Objective

To define the differential, interpret it geometrically, and use it in approximations. Also, to restate the reciprocal relationship between dx/dy and dy/dx.

14.1 Differentials

We will soon give a reason for using the symbol dy/dx to denote the derivative of y with respect to x. To do this, we introduce the notion of the *differential* of a function.

Definition

Let $y = f(x)$ be a differentiable function of x, and let Δx denote a change in x, where Δx can be any real number. Then the ***differential*** of y, denoted dy or $d(f(x))$, is given by

$$dy = f'(x)\,\Delta x$$

TO REVIEW functions of several variables, see Section 2.8.

Note that dy depends on two variables, namely, x and Δx. In fact, dy is a function of two variables.

EXAMPLE 1 Computing a Differential

Find the differential of $y = x^3 - 2x^2 + 3x - 4$, and evaluate it when $x = 1$ and $\Delta x = 0.04$.

Solution: The differential is

$$dy = \frac{d}{dx}(x^3 - 2x^2 + 3x - 4)\,\Delta x$$
$$= (3x^2 - 4x + 3)\,\Delta x$$

When $x = 1$ and $\Delta x = 0.04$,

$$dy = [3(1)^2 - 4(1) + 3](0.04) = 0.08$$

Now Work Problem 1 ◁

If $y = x$, then $dy = d(x) = 1\,\Delta x = \Delta x$. Hence, the differential of x is Δx. We abbreviate $d(x)$ by dx. Thus, $dx = \Delta x$. From now on, it will be our practice to write dx for Δx when finding a differential. For example,

$$d(x^2 + 5) = \frac{d}{dx}(x^2 + 5)\,dx = 2x\,dx$$

Summarizing, we say that if $y = f(x)$ defines a differentiable function of x, then

$$dy = f'(x)\,dx$$

where dx is any real number. Provided that $dx \neq 0$, we can divide both sides by dx:

$$\frac{dy}{dx} = f'(x)$$

That is, dy/dx can be viewed either as the quotient of two differentials, namely, dy divided by dx, or as one symbol for the derivative of f at x. It is for this reason that we introduced the symbol dy/dx to denote the derivative.

EXAMPLE 2 Finding a Differential in Terms of dx

a. If $f(x) = \sqrt{x}$, then

$$d(\sqrt{x}) = \frac{d}{dx}(\sqrt{x})\,dx = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$$

b. If $u = (x^2 + 3)^5$, then $du = 5(x^2 + 3)^4(2x)\,dx = 10x(x^2 + 3)^4\,dx$.

Now Work Problem 3 ◁

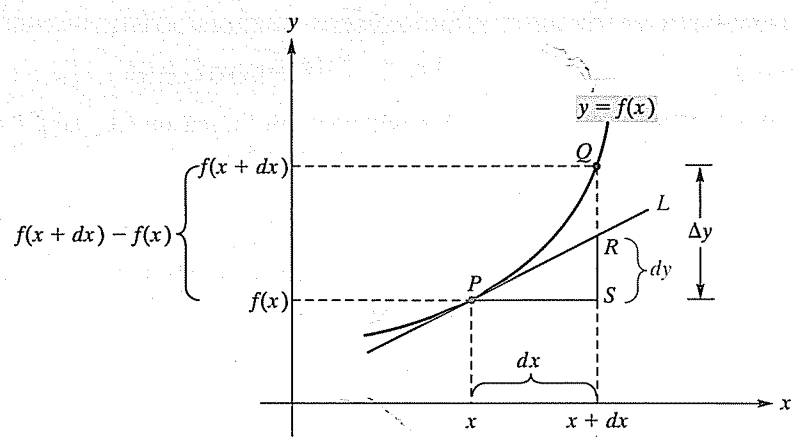


FIGURE 14.1 Geometric interpretation of dy and Δx.

The differential can be interpreted geometrically. In Figure 14.1, the point $P(x,f(x))$ is on the curve $y = f(x)$. Suppose x changes by dx, a real number, to the new value $x + dx$. Then the new function value is $f(x + dx)$, and the corresponding point on the curve is $Q(x + dx, f(x + dx))$. Passing through P and Q are horizontal and vertical lines, respectively, that intersect at S. A line L tangent to the curve at P intersects segment QS at R, forming the right triangle PRS. Observe that the graph of f near P is approximated by the tangent line at P. The slope of L is $f'(x)$ but it is also given by $\overline{SR}/\overline{PS}$ so that

$$f'(x) = \frac{\overline{SR}}{\overline{PS}}$$

Since $dy = f'(x)\,dx$ and $dx = \overline{PS}$,

$$dy = f'(x)\,dx = \frac{\overline{SR}}{\overline{PS}} \cdot \overline{PS} = \overline{SR}$$

Thus, if dx is a change in x at P, then dy is the corresponding vertical change along the **tangent line** at P. Note that for the same dx, the vertical change along the **curve** is $\Delta y = \overline{SQ} = f(x+dx) - f(x)$. Do not confuse Δy with dy. However, from Figure 14.1, the following is apparent:

When dx is close to 0, dy is an approximation to Δy. Therefore,

$$\Delta y \approx dy$$

This fact is useful in estimating Δy, a change in y, as Example 3 shows.

EXAMPLE 3 Using the Differential to Estimate a Change in a Quantity

A governmental health agency examined the records of a group of individuals who were hospitalized with a particular illness. It was found that the total proportion P that are discharged at the end of t days of hospitalization is given by

$$P = P(t) = 1 - \left(\frac{300}{300 + t}\right)^3$$

Use differentials to approximate the change in the proportion discharged if t changes from 300 to 305.

Solution: The change in t from 300 to 305 is $\Delta t = dt = 305 - 300 = 5$. The change in P is $\Delta P = P(305) - P(300)$. We approximate ΔP by dP:

$$\Delta P \approx dP = P'(t)\,dt = -3\left(\frac{300}{300 + t}\right)^2 \left(-\frac{300}{(300 + t)^2}\right) dt = 3\frac{300^3}{(300 + t)^4}\,dt$$

When $t = 300$ and $dt = 5$,

$$dP = 3\frac{300^3}{600^4}5 = \frac{15}{2^3 600} = \frac{1}{2^3 40} = \frac{1}{320} \approx 0.0031$$

For a comparison, the true value of ΔP is

$$P(305) - P(300) = 0.87807 - 0.87500 = 0.00307$$

(to five decimal places).

Now Work Problem 11 ◁

We said that if $y = f(x)$, then $\Delta y \approx dy$ if dx is close to zero. Thus,

$$\Delta y = f(x + dx) - f(x) \approx dy$$

Formula (1) is used to approximate a function value, whereas the formula $\Delta y \approx dy$ is used to approximate a change in function values.

so that

$$f(x + dx) \approx f(x) + dy \qquad (1)$$

This formula gives us a way of estimating a function value $f(x + dx)$. For example, suppose we estimate ln(1.06). Letting $y = f(x) = \ln x$, we need to estimate $f(1.06)$. Since $d(\ln x) = (1/x)\,dx$, we have, from Formula (1),

$$f(x + dx) \approx f(x) + dy$$

$$\ln(x + dx) \approx \ln x + \frac{1}{x}\,dx$$

We know the exact value of ln 1, so we will let $x = 1$ and $dx = 0.06$. Then $x + dx = 1.06$, and dx is close to zero. Therefore,

$$\ln(1 + 0.06) \approx \ln(1) + \frac{1}{1}(0.06)$$

$$\ln(1.06) \approx 0 + 0.06 = 0.06$$

The true value of ln(1.06) to five decimal places is 0.05827.

