so we will try integration by parts. Let $u = \ln y$ and $dv = dy$. Then $du = (1/y)\, dy$ and $v = y$. So we have

$$\begin{aligned}\int \ln y\, dy &= (\ln y)(y) - \int y\left(\frac{1}{y}\, dy\right)\\ &= y\ln y - \int dy = y\ln y - y + C\\ &= y[\ln(y) - 1] + C\end{aligned}$$

Now Work Problem 37 ◁

Before trying integration by parts, see whether the technique is really needed. Sometimes the integral can be handled by a basic technique, as Example 4 shows.

EXAMPLE 4 Basic Integration Form

Determine $\int xe^{x^2}\, dx$.

Solution: This integral can be fit to the form $\int e^u\, du$.

CAUTION!

Remember the simpler integration forms too. Integration by parts is not needed here.

$$\begin{aligned}\int xe^{x^2}\, dx &= \frac{1}{2}\int e^{x^2}(2x\, dx)\\ &= \frac{1}{2}\int e^u\, du \qquad \text{where } u = x^2\\ &= \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C\end{aligned}$$

Now Work Problem 17 ◁

Sometimes integration by parts must be used more than once, as shown in the following example.

APPLY IT ▷

2. Suppose a population of bacteria grows at a rate of

$$P'(t) = 0.1t(\ln t)^2$$

Find the general form of $P(t)$.

EXAMPLE 5 Applying Integration by Parts Twice

Determine $\int x^2 e^{2x+1}\, dx$.

Solution: Let $u = x^2$ and $dv = e^{2x+1}\, dx$. Then $du = 2x\, dx$ and $v = e^{2x+1}/2$.

$$\begin{aligned}\int x^2 e^{2x+1}\, dx &= \frac{x^2 e^{2x+1}}{2} - \int \frac{e^{2x+1}}{2}(2x\, dx)\\ &= \frac{x^2 e^{2x+1}}{2} - \int xe^{2x+1}\, dx\end{aligned}$$

To find $\int xe^{2x+1}\, dx$, we will again use integration by parts. Here, let $u = x$ and $dv = e^{2x+1}\, dx$. Then $du = dx$ and $v = e^{2x+1}/2$, and we have

$$\begin{aligned}\int xe^{2x+1}\, dx &= \frac{xe^{2x+1}}{2} - \int \frac{e^{2x+1}}{2}\, dx\\ &= \frac{xe^{2x+1}}{2} - \frac{e^{2x+1}}{4} + C_1\end{aligned}$$

Thus,

$$\int x^2 e^{2x+1}\,dx = \frac{x^2 e^{2x+1}}{2} - \frac{xe^{2x+1}}{2} + \frac{e^{2x+1}}{4} + C \qquad \text{where } C = -C_1$$
$$= \frac{e^{2x+1}}{2}\left(x^2 - x + \frac{1}{2}\right) + C$$

Now Work Problem 23 ◁

PROBLEMS 15.1

1. In applying integration by parts to

$$\int f(x)\,dx$$

a student found that $u = x$, $du = dx$, $dv = (x+5)^{1/2}$, and $v = \frac{2}{3}(x+5)^{3/2}$. Use this information to find $\int f(x)\,dx$.

2. Use integration by parts to find

$$\int xe^{3x+1}\,dx$$

by choosing $u = x$ and $dv = e^{3x+1}\,dx$.

In Problems 3–29, find the integrals.