EXAMPLE 4 Using the Differential to Estimate a Function Value

The demand function for a product is given by

$$p = f(q) = 20 - \sqrt{q}$$

where p is the price per unit in dollars for q units. By using differentials, approximate the price when 99 units are demanded.

Solution: We want to approximate $f(99)$. By Formula (1),

$$f(q + dq) \approx f(q) + dp$$

where

$$dp = -\frac{1}{2\sqrt{q}}\,dq \qquad \frac{dp}{dq} = -\frac{1}{2}q^{-1/2}$$

We choose $q = 100$ and $dq = -1$ because $q + dq = 99$, dq is small, and it is easy to compute $f(100) = 20 - \sqrt{100} = 10$. We thus have

$$f(99) = f[100 + (-1)] \approx f(100) - \frac{1}{2\sqrt{100}}(-1)$$

$$f(99) \approx 10 + 0.05 = 10.05$$

Hence, the price per unit when 99 units are demanded is approximately \$10.05.

Now Work Problem 17 ◁

The equation $y = x^3 + 4x + 5$ defines y as a function of x. We could write $f(x) = x^3 + 4x + 5$. However, the equation also defines x implicitly as a function of y. In fact, if we restrict the domain of f to some set of real numbers x so that $y = f(x)$ is a one-to-one function, then in principle we could solve for x in terms of y and get $x = f^{-1}(y)$. [Actually, no restriction of the domain is necessary here. Since $f'(x) = 3x^2 + 4 > 0$, for all x, we see that f is strictly increasing on $(-\infty, \infty)$ and is thus one-to-one on $(-\infty, \infty)$.] As we did in Section 12.2, we can look at the derivative of x with respect to y, dx/dy and we have seen that it is given by

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \qquad \text{provided that } dy/dx \neq 0$$

Since dx/dy can be considered a quotient of differentials, we now see that it is the reciprocal of the quotient of differentials dy/dx. Thus

$$\frac{dx}{dy} = \frac{1}{3x^2 + 4}$$

It is important to understand that it is not necessary to be able to solve $y = x^3 + 4x + 5$ for x in terms of y, and the equation $\frac{dx}{dy} = \frac{1}{3x^2+4}$ holds for all x.

EXAMPLE 5 **Finding dp/dq from dq/dp**

Find $\frac{dp}{dq}$ if $q = \sqrt{2500 - p^2}$.

Solution:

Strategy There are a number of ways to find dp/dq. One approach is to solve the given equation for p explicitly in terms of q and then differentiate directly. Another approach to find dp/dq is to use implicit differentiation. However, since q is given explicitly as a function of p, we can easily find dq/dp and then use the preceding reciprocal relation to find dp/dq. We will take this approach.

We have

$$\frac{dq}{dp} = \frac{1}{2}(2500 - p^2)^{-1/2}(-2p) = -\frac{p}{\sqrt{2500 - p^2}}$$

Hence,

$$\frac{dp}{dq} = \frac{1}{\frac{dq}{dp}} = -\frac{\sqrt{2500 - p^2}}{p}$$

Now Work Problem 27 ◁

PROBLEMS 14.1

In Problems 1–10, find the differential of the function in terms of x and dx.

1. $y = ax + b$
2. $y = 2$
3. $f(x) = \sqrt{x^4 - 9}$
4. $f(x) = (4x^2 - 5x + 2)^3$
5. $u = \frac{1}{x^2}$
6. $u = \sqrt{x}$
7. $p = \ln(x^2 + 7)$
8. $p = e^{x^3+2x-5}$
9. $y = (9x + 3)e^{2x^2+3}$
10. $y = \ln\sqrt{x^2 + 12}$

In Problems 11–16, find Δy and dy for the given values of x and dx.

11. $y = ax + b$; for any x and any dx
12. $y = 5x^2$; $x = -1$, $dx = -0.02$
13. $y = 2x^2 + 5x - 7$; $x = -2$, $dx = 0.1$
14. $y = (3x + 2)^2$; $x = -1$, $dx = -0.03$
15. $y = \sqrt{32 - x^2}$; $x = 4$, $dx = -0.05$ Round your answer to three decimal places.
16. $y = \ln x$; $x = 1$, $dx = 0.01$
17. Let $f(x) = \frac{x+5}{x+1}$.
    (a) Evaluate $f'(1)$.
    (b) Use differentials to estimate the value of $f(1.1)$.
18. Let $f(x) = x^{3x}$.
    (a) Evaluate $f'(1)$.
    (b) Use differentials to estimate the value of $f(0.98)$.

In Problems 19–26, approximate each expression by using differentials.

19. $\sqrt{288}$ (*Hint:* $17^2 = 289$.)
20. $\sqrt{122}$
21. $\sqrt[3]{9}$
22. $\sqrt[4]{16.3}$
23. $\ln 0.97$
24. $\ln 1.01$
25. $e^{0.001}$
26. $e^{-0.002}$

In Problems 27–32, find dx/dy or dp/dq.

27. $y = 2x - 1$
28. $y = 5x^2 + 3x + 2$
29. $q = (p^2 + 5)^3$
30. $q = \sqrt{p+5}$
31. $q = \dfrac{1}{p^2}$
32. $q = e^{4-2p}$

33. If $y = 7x^2 - 6x + 3$, find the value of dx/dy when $x = 3$.

34. If $y = \ln x^2$, find the value of dx/dy when $x = 3$.

In Problems 35 and 36, find the rate of change of q with respect to p for the indicated value of q.

35. $p = \dfrac{500}{q+2}; q = 18$
36. $p = 60 - \sqrt{2q}; q = 50$

37. **Profit** Suppose that the profit (in dollars) of producing q units of a product is

$$P = 397q - 2.3q^2 - 400$$

Using differentials, find the approximate change in profit if the level of production changes from $q = 90$ to $q = 91$. Find the true change.

38. **Revenue** Given the revenue function

$$r = 250q + 45q^2 - q^3$$

use differentials to find the approximate change in revenue if the number of units increases from $q = 40$ to $q = 41$. Find the true change.

39. **Demand** The demand equation for a product is

$$p = \frac{10}{\sqrt{q}}$$

Using differentials, approximate the price when 24 units are demanded.

40. **Demand** Given the demand function

$$p = \frac{200}{\sqrt{q+8}}$$

use differentials to estimate the price per unit when 40 units are demanded.

41. If $y = f(x)$, then the *proportional change in y* is defined to be $\Delta y/y$, which can be approximated with differentials by dy/y. Use this last form to approximate the proportional change in the cost function

$$c = f(q) = \frac{q^2}{2} + 5q + 300$$

when $q = 10$ and $dq = 2$. Round your answer to one decimal place.

42. **Status/Income** Suppose that S is a numerical value of status based on a person's annual income I (in thousands of dollars). For a certain population, suppose $S = 20\sqrt{I}$. Use differentials to approximate the change in S if annual income decreases from \$45,000 to \$44,500.