3. $\int xe^{-x}\,dx$

4. $\int xe^{ax}\,dx$ for $a \neq 0$

5. $\int y^3 \ln y\,dy$

6. $\int x^2 \ln x\,dx$

7. $\int \ln(4x)\,dx$

8. $\int \frac{t}{e^t}\,dt$

9. $\int x\sqrt{ax+b}\,dx$

10. $\int \frac{12x}{\sqrt{1+4x}}\,dx$

11. $\int \frac{x}{(5x+2)^3}\,dx$

12. $\int \frac{\ln(x+1)}{2(x+1)}\,dx$

13. $\int \frac{\ln x}{x^2}\,dx$

14. $\int \frac{2x+7}{e^{3x}}\,dx$

15. $\int_1^2 4xe^{2x}\,dx$

16. $\int_1^2 2xe^{-3x}\,dx$

17. $\int_0^1 xe^{-x^2}\,dx$

18. $\int \frac{3x^3}{\sqrt{4-x^2}}\,dx$

19. $\int_5^8 \frac{4x}{\sqrt{9-x}}\,dx$

20. $\int (\ln x)^2\,dx$

21. $\int 3(2x-2)\ln(x-2)\,dx$

22. $\int \frac{xe^x}{(x+1)^2}\,dx$

23. $\int x^2 e^x\,dx$

24. $\int_1^4 \sqrt{x}\ln(x^9)\,dx$

25. $\int (x - e^{-x})^2\,dx$

26. $\int x^2 e^{3x}\,dx$

27. $\int x^3 e^{x^2}\,dx$

28. $\int x^5 e^{x^2}\,dx$

29. $\int (e^x + x)^2\,dx$

30. Find $\int \ln(x + \sqrt{x^2+1})\,dx$. *Hint:* Show that

$$\frac{d}{dx}[\ln(x+\sqrt{x^2+1})] = \frac{1}{\sqrt{x^2+1}}$$

31. Find the area of the region bounded by the x-axis, the curve $y = \ln x$, and the line $x = e^3$.

32. Find the area of the region bounded by the x-axis and the curve $y = x^2e^x$ between $x = 0$ and $x = 1$.

33. Find the area of the region bounded by the x-axis and the curve $y = x^2 \ln x$ between $x = 1$ and $x = 2$.

34. Consumers' Surplus Suppose the demand equation for a manufacturer's product is given by

$$p = 5(q+5)e^{-(q+5)/5}$$

where p is the price per unit (in dollars) when q units are demanded. Assume that market equilibrium occurs when $q = 7$. Determine the consumers' surplus at market equilibrium.

35. Revenue Suppose total revenue r and price per unit p are differentiable functions of output q.

(a) Use integration by parts to show that

$$\int p\,dq = r - \int q\frac{dp}{dq}\,dq$$

(b) Using part (a), show that

$$r = \int\left(p + q\frac{dp}{dq}\right)dq$$

(c) Using part (b), prove that

$$r(q_0) = \int_0^{q_0}\left(p + q\frac{dp}{dq}\right)dq$$

(*Hint:* Refer to Section 14.7.)

36. Suppose f is a differentiable function. Apply integration by parts to $\int f(x)e^x\,dx$ to prove that

$$\int f(x)e^x\,dx + \int f'(x)e^x\,dx = f(x)e^x + C$$

$$\left(\text{Hence, } \int [f(x) + f'(x)]e^x\,dx = f(x)e^x + C\right)$$

37. Suppose that f has an inverse and that $F' = f$. Use integration by parts to develop a useful formula for $\int f^{-1}(x)\,dx$ in terms of F and f^{-1}. [*Hint:* Review Example 3. It used the idea required here, for the special case of $f(x) = e^x$.] If $f^{-1}(a) = c$ and $f^{-1}(b) = d$, show that

$$\int_a^b f^{-1}(x)\,dx = bd - ac - \int_c^d f(x)\,dx$$

For $0 < a < b$ and $f^{-1} > 0$ on $[a, b]$, draw a diagram that illustrates the last equation.

Objective

To show how to integrate a proper rational function by first expressing it as a sum of its partial fractions.

15.2 Integration by Partial Fractions[2]

Recall that a *rational function* is a quotient of polynomials $N(x)/D(x)$ and that it is *proper* if N and D have no common polynomial factor and the degree of the numerator N is less than the degree of the denominator D. If N/D is not proper, then we can use long division to divide $N(x)$ by $D(x)$:

$$D(x)\overline{)N(x)}^{\,Q(x)} \quad \vdots \quad \overline{R(x)} \qquad \text{thus} \qquad \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