43. **Biology** The volume of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Estimate the change in volume when the radius changes from 6.5×10^{-4} cm to 6.6×10^{-4} cm.

44. **Muscle Contraction** The equation

$$(P + a)(v + b) = k$$

is called the "fundamental equation of muscle contraction."[1] Here P is the load imposed on the muscle, v is the velocity of the shortening of the muscle fibers, and a, b, and k are positive constants. Find P in terms of v, and then use the differential to approximate the change in P due to a small change in v.

45. **Demand** The demand, q, for a monopolist's product is related to the price per unit, p, according to the equation

$$2 + \frac{q^2}{200} = \frac{4000}{p^2}$$

(a) Verify that 40 units will be demanded when the price per unit is \$20.
(b) Show that $\dfrac{dq}{dp} = -2.5$ when the price per unit is \$20.
(c) Use differentials and the results of parts (a) and (b) to approximate the number of units that will be demanded if the price per unit is reduced to \$19.20.

46. **Profit** The demand equation for a monopolist's product is

$$p = \frac{1}{2}q^2 - 66q + 7000$$

and the average-cost function is

$$\bar{c} = 500 - q + \frac{80{,}000}{2q}$$

(a) Find the profit when 100 units are demanded.
(b) Use differentials and the result of part (a) to estimate the profit when 101 units are demanded.

Objective

To define the antiderivative and the indefinite integral and to apply basic integration formulas.

14.2 The Indefinite Integral

Given a function f, if F is a function such that

$$F'(x) = f(x) \quad (1)$$

then F is called an *antiderivative* of f. Thus,

An antiderivative of f is simply a function whose derivative is f.

[1] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

Multiplying both sides of Equation (1) by the differential dx gives $F'(x)\,dx = f(x)\,dx$. However, because $F'(x)\,dx$ is the differential of F, we have $dF = f(x)\,dx$. Hence, we can think of an antiderivative of f as a function whose differential is $f(x)\,dx$.

Definition

An ***antiderivative*** of a function f is a function F such that

$$F'(x) = f(x)$$

Equivalently, in differential notation,

$$dF = f(x)\,dx$$

For example, because the derivative of x^2 is $2x$, x^2 is an antiderivative of $2x$. However, it is not the only antiderivative of $2x$: Since

$$\frac{d}{dx}(x^2 + 1) = 2x \qquad \text{and} \qquad \frac{d}{dx}(x^2 - 5) = 2x$$

both $x^2 + 1$ and $x^2 - 5$ are also antiderivatives of $2x$. In fact, it is obvious that because the derivative of a constant is zero, $x^2 + C$ is also an antiderivative of $2x$ for *any* constant C. Thus, $2x$ has infinitely many antiderivatives. More importantly, *all* antiderivatives of $2x$ must be functions of the form $x^2 + C$, because of the following fact:

Any two antiderivatives of a function differ only by a constant.

Since $x^2 + C$ describes all antiderivatives of $2x$, we can refer to it as being the *most general antiderivative* of $2x$, denoted by $\int 2x\,dx$, which is read "the *indefinite integral* of $2x$ with respect to x." Thus, we write

$$\int 2x\,dx = x^2 + C$$

The symbol $\int$ is called the **integral sign,** $2x$ is the **integrand,** and C is the **constant of integration.** The dx is part of the integral notation and indicates the variable involved. Here x is the **variable of integration.**

More generally, the **indefinite integral** of any function f with respect to x is written $\int f(x)\,dx$ and denotes the most general antiderivative of f. Since all antiderivatives of f differ only by a constant, if F is any antiderivative of f, then

$$\int f(x)\,dx = F(x) + C, \quad \text{where } C \text{ is a constant}$$

To *integrate* f means to find $\int f(x)\,dx$. In summary,

$$\int f(x)\,dx = F(x) + C \quad \text{if and only if} \quad F'(x) = f(x)$$

Thus we have

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x) \quad \text{and} \quad \int \frac{d}{dx}(F(x))\,dx = F(x) + C$$

which shows the extent to which differentiation and indefinite integration are inverse procedures.

APPLY IT ▷

1. If the marginal cost for a company is $f(q) = 28.3$, find $\int 28.3\,dq$, which gives the form of the cost function.

EXAMPLE 1 Finding an Indefinite Integral

Find $\int 5\,dx$.

Solution:

Strategy First we must find (perhaps better words are *guess at*) a function whose derivative is 5. Then we add the constant of integration.

CAUTION!

A common mistake is to omit C, the constant of integration.

Since we know that the derivative of $5x$ is 5, $5x$ is an antiderivative of 5. Therefore,

$$\int 5\,dx = 5x + C$$

Now Work Problem 1 ◁

Table 14.1 Elementary Integration Formulas

1. $\int k\,dx = kx + C$	k is a constant
2. $\int x^a\,dx = \dfrac{x^{a+1}}{a+1} + C$	$a \neq -1$
3. $\int x^{-1}\,dx = \int \dfrac{1}{x}dx = \int \dfrac{dx}{x} = \ln x + C$	for $x > 0$
4. $\int e^x\,dx = e^x + C$	
5. $\int kf(x)\,dx = k\int f(x)\,dx$	k is a constant
6. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	

Using differentiation formulas from Chapters 11 and 12, we have compiled a list of elementary integration formulas in Table 14.1. These formulas are easily verified. For example, Formula (2) is true because the derivative of $x^{a+1}/(a+1)$ is x^a for $a \neq -1$. (We must have $a \neq -1$ because the denominator is 0 when $a = -1$.) Formula (2) states that the indefinite integral of a power of x, other than x^{-1}, is obtained by increasing the exponent of x by 1, dividing by the new exponent, and adding a constant of integration. The indefinite integral of x^{-1} will be discussed in Section 14.4.

To verify Formula (5), we must show that the derivative of $k\int f(x)\,dx$ is $kf(x)$. Since the derivative of $k\int f(x)\,dx$ is simply k times the derivative of $\int f(x)\,dx$, and the derivative of $\int f(x)\,dx$ is $f(x)$, Formula (5) is verified. The reader should verify the other formulas. Formula (6) can be extended to any number of terms.

EXAMPLE 2 Indefinite Integrals of a Constant and of a Power of x

a. Find $\int 1\,dx$.

Solution: By Formula (1) with $k = 1$

$$\int 1\,dx = 1x + C = x + C$$

Usually, we write $\int 1\,dx$ as $\int dx$. Thus, $\int dx = x + C$.

b. Find $\int x^5\,dx$.

Solution: By Formula (2) with $n = 5$,

$$\int x^5\,dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$$

Now Work Problem 3 ◁

APPLY IT ▷

2. If the rate of change of a company's revenues can be modeled by $\frac{dR}{dt} = 0.12t^2$, then find $\int 0.12t^2\,dt$, which gives the form of the company's revenue function.

EXAMPLE 3 Indefinite Integral of a Constant Times a Function

Find $\int 7x\,dx$.