Here the quotient $Q(x)$ and the remainder $R(x)$ are also polynomials and either $R(x)$ is the constant 0-polynomial or the degree of $R(x)$ is less than that of $D(x)$. Thus R/D is a proper rational function. Since

$$\int \frac{N(x)}{D(x)}\,dx = \int \left(Q(x) + \frac{R(x)}{D(x)}\right) dx = \int Q(x)\,dx + \int \frac{R(x)}{D(x)}\,dx$$

and we already know how to integrate a polynomial, it follows that the task of integrating rational functions reduces to that of integrating *proper* rational functions. We emphasize that the technique we are about to explain requires that a rational function be proper so that the long divison step is not optional. For example,

$$\int \frac{2x^4 - 3x^3 - 4x^2 - 17x - 6}{x^3 - 2x^2 - 3x}\,dx = \int \left(2x + 1 + \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x}\right) dx$$

$$= x^2 + x + \int \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x}\,dx$$

Distinct Linear Factors

We now consider

$$\int \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x}\,dx$$

It is essential that the denominator be expressed in factored form:

$$\int \frac{4x^2 - 14x - 6}{x(x+1)(x-3)}\,dx$$

Observe that in this example the denominator consists only of **linear factors** and that each factor occurs exactly once. It can be shown that, to each such factor $x - a$, there corresponds a *partial fraction* of the form

$$\frac{A}{x-a} \qquad A \text{ a constant}$$

such that the integrand is the sum of the partial fractions. If there are n such *distinct* linear factors, there will be n such partial fractions, each of which is easily integrated. Applying these facts, we can write

$$\frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3} \tag{1}$$

To determine the constants A, B, and C, we first combine the terms on the right side:

$$\frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{A(x+1)(x-3) + Bx(x-3) + Cx(x+1)}{x(x+1)(x-3)}$$

[2]This section can be omitted without loss of continuity.

Since the denominators of both sides are equal, we can equate their numerators:

$$4x^2 - 14x - 6 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1) \tag{2}$$

Although Equation (1) is not defined for $x = 0, x = -1$, and $x = 3$, we want to find values for A, B, and C that will make Equation (2) true for all values of x, so that the two sides of the equality provide equal functions. By successively setting x in Equation (2) equal to any three different numbers, we can obtain a system of equations that can be solved for A, B, and C. In particular, the work can be simplified by letting x be the roots of $D(x) = 0$; in our case, $x = 0, x = -1$, and $x = 3$. Using Equation (2), we have, for $x = 0$,

$$-6 = A(1)(-3) + B(0) + C(0) = -3A, \quad \text{so } A = 2$$

If $x = -1$,

$$12 = A(0) + B(-1)(-4) + C(0) = 4B, \quad \text{so } B = 3$$

If $x = 3$,

$$-12 = A(0) + B(0) + C(3)(4) = 12C, \quad \text{so } C = -1$$

Thus Equation (1) becomes

$$\frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{x-3}$$

Hence,

$$\begin{aligned}\int \frac{4x^2 - 14x - 6}{x(x+1)(x-3)}\,dx &= \int \left(\frac{2}{x} + \frac{3}{x+1} - \frac{1}{x-3}\right) dx \\ &= 2\int \frac{dx}{x} + 3\int \frac{dx}{x+1} - \int \frac{dx}{x-3} \\ &= 2\ln|x| + 3\ln|x+1| - \ln|x-3| + C\end{aligned}$$

For the *original* integral, we can now state that

$$\int \frac{2x^4 - 3x^3 - 4x^2 - 17x - 6}{x^3 - 2x^2 - 3x}\,dx = x^2 + x + 2\ln|x| + 3\ln|x+1| - \ln|x-3| + C$$

An alternative method of determining A, B, and C involves expanding the right side of Equation (2) and combining like terms:

$$\begin{aligned}4x^2 - 14x - 6 &= A(x^2 - 2x - 3) + B(x^2 - 3x) + C(x^2 + x) \\ &= Ax^2 - 2Ax - 3A + Bx^2 - 3Bx + Cx^2 + Cx \\ 4x^2 - 14x - 6 &= (A + B + C)x^2 + (-2A - 3B + C)x + (-3A)\end{aligned}$$

For this last equation to express an equality of functions, the coefficients of corresponding powers of x on the left and right sides must be equal:

$$\begin{cases} 4 = A + B + C \\ -14 = -2A - 3B + C \\ -6 = -3A \end{cases}$$

Solving gives $A = 2, B = 3$, and $C = -1$ as before.