Solution: By Formula (5) with $k = 7$ and $f(x) = x$,

$$\int 7x\,dx = 7\int x\,dx$$

Since x is x^1, by Formula (2) we have

$$\int x^1\,dx = \frac{x^{1+1}}{1+1} + C_1 = \frac{x^2}{2} + C_1$$

where C_1 is the constant of integration. Therefore,

$$\int 7x\,dx = 7\int x\,dx = 7\left(\frac{x^2}{2} + C_1\right) = \frac{7}{2}x^2 + 7C_1$$

CAUTION!

Only a *constant* factor of the integrand can pass through an integral sign.

Since $7C_1$ is just an arbitrary constant, we will replace it by C for simplicity. Thus,

$$\int 7x\,dx = \frac{7}{2}x^2 + C$$

It is not necessary to write all intermediate steps when integrating. More simply, we write

$$\int 7x\,dx = (7)\frac{x^2}{2} + C = \frac{7}{2}x^2 + C$$

Now Work Problem 5 ◁

EXAMPLE 4 Indefinite Integral of a Constant Times a Function

Find $\int -\frac{3}{5}e^x\,dx$.

Solution:

$$\begin{aligned}\int -\frac{3}{5}e^x\,dx &= -\frac{3}{5}\int e^x\,dx && \text{Formula (5)}\\ &= -\frac{3}{5}e^x + C && \text{Formula (4)}\end{aligned}$$

Now Work Problem 21 ◁

APPLY IT ▷

3. Due to new competition, the number of subscriptions to a certain magazine is declining at a rate of $\frac{dS}{dt} = -\frac{480}{t^3}$ subscriptions per month, where t is the number of months since the competition entered the market. Find the form of the equation for the number of subscribers to the magazine.

EXAMPLE 5 Finding Indefinite Integrals

a. Find $\int \frac{1}{\sqrt{t}}\,dt$.

Solution: Here t is the variable of integration. We rewrite the integrand so that a basic formula can be used. Since $1/\sqrt{t} = t^{-1/2}$, applying Formula (2) gives

$$\int \frac{1}{\sqrt{t}}\,dt = \int t^{-1/2}\,dt = \frac{t^{(-1/2)+1}}{-\frac{1}{2}+1} + C = \frac{t^{1/2}}{\frac{1}{2}} + C = 2\sqrt{t} + C$$

b. Find $\int \frac{1}{6x^3}\,dx$.

Solution:

$$\int \frac{1}{6x^3}\,dx = \frac{1}{6}\int x^{-3}\,dx = \left(\frac{1}{6}\right)\frac{x^{-3+1}}{-3+1} + C$$
$$= -\frac{x^{-2}}{12} + C = -\frac{1}{12x^2} + C$$

Now Work Problem 9 ◁

APPLY IT ▷

4. The rate of growth of the population of a new city is estimated by $\frac{dN}{dt} = 500 + 300\sqrt{t}$, where t is in years. Find

$$\int (500 + 300\sqrt{t})\,dt$$

EXAMPLE 6 Indefinite Integral of a Sum

Find $\int (x^2 + 2x)\,dx$.

Solution: By Formula (6),

$$\int (x^2 + 2x)\,dx = \int x^2\,dx + \int 2x\,dx$$

Now,

$$\int x^2\,dx = \frac{x^{2+1}}{2+1} + C_1 = \frac{x^3}{3} + C_1$$

and

$$\int 2x\,dx = 2\int x\,dx = (2)\frac{x^{1+1}}{1+1} + C_2 = x^2 + C_2$$

Thus,

$$\int (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 + C_1 + C_2$$

When integrating an expression involving more than one term, only one constant of integration is needed.

For convenience, we will replace the constant $C_1 + C_2$ by C. We then have

$$\int (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 + C$$

Omitting intermediate steps, we simply integrate term by term and write

$$\int (x^2 + 2x)\,dx = \frac{x^3}{3} + (2)\frac{x^2}{2} + C = \frac{x^3}{3} + x^2 + C$$

Now Work Problem 11 ◁

APPLY IT ▷

5. Suppose the rate of savings in the United States is given by $\frac{dS}{dt} = 2.1t^2 - 65.4t + 491.6$, where t is the time in years and S is the amount of money saved in billions of dollars. Find the form of the equation for the amount of money saved.

EXAMPLE 7 Indefinite Integral of a Sum and Difference

Find $\int (2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1)\,dx$.

Solution:

$$\int (2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1)\,dx$$
$$= 2\int x^{4/5}\,dx - 7\int x^3\,dx + 10\int e^x\,dx - \int 1\,dx \qquad \text{Formulas (5) and (6)}$$
$$= (2)\frac{x^{9/5}}{\frac{9}{5}} - (7)\frac{x^4}{4} + 10e^x - x + C \qquad \text{Formulas (1), (2), and (4)}$$
$$= \frac{10}{9}x^{9/5} - \frac{7}{4}x^4 + 10e^x - x + C$$

Now Work Problem 15 ◁

Sometimes, in order to apply the basic integration formulas, it is necessary first to perform algebraic manipulations on the integrand, as Example 8 shows.

EXAMPLE 8 **Using Algebraic Manipulation to Find an Indefinite Integral**

Find $\int y^2\left(y+\frac{2}{3}\right)dy$.

Solution: The integrand does not fit a familiar integration form. However, by multiplying the integrand we get

$$\int y^2\left(y+\frac{2}{3}\right)dy = \int\left(y^3+\frac{2}{3}y^2\right)dy$$
$$= \frac{y^4}{4}+\left(\frac{2}{3}\right)\frac{y^3}{3}+C = \frac{y^4}{4}+\frac{2y^3}{9}+C$$

Now Work Problem 41 ◁

CAUTION!

In Example 8, we first multiplied the factors in the integrand. The answer could not have been found simply in terms of $\int y^2\,dy$ and $\int (y+\frac{2}{3})\,dy$. There is not a formula for the integral of a general product of functions.

EXAMPLE 9 **Using Algebraic Manipulation to Find an Indefinite Integral**

a. Find $\int \frac{(2x-1)(x+3)}{6}\,dx$.

Solution: By factoring out the constant $\frac{1}{6}$ and multiplying the binomials, we get

$$\int \frac{(2x-1)(x+3)}{6}\,dx = \frac{1}{6}\int (2x^2+5x-3)\,dx$$
$$= \frac{1}{6}\left((2)\frac{x^3}{3}+(5)\frac{x^2}{2}-3x\right)+C$$
$$= \frac{x^3}{9}+\frac{5x^2}{12}-\frac{x}{2}+C$$

b. Find $\int \frac{x^3-1}{x^2}\,dx$.

Solution: We can break up the integrand into fractions by dividing each term in the numerator by the denominator:

$$\int \frac{x^3-1}{x^2}\,dx = \int\left(\frac{x^3}{x^2}-\frac{1}{x^2}\right)dx = \int (x-x^{-2})\,dx$$
$$= \frac{x^2}{2}-\frac{x^{-1}}{-1}+C = \frac{x^2}{2}+\frac{1}{x}+C$$

Now Work Problem 49 ◁

Another algebraic approach to part (b) is

$$\int \frac{x^3-1}{x^2}\,dx = \int (x^3-1)x^{-2}\,dx$$
$$= \int (x-x^{-2})\,dx$$

and so on.