APPLY IT ▷

3. The marginal revenue for a company manufacturing q radios per week is given by $r'(q) = \dfrac{5(q+4)}{q^2+4q+3}$, where $r(q)$ is the revenue in thousands of dollars. Find the equation for $r(q)$.

EXAMPLE 1 **Distinct Linear Factors**

Determine $\int \frac{2x+1}{3x^2-27}\,dx$ by using partial fractions.

Solution: Since the degree of the numerator is less than the degree of the denominator, no long division is necessary. The integral can be written as

$$\frac{1}{3}\int \frac{2x+1}{x^2-9}\,dx$$

Expressing $(2x+1)/(x^2-9)$ as a sum of partial fractions, we have

$$\frac{2x+1}{x^2-9} = \frac{2x+1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

Combining terms and equating numerators gives

$$2x+1 = A(x-3) + B(x+3)$$

If $x = 3$, then

$$7 = 6B, \quad \text{so } B = \frac{7}{6}$$

If $x = -3$, then

$$-5 = -6A, \quad \text{so } A = \frac{5}{6}$$

Thus,

$$\begin{aligned}\int \frac{2x+1}{3x^2-27}\,dx &= \frac{1}{3}\left(\int \frac{\frac{5}{6}\,dx}{x+3} + \int \frac{\frac{7}{6}\,dx}{x-3}\right)\\ &= \frac{1}{3}\left(\frac{5}{6}\ln|x+3| + \frac{7}{6}\ln|x-3|\right) + C\end{aligned}$$

Now Work Problem 1 ◁

Repeated Linear Factors

If the denominator of $N(x)/D(x)$ contains only linear factors, some of which are repeated, then, for each factor $(x-a)^k$, where k is the maximum number of times $x-a$ occurs as a factor, there will correspond the sum of k partial fractions:

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \cdots + \frac{K}{(x-a)^k}$$

EXAMPLE 2 **Repeated Linear Factors**

Determine $\int \frac{6x^2+13x+6}{(x+2)(x+1)^2}\,dx$ by using partial fractions.

Solution: Since the degree of the numerator, namely 2, is less than that of the denominator, namely 3, no long division is necessary. In the denominator, the linear factor $x+2$ occurs once and the linear factor $x+1$ occurs twice. There will thus be three partial fractions and three constants to determine, and we have

$$\frac{6x^2+13x+6}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$6x^2+13x+6 = A(x+1)^2 + B(x+2)(x+1) + C(x+2)$$

Let us choose $x = -2$, $x = -1$, and, for convenience, $x = 0$. For $x = -2$, we have

$$4 = A$$

If $x = -1$, then

$$-1 = C$$

If $x = 0$, then

$$6 = A + 2B + 2C = 4 + 2B - 2 = 2 + 2B$$
$$4 = 2B$$
$$2 = B$$

Therefore,

$$\int \frac{6x^2 + 13x + 6}{(x+2)(x+1)^2}\,dx = 4\int \frac{dx}{x+2} + 2\int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2}$$
$$= 4\ln|x+2| + 2\ln|x+1| + \frac{1}{x+1} + C$$
$$= \ln[(x+2)^4(x+1)^2] + \frac{1}{x+1} + C$$

The last line above is somewhat optional (depending on what you need the integral for). It merely illustrates that in problems of this kind the logarithms that arise can often be combined.

Now Work Problem 5 ◁

Distinct Irreducible Quadratic Factors

Suppose a quadratic factor $x^2 + bx + c$ occurs in $D(x)$ and it cannot be expressed as a product of two linear factors with real coefficients. Such a factor is said to be an *irreducible quadratic factor over the real numbers*. To each distinct irreducible quadratic factor that occurs exactly once in $D(x)$, there will correspond a partial fraction of the form

$$\frac{Ax + B}{x^2 + bx + c}$$

Note that even after a rational function has been expressed in terms of partial fractions, it may still be impossible to integrate using only the basic functions we have covered in this book. For example, a very simple irreducible quadratic factor is $x^2 + 1$ and yet

$$\int \frac{1}{x^2+1}\,dx = \int \frac{dx}{x^2+1} = \tan^{-1}x + C$$

where $\tan^{-1}$ is the inverse of the trigonometric function tan when tan is restricted to $(-\pi/2, \pi/2)$. We do not discuss trigonometric functions in this book, but note that any good calculator has a $\tan^{-1}$ key.