PROBLEMS 14.2

In Problems 1–52, find the indefinite integrals.

1. $\int 7\,dx$
2. $\int \frac{1}{x}\,dx$
3. $\int x^8\,dx$
4. $\int 5x^{24}\,dx$
5. $\int 5x^{-7}\,dx$
6. $\int \frac{z^{-3}}{3}\,dz$
7. $\int \frac{5}{x^7}\,dx$
8. $\int \frac{7}{x^4\,dx}$
9. $\int \frac{1}{t^{7/4}}\,dt$
10. $\int \frac{7}{2x^{9/4}}\,dx$
11. $\int (4+t)\,dt$
12. $\int (7r^5+4r^2+1)\,dr$
13. $\int (y^5-5y)\,dy$
14. $\int (5-2w-6w^2)\,dw$
15. $\int (3t^2-4t+5)\,dt$
16. $\int (1+t^2+t^4+t^6)\,dt$

17. $\int (\sqrt{2} + e)\,dx$

18. $\int (5 - 2^{-1})\,dx$

19. $\int \left(\frac{x}{7} - \frac{3}{4}x^4\right) dx$

20. $\int \left(\frac{2x^2}{7} - \frac{8}{3}x^4\right) dx$

21. $\int \pi e^x\,dx$

22. $\int (e^x + 3x^2 + 2x)\,dx$

23. $\int (x^{8.3} - 9x^6 + 3x^{-4} + x^{-3})\,dx$

24. $\int (0.7y^3 + 10 + 2y^{-3})\,dy$

25. $\int \frac{-2\sqrt{x}}{3}\,dx$

26. $\int dz$

27. $\int \frac{5}{3\sqrt[3]{x^2}}\,dx$

28. $\int \frac{-4}{(3x)^3}\,dx$

29. $\int \left(\frac{x^3}{3} - \frac{3}{x^3}\right) dx$

30. $\int \left(\frac{1}{2x^3} - \frac{1}{x^4}\right) dx$

31. $\int \left(\frac{3w^2}{2} - \frac{2}{3w^2}\right) dw$

32. $\int 7e^{-s}\,ds$

33. $\int \frac{3u - 4}{5}\,du$

34. $\int \frac{1}{12}\left(\frac{1}{3}e^x\right) dx$

35. $\int (u^e + e^u)\,du$

36. $\int \left(3y^3 - 2y^2 + \frac{e^y}{6}\right) dy$

37. $\int \left(\frac{3}{\sqrt{x}} - 12\sqrt[3]{x}\right) dx$

38. $\int 0\,dt$

39. $\int \left(-\frac{\sqrt[3]{x^2}}{5} - \frac{7}{2\sqrt{x}} + 6x\right) dx$

40. $\int \left(\sqrt[3]{u} + \frac{1}{\sqrt{u}}\right) du$

41. $\int (x^2 + 5)(x - 3)\,dx$

42. $\int x^3(x^2 + 5x + 2)\,dx$

43. $\int \sqrt{x}(x + 3)\,dx$

44. $\int (z + 2)^2\,dz$

45. $\int (3u + 2)^3\,du$

46. $\int \left(\frac{2}{\sqrt[5]{x}} - 1\right)^2 dx$

47. $\int x^{-2}(3x^4 + 4x^2 - 5)\,dx$

48. $\int (6e^u - u^3(\sqrt{u} + 1))\,du$

49. $\int \frac{z^4 + 10z^3}{2z^2}\,dz$

50. $\int \frac{x^4 - 5x^2 + 2x}{5x^2}\,dx$

51. $\int \frac{e^x + e^{2x}}{e^x}\,dx$

52. $\int \frac{(x^2 + 1)^3}{x}\,dx$

53. If $F(x)$ and $G(x)$ are such that $F'(x) = G'(x)$, is it true that $F(x) - G(x)$ must be zero?

54. **(a)** Find a function F such that $\int F(x)\,dx = xe^x + C$.
(b) Is there only one function F satisfying the equation given in part (a), or are there many such functions?

55. Find $\int \frac{d}{dx}\left(\frac{1}{\sqrt{x^2 + 1}}\right) dx$.

Objective

To find a particular antiderivative of a function that satisfies certain conditions. This involves evaluating constants of integration.

14.3 Integration with Initial Conditions

If we know the rate of change, f', of the function f, then the function f itself is an antiderivative of f' (since the derivative of f is f'). Of course, there are many antiderivatives of f', and the most general one is denoted by the indefinite integral. For example, if

$$f'(x) = 2x$$

then

$$f(x) = \int f'(x)\,dx = \int 2x\,dx = x^2 + C. \tag{1}$$

That is, *any* function of the form $f(x) = x^2 + C$ has its derivative equal to $2x$. Because of the constant of integration, notice that we do not know $f(x)$ specifically. However, if f must assume a certain function value for a particular value of x, then we can determine the value of C and thus determine $f(x)$ specifically. For instance, if $f(1) = 4$, then, from Equation (1),

$$f(1) = 1^2 + C$$
$$4 = 1 + C$$
$$C = 3$$

Thus,

$$f(x) = x^2 + 3$$

That is, we now know the particular function $f(x)$ for which $f'(x) = 2x$ and $f(1) = 4$. The condition $f(1) = 4$, which gives a function value of f for a specific value of x, is called an *initial condition*.

APPLY IT ▷

6. The rate of growth of a species of bacteria is estimated by $\frac{dN}{dt} = 800+200e^t$, where N is the number of bacteria (in thousands) after t hours. If $N(5) = 40{,}000$, find $N(t)$.

EXAMPLE 1 Initial-Condition Problem

If y is a function of x such that $y' = 8x - 4$ and $y(2) = 5$, find y. [Note: $y(2) = 5$ means that $y = 5$ when $x = 2$.] Also, find $y(4)$.

Solution: Here $y(2) = 5$ is the initial condition. Since $y' = 8x-4$, y is an antiderivative of $8x - 4$:

$$y = \int (8x-4)\,dx = 8 \cdot \frac{x^2}{2} - 4x + C = 4x^2 - 4x + C \tag{2}$$

We can determine the value of C by using the initial condition. Because $y = 5$ when $x = 2$, from Equation (2), we have

$$5 = 4(2)^2 - 4(2) + C$$
$$5 = 16 - 8 + C$$
$$C = -3$$

Replacing C by -3 in Equation (2) gives the function that we seek:

$$y = 4x^2 - 4x - 3 \tag{3}$$

To find $y(4)$, we let $x = 4$ in Equation (3):

$$y(4) = 4(4)^2 - 4(4) - 3 = 64 - 16 - 3 = 45$$

Now Work Problem 1 ◁

APPLY IT ▷

7. The acceleration of an object after t seconds is given by $y'' = 84t + 24$, the velocity at 8 seconds is given by $y'(8) = 2891$ ft/s, and the position at 2 seconds is given by $y(2) = 185$ ft. Find $y(t)$.

EXAMPLE 2 Initial-Condition Problem Involving y''

Given that $y'' = x^2 - 6$, $y'(0) = 2$, and $y(1) = -1$, find y.

Solution:

Strategy To go from y'' to y, two integrations are needed: the first to take us from y'' to y' and the other to take us from y' to y. Hence, there will be two constants of integration, which we will denote by C_1 and C_2.

Since $y'' = \frac{d}{dx}(y') = x^2 - 6$, y' is an antiderivative of $x^2 - 6$. Thus,

$$y' = \int (x^2 - 6)\,dx = \frac{x^3}{3} - 6x + C_1 \tag{4}$$

Now, $y'(0) = 2$ means that $y' = 2$ when $x = 0$; therefore, from Equation (4), we have

$$2 = \frac{0^3}{3} - 6(0) + C_1$$

Hence, $C_1 = 2$, so

$$y' = \frac{x^3}{3} - 6x + 2$$

By integration, we can find y:

$$y = \int \left(\frac{x^3}{3} - 6x + 2\right) dx$$
$$= \left(\frac{1}{3}\right)\frac{x^4}{4} - (6)\frac{x^2}{2} + 2x + C_2$$

so

$$y = \frac{x^4}{12} - 3x^2 + 2x + C_2 \tag{5}$$

Now, since $y = -1$ when $x = 1$, we have, from Equation (5),

$$-1 = \frac{1^4}{12} - 3(1)^2 + 2(1) + C_2$$

Thus, $C_2 = -\frac{1}{12}$, so

$$y = \frac{x^4}{12} - 3x^2 + 2x - \frac{1}{12}$$

Now Work Problem 5 ◁

Integration with initial conditions is applicable to many applied situations, as the next three examples illustrate.

EXAMPLE 3 Income and Education

For a particular urban group, sociologists studied the current average yearly income y (in dollars) that a person can expect to receive with x years of education before seeking regular employment. They estimated that the rate at which income changes with respect to education is given by

$$\frac{dy}{dx} = 100x^{3/2} \quad 4 \leq x \leq 16$$

where $y = 28{,}720$ when $x = 9$. Find y.

Solution: Here y is an antiderivative of $100x^{3/2}$. Thus,

$$y = \int 100x^{3/2}\,dx = 100 \int x^{3/2}\,dx$$

$$= (100)\frac{x^{5/2}}{\frac{5}{2}} + C$$

$$y = 40x^{5/2} + C \tag{6}$$

The initial condition is that $y = 28{,}720$ when $x = 9$. By putting these values into Equation (6), we can determine the value of C:

$$28{,}720 = 40(9)^{5/2} + C$$

$$= 40(243) + C$$

$$28{,}720 = 9720 + C$$

Therefore, $C = 19{,}000$, and

$$y = 40x^{5/2} + 19{,}000$$

Now Work Problem 17 ◁

EXAMPLE 4 Finding the Demand Function from Marginal Revenue

If the marginal-revenue function for a manufacturer's product is

$$\frac{dr}{dq} = 2000 - 20q - 3q^2$$

find the demand function.

Solution:

Strategy By integrating dr/dq and using an initial condition, we can find the revenue function r. But revenue is also given by the general relationship $r = pq$, where p is the price per unit. Thus, $p = r/q$. Replacing r in this equation by the revenue function yields the demand function.

Since dr/dq is the derivative of total revenue r,

$$r = \int (2000 - 20q - 3q^2)\,dq$$
$$= 2000q - (20)\frac{q^2}{2} - (3)\frac{q^3}{3} + C$$

so that

$$r = 2000q - 10q^2 - q^3 + C \tag{7}$$

Revenue is 0 when q is 0.

We assume that **when no units are sold, there is no revenue;** that is, $r = 0$ when $q = 0$. This is our initial condition. Putting these values into Equation (7) gives

$$0 = 2000(0) - 10(0)^2 - 0^3 + C$$

Although $q = 0$ gives $C = 0$, this is not true in general. It occurs in this section because the revenue functions are polynomials. In later sections, evaluating at $q = 0$ may produce a nonzero value for C.

Hence, $C = 0$, and

$$r = 2000q - 10q^2 - q^3$$

To find the demand function, we use the fact that $p = r/q$ and substitute for r:

$$p = \frac{r}{q} = \frac{2000q - 10q^2 - q^3}{q}$$
$$p = 2000 - 10q - q^2$$

Now Work Problem 11 ◁

EXAMPLE 5 Finding Cost from Marginal Cost

In the manufacture of a product, fixed costs per week are \$4000. (Fixed costs are costs, such as rent and insurance, that remain constant at all levels of production during a given time period.) If the marginal-cost function is

$$\frac{dc}{dq} = 0.000001(0.002q^2 - 25q) + 0.2$$

where c is the total cost (in dollars) of producing q pounds of product per week, find the cost of producing 10,000 lb in 1 week.

Solution: Since dc/dq is the derivative of the total cost c,

$$c(q) = \int [0.000001(0.002q^2 - 25q) + 0.2]\,dq$$
$$= 0.000001 \int (0.002q^2 - 25q)\,dq + \int 0.2\,dq$$
$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + C$$

When q is 0, total cost is equal to fixed cost.

Fixed costs are constant regardless of output. Therefore, when $q = 0, c = 4000$, which is our initial condition. Putting $c(0) = 4000$ in the last equation, we find that $C = 4000$, so

Although $q = 0$ gives C a value equal to fixed costs, this is not true in general. It occurs in this section because the cost functions are polynomials. In later sections, evaluating at $q = 0$ may produce a value for C that is different from fixed cost.

$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + 4000 \tag{8}$$

From Equation (8), we have $c(10{,}000) = 5416\frac{2}{3}$. Thus, the total cost for producing 10,000 pounds of product in 1 week is \$5416.67.

Now Work Problem 15 ◁

PROBLEMS 14.3

In Problems 1 and 2, find y subject to the given conditions.

1. $dy/dx = 3x - 4; \quad y(-1) = \frac{13}{2}$
2. $dy/dx = x^2 - x; \quad y(3) = \frac{19}{2}$

In Problems 3 and 4, if y satisfies the given conditions, find y(x) for the given value of x.

3. $y' = \dfrac{9}{8\sqrt{x}}, \; y(16) = 10; \quad x = 9$
4. $y' = -x^2 + 2x, \; y(2) = 1; \quad x = 1$

In Problems 5–8, find y subject to the given conditions.

5. $y'' = -3x^2 + 4x; \quad y'(1) = 2, \; y(1) = 3$
6. $y'' = x + 1; \quad y'(0) = 0, \; y(0) = 5$
7. $y''' = 2x; \quad y''(-1) = 3, \; y'(3) = 10, \; y(0) = 13$
8. $y''' = 2e^{-x} + 3; \quad y''(0) = 7, \; y'(0) = 5, \; y(0) = 1$

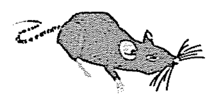
In Problems 9–12, dr/dq is a marginal-revenue function. Find the demand function.

9. $dr/dq = 0.7$
10. $dr/dq = 10 - \dfrac{1}{16}q$
11. $dr/dq = 275 - q - 0.3q^2$
12. $dr/dq = 5{,}000 - 3(2q + 2q^3)$

In Problems 13–16, dc/dq is a marginal-cost function and fixed costs are indicated in braces. For Problems 13 and 14, find the total-cost function. For Problems 15 and 16, find the total cost for the indicated value of q.