EXAMPLE 3 An Integral with a Distinct Irreducible Quadratic Factor

Determine $\displaystyle\int \frac{-2x - 4}{x^3 + x^2 + x}\,dx$ by using partial fractions.

Solution: Since $x^3+x^2+x = x(x^2+x+1)$, we have the linear factor x and the quadratic factor $x^2 + x + 1$, which does not seem factorable on inspection. If it were factorable as $(x - r_1)(x - r_2)$, with r_1 and r_2 real, then r_1 and r_2 would be roots of the equation $x^2 + x + 1 = 0$. By the quadratic formula, the roots are

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

Since there are no real roots, we conclude that $x^2 + x + 1$ is irreducible. Thus there will be two partial fractions and *three* constants to determine. We have

$$\frac{-2x - 4}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+x+1}$$
$$-2x - 4 = A(x^2 + x + 1) + (Bx + C)x$$
$$= Ax^2 + Ax + A + Bx^2 + Cx$$
$$0x^2 - 2x - 4 = (A + B)x^2 + (A + C)x + A$$

Equating coefficients of like powers of x, we obtain

$$\begin{cases} 0 = A + B \\ -2 = A + C \\ -4 = A \end{cases}$$

Solving gives $A = -4, B = 4$, and $C = 2$. Hence,

$$\int \frac{-2x-4}{x(x^2+x+1)}\,dx = \int \left(\frac{-4}{x} + \frac{4x+2}{x^2+x+1}\right)dx$$
$$= -4\int \frac{dx}{x} + 2\int \frac{2x+1}{x^2+x+1}\,dx$$

Both integrals have the form $\int \frac{du}{u}$, so

$$\int \frac{-2x-4}{x(x^2+x+1)}\,dx = -4\ln|x| + 2\ln|x^2+x+1| + C$$
$$= \ln\left[\frac{(x^2+x+1)^2}{x^4}\right] + C$$

Now Work Problem 7 ◁

Repeated Irreducible Quadratic Factors

Suppose $D(x)$ contains factors of the form $(x^2 + bx + c)^k$, where k is the maximum number of times the irreducible factor $x^2 + bx + c$ occurs. Then, to each such factor there will correspond a sum of k partial fractions of the form

$$\frac{A+Bx}{x^2+bx+c} + \frac{C+Dx}{(x^2+bx+c)^2} + \cdots + \frac{M+Nx}{(x^2+bx+c)^k}$$

EXAMPLE 4 Repeated Irreducible Quadratic Factors

Determine $\int \frac{x^5}{(x^2+4)^2}\,dx$ by using partial fractions.

Solution: Since the numerator has degree 5 and the denominator has degree 4, we first use long division, which gives

$$\frac{x^5}{x^4+8x^2+16} = x - \frac{8x^3+16x}{(x^2+4)^2}$$

The quadratic factor $x^2 + 4$ in the denominator of $(8x^3 + 16x)/(x^2 + 4)^2$ is irreducible and occurs as a factor twice. Thus, to $(x^2 + 4)^2$ there correspond two partial fractions and *four* coefficients to be determined. Accordingly, we set

$$\frac{8x^3+16x}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

and obtain

$$8x^3 + 16x = (Ax+B)(x^2+4) + Cx + D$$
$$8x^3 + 0x^2 + 16x + 0 = Ax^3 + Bx^2 + (4A+C)x + 4B + D$$

Equating like powers of x yields

$$\begin{cases} 8 = A \\ 0 = B \\ 16 = 4A + C \\ 0 = 4B + D \end{cases}$$

Solving gives $A = 8, B = 0, C = -16$, and $D = 0$. Therefore,

$$\int \frac{x^5}{(x^2+4)^2}\,dx = \int \left(x - \left(\frac{8x}{x^2+4} - \frac{16x}{(x^2+4)^2}\right)\right) dx$$
$$= \int x\,dx - 4\int \frac{2x}{x^2+4}\,dx + 8\int \frac{2x}{(x^2+4)^2}\,dx$$

The second integral on the preceding line has the form $\int \frac{du}{u}$, and the third integral has the form $\int \frac{du}{u^2}$. So

$$\int \frac{x^5}{(x^2+4)^2} = \frac{x^2}{2} - 4\ln(x^2+4) - \frac{8}{x^2+4} + C$$

Now Work Problem 27 ◁

CAUTION!