13. $dc/dq = 2.47; \quad \{159\}$
14. $dc/dq = 2q + 75; \quad \{2000\}$
15. $dc/dq = 0.08q^2 - 1.6q + 6.5; \quad \{8000\}; \quad q = 25$
16. $dc/dq = 0.000204q^2 - 0.046q + 6; \quad \{15{,}000\}; \quad q = 200$

17. **Diet for Rats** A group of biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.[2] The protein consisted of yeast and corn flour.



Over a period of time, the group found that the (approximate) rate of change of the average weight gain G (in grams) of a rat with respect to the percentage P of yeast in the protein mix was

$$\frac{dG}{dP} = -\frac{P}{25} + 2 \qquad 0 \le P \le 100$$

If $G = 38$ when $P = 10$, find G.

18. **Winter Moth** A study of the winter moth was made in Nova Scotia.[3] The prepupae of the moth fall onto the ground from host trees. It was found that the (approximate) rate at which prepupal density y (the number of prepupae per square foot of soil) changes with respect to distance x (in feet) from the base of a host tree is

$$\frac{dy}{dx} = -1.5 - x \quad 1 \le x \le 9$$

If $y = 59.6$ when $x = 1$, find y.

19. **Fluid Flow** In the study of the flow of fluid in a tube of constant radius R, such as blood flow in portions of the body, one can think of the tube as consisting of concentric tubes of radius r, where $0 \le r \le R$. The velocity v of the fluid is a function of r and is given by[4]

$$v = \int -\frac{(P_1 - P_2)r}{2l\eta}\, dr$$

where P_1 and P_2 are pressures at the ends of the tube, η (a Greek letter read "eta") is fluid viscosity, and l is the length of the tube. If $v = 0$ when $r = R$, show that

$$v = \frac{(P_1 - P_2)(R^2 - r^2)}{4l\eta}$$

20. **Elasticity of Demand** The sole producer of a product has determined that the marginal-revenue function is

$$\frac{dr}{dq} = 100 - 3q^2$$

Determine the point elasticity of demand for the product when $q = 5$. (*Hint:* First find the demand function.)

21. **Average Cost** A manufacturer has determined that the marginal-cost function is

$$\frac{dc}{dq} = 0.003q^2 - 0.4q + 40$$

where q is the number of units produced. If marginal cost is \$27.50 when $q = 50$ and fixed costs are \$5000, what is the *average* cost of producing 100 units?

22. If $f''(x) = 30x^4 + 12x$ and $f'(1) = 10$, evaluate

$$f(965.335245) - f(-965.335245)$$

Objective

To learn and apply the formulas for $\int u^a\,du$, $\int e^u\,du$, and $\int \frac{1}{u}\,du$.

14.4 More Integration Formulas

Power Rule for Integration

The formula

$$\int x^a\,dx = \frac{x^{a+1}}{n+1} + C \qquad \text{if } a \neq -1$$

[2]Adapted from R. Bressani, "The Use of Yeast in Human Foods," in *Single-Cell Protein,* eds. R. I. Mateles and S. R. Tannenbaum (Cambridge, MA: MIT Press, 1968).

[3]Adapted from D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada,* no. 46 (1965).

[4]R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

which applies to a power of x, can be generalized to handle a power of a *function* of x. Let u be a differentiable function of x. By the power rule for differentiation, if $a \neq -1$, then

$$\frac{d}{dx}\left(\frac{(u(x))^{a+1}}{a+1}\right) = \frac{(a+1)(u(x))^a \cdot u'(x)}{a+1} = (u(x))^a \cdot u'(x)$$

Thus,

$$\int (u(x))^a \cdot u'(x)\, dx = \frac{(u(x))^{a+1}}{a+1} + C \quad a \neq -1$$

We call this the *power rule for integration*. Note that $u'(x)dx$ is the differential of u, namely du. In mathematical shorthand, we can replace $u(x)$ by u and $u'(x)\, dx$ by du:

Power Rule for Integration

If u is differentiable, then

$$\int u^a\, du = \frac{u^{a+1}}{a+1} + C \qquad \text{if } a \neq -1 \qquad (1)$$

It is important to appreciate the difference between the power rule for integration and the formula for $\int x^a\, dx$. In the power rule, u represents a function, whereas in $\int x^a\, dx$, x is a variable.

EXAMPLE 1 Applying the Power Rule for Integration

a. Find $\displaystyle\int (x+1)^{20}\, dx$.

Solution: Since the integrand is a power of the function $x+1$, we will set $u = x+1$. Then $du = dx$, and $\int (x+1)^{20}\, dx$ has the form $\int u^{20}\, du$. By the power rule for integration,

$$\int (x+1)^{20}\, dx = \int u^{20}\, du = \frac{u^{21}}{21} + C = \frac{(x+1)^{21}}{21} + C$$

Note that we give our answer not in terms of u, but explicitly in terms of x.

b. Find $\displaystyle\int 3x^2(x^3+7)^3\, dx$.

Solution: We observe that the integrand contains a power of the function x^3+7. Let $u = x^3+7$. Then $du = 3x^2\, dx$. Fortunately, $3x^2$ appears as a factor in the integrand and we have

$$\int 3x^2(x^3+7)^3\, dx = \int (x^3+7)^3[3x^2\, dx] = \int u^3\, du$$

$$= \frac{u^4}{4} + C = \frac{(x^3+7)^4}{4} + C$$

After integrating, you may wonder what happened to $3x^2$. We note again that $du = 3x^2\, dx$.

Now Work Problem 3 ◁

In order to apply the power rule for integration, sometimes an adjustment must be made to obtain du in the integrand, as Example 2 illustrates.

EXAMPLE 2 Adjusting for *du*

Find $\displaystyle\int x\sqrt{x^2+5}\, dx$.

Solution: We can write this as $\int x(x^2+5)^{1/2}\, dx$. Notice that the integrand contains a power of the function x^2+5. If $u = x^2+5$, then $du = 2x\, dx$. Since the *constant* factor 2 in du does *not* appear in the integrand, this integral does not have the form $\int u^n\,du$. However, from $du = 2x\,dx$ we can write $x\,dx = \frac{du}{2}$ so that the integral becomes

$$\int x(x^2+5)^{1/2}\,dx = \int (x^2+5)^{1/2}[x\,dx] = \int u^{1/2}\frac{du}{2}$$

Moving the *constant* factor $\frac{1}{2}$ in front of the integral sign, we have

$$\int x(x^2+5)^{1/2}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C = \frac{1}{3}u^{3/2} + C$$

which in terms of x (as is required) gives

$$\int x\sqrt{x^2+5}\,dx = \frac{(x^2+5)^{3/2}}{3} + C$$

CAUTION!

The answer to an integration problem must be expressed in terms of the original variable.

Now Work Problem 15 ◁

In Example 2, the integrand $x\sqrt{x^2+5}$ missed being of the form $(u(x))^{1/2}u'(x)$ by the *constant factor* of 2. In general, if we have $\int (u(x))^a \frac{u'(x)}{k}dx$, for k a nonzero constant, then we can write

$$\int (u(x))^a \frac{u'(x)}{k}\,dx = \int u^a \frac{du}{k} = \frac{1}{k}\int u^a\,du$$

CAUTION!