Be on the lookout for simple solutions too.

From our examples, you may have deduced that the number of constants needed to express $N(x)/D(x)$ by partial fractions is equal to the degree of $D(x)$, if it is assumed that $N(x)/D(x)$ defines a proper rational function. This is indeed the case. Note also that the representation of a proper rational function by partial fractions is unique; that is, there is only one choice of constants that can be made. Furthermore, regardless of the complexity of the polynomial $D(x)$, it can always (theoretically) be expressed as a product of linear and irreducible quadratic factors with real coefficients.

APPLY IT ▷

4. The rate of change of the voting population of a city with respect to time t (in years) is estimated to be $V'(t) = \frac{300t^3}{t^2+6}$. Find the general form of $V(t)$.

EXAMPLE 5 An Integral Not Requiring Partial Fractions

Find $\int \frac{2x+3}{x^2+3x+1}\,dx$.

Solution: This integral has the form $\int \frac{1}{u}\,du$. Thus,

$$\int \frac{2x+3}{x^2+3x+1}\,dx = \ln|x^2+3x+1| + C$$

Now Work Problem 17 ◁

PROBLEMS 15.2

In Problems 1–8, express the given rational function in terms of partial fractions. Watch out for any preliminary divisions.

1. $f(x) = \frac{10x}{x^2+7x+6}$
2. $f(x) = \frac{x+5}{x^2-1}$
3. $f(x) = \frac{2x^2}{x^2+5x+6}$
4. $f(x) = \frac{2x^2-15}{x^2+5x}$
5. $f(x) = \frac{3x-1}{x^2-2x+1}$
6. $f(x) = \frac{2x+3}{x^2(x-1)}$
7. $f(x) = \frac{x^2+3}{x^3+x}$
8. $f(x) = \frac{3x^2+5}{(x^2+4)^2}$

In Problems 9–30, determine the integrals.

9. $\int \frac{5x-2}{x^2-x}\,dx$
10. $\int \frac{15x+5}{x^2+5x}\,dx$
11. $\int \frac{x+10}{x^2-x-2}\,dx$
12. $\int \frac{2x-1}{x^2-x-12}\,dx$
13. $\int \frac{3x^3-3x+4}{4x^2-4}\,dx$
14. $\int \frac{7(4-x^2)}{(x-4)(x-2)(x+3)}\,dx$
15. $\int \frac{19x^2-5x-36}{2x^3-2x^2-12x}\,dx$
16. $\int \frac{4-x}{x^4-x^2}\,dx$
17. $\int \frac{2(3x^5+4x^3-x)}{x^6+2x^4-x^2-2}\,dx$
18. $\int \frac{x^4-2x^3+6x^2-11x+2}{x^3-3x^2+2x}\,dx$
19. $\int \frac{2x^2-5x-2}{(x-2)^2(x-1)}\,dx$
20. $\int \frac{5x^3+x^2+x-3}{x^4-x^3}\,dx$
21. $\int \frac{2(x^2+8)}{x^3+4x}\,dx$
22. $\int \frac{4x^3-3x^2+2x-3}{(x^2+3)(x+1)(x-2)}\,dx$
23. $\int \frac{-x^3+8x^2-9x+2}{(x^2+1)(x-3)^2}\,dx$
24. $\int \frac{5x^4+9x^2+3}{x(x^2+1)^2}\,dx$
25. $\int \frac{7x^3+24x}{(x^2+3)(x^2+4)}\,dx$
26. $\int \frac{12x^3+20x^2+28x+4}{3(x^2+2x+3)(x^2+1)}\,dx$
27. $\int \frac{3x^3+8x}{(x^2+2)^2}\,dx$
28. $\int \frac{3x^2-8x+4}{x^3-4x^2+4x-6}\,dx$