We can adjust for constant factors, but not variable factors.

to simplify the integral, but such *adjustments* of the integrand are *not possible for variable factors.*

When using the form $\int u^a\,du$, do not neglect du. For example,

$$\int (4x+1)^2\,dx \neq \frac{(4x+1)^3}{3} + C$$

The correct way to do this problem is as follows. Let $u = 4x+1$, from which it follows that $du = 4\,dx$. Thus $dx = \frac{du}{4}$ and

$$\int (4x+1)^2\,dx = \int u^2\left[\frac{du}{4}\right] = \frac{1}{4}\int u^2\,du = \frac{1}{4}\cdot\frac{u^3}{3} + C = \frac{(4x+1)^3}{12} + C$$

EXAMPLE 3 Applying the Power Rule for Integration

a. Find $\int \sqrt[3]{6y}\,dy$.

Solution: The integrand is $(6y)^{1/3}$, a power of a function. However, in this case the obvious substitution $u = 6y$ can be avoided. More simply, we have

$$\int \sqrt[3]{6y}\,dy = \int 6^{1/3}y^{1/3}\,dy = \sqrt[3]{6}\int y^{1/3}\,dy = \sqrt[3]{6}\frac{y^{4/3}}{\frac{4}{3}} + C = \frac{3\sqrt[3]{6}}{4}y^{4/3} + C$$

b. Find $\int \frac{2x^3+3x}{(x^4+3x^2+7)^4}\,dx$.

Solution: We can write this as $\int (x^4+3x^2+7)^{-4}(2x^3+3x)\,dx$. Let us try to use the power rule for integration. If $u = x^4+3x^2+7$, then $du = (4x^3+6x)\,dx$, which is two times the quantity $(2x^3+3x)\,dx$ in the integral. Thus $(2x^3+3x)\,dx = \frac{du}{2}$ and we again illustrate the *adjustment* technique:

$$\int (x^4+3x^2+7)^{-4}[(2x^3+3x)\,dx] = \int u^{-4}\left[\frac{du}{2}\right] = \frac{1}{2}\int u^{-4}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{-3}}{-3} + C = -\frac{1}{6u^3} + C = -\frac{1}{6(x^4+3x^2+7)^3} + C$$

Now Work Problem 5 ◁

In using the power rule for integration, take care when making a choice for u. In Example 3(b), letting $u = 2x^3 + 3x$ does not lead very far. At times it may be necessary to try many different choices. Sometimes a wrong choice will provide a hint as to what does work. **Skill at integration comes only after many hours of practice and conscientious study.**

EXAMPLE 4 An Integral to Which the Power Rule Does Not Apply

Find $\int 4x^2(x^4 + 1)^2\,dx$.

Solution: If we set $u = x^4 + 1$, then $du = 4x^3\,dx$. To get du in the integral, we need an additional factor of the *variable* x. However, we can adjust only for **constant** factors. Thus, we cannot use the power rule. Instead, to find the integral, we will first expand $(x^4 + 1)^2$:

$$\int 4x^2(x^4 + 1)^2\,dx = 4\int x^2(x^8 + 2x^4 + 1)\,dx$$
$$= 4\int (x^{10} + 2x^6 + x^2)\,dx$$
$$= 4\left(\frac{x^{11}}{11} + \frac{2x^7}{7} + \frac{x^3}{3}\right) + C$$

Now Work Problem 67 ◁

Integrating Natural Exponential Functions

We now turn our attention to integrating exponential functions. If u is a differentiable function of x, then

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

CAUTION!

Do not apply the power-rule formula for $\int u^a\,du$ to $\int e^u\,du$.

Corresponding to this differentiation formula is the integration formula

$$\int e^u\frac{du}{dx}\,dx = e^u + C$$

But $\frac{du}{dx}\,dx$ is the differential of u, namely, du. Thus,

$$\int e^u\,du = e^u + C \qquad (2)$$

APPLY IT ▷

8. When an object is moved from one environment to another, its temperature T changes at a rate given by $\frac{dT}{dt} = kCe^{kt}$, where t is the time (in hours) after changing environments, C is the temperature difference (original minus new) between the environments, and k is a constant. If the original environment is 70°, the new environment is 60°, and $k = -0.5$, find the general form of $T(t)$.

EXAMPLE 5 Integrals Involving Exponential Functions

a. Find $\int 2xe^{x^2}\,dx$.

Solution: Let $u = x^2$. Then $du = 2x\,dx$, and, by Equation (2),

$$\int 2xe^{x^2}\,dx = \int e^{x^2}[2x\,dx] = \int e^u\,du$$
$$= e^u + C = e^{x^2} + C$$

b. Find $\int (x^2+1)e^{x^3+3x}\,dx$.

Solution: If $u = x^3 + 3x$, then $du = (3x^2+3)\,dx = 3(x^2+1)\,dx$. If the integrand contained a factor of 3, the integral would have the form $\int e^u\,du$. Thus, we write

$$\begin{aligned}\int (x^2+1)e^{x^3+3x}\,dx &= \int e^{x^3+3x}[(x^2+1)\,dx]\\ &= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C\\ &= \frac{1}{3}e^{x^3+3x} + C\end{aligned}$$

where in the second step we replaced $(x^2+1)\,dx$ by $\frac{1}{3}\,du$ but wrote $\frac{1}{3}$ outside the integral.

Now Work Problem 41 ◁

Integrals Involving Logarithmic Functions

As we know, the power-rule formula $\int u^a\,du = u^{a+1}/(a+1) + C$ does not apply when $a = -1$. To handle that situation, namely, $\int u^{-1}\,du = \int \frac{1}{u}\,du$, we first recall from Section 12.1 that

$$\frac{d}{dx}(\ln|u|) = \frac{1}{u}\frac{du}{dx} \quad \text{for } u \neq 0$$

which gives us the integration formula

$$\int \frac{1}{u}\,du = \ln|u| + C \quad \text{for } u \neq 0 \tag{3}$$

In particular, if $u = x$, then $du = dx$, and

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{for } x \neq 0 \tag{4}$$

APPLY IT ▷

9. If the rate of vocabulary memorization of the average student in a foreign language is given by $\frac{dv}{dt} = \frac{35}{t+1}$, where v is the number of vocabulary words memorized in t hours of study, find the general form of $v(t)$.

EXAMPLE 6 Integrals Involving $\frac{1}{u}\,du$

a. Find $\int \frac{7}{x}\,dx$.

Solution: From Equation (4),

$$\int \frac{7}{x}\,dx = 7\int \frac{1}{x}\,dx = 7\ln|x| + C$$

Using properties of logarithms, we can write this answer another way:

$$\int \frac{7}{x}\,dx = \ln|x^7| + C$$

b. Find $\int \frac{2x}{x^2+5}\,dx$.

Solution: Let $u = x^2 + 5$. Then $du = 2x\,dx$. From Equation (3),

$$\begin{aligned}\int \frac{2x}{x^2+5}\,dx &= \int \frac{1}{x^2+5}[2x\,dx] = \int \frac{1}{u}\,du\\ &= \ln|u| + C = \ln|x^2+5| + C\end{aligned}$$

Since $x^2 + 5$ is always positive, we can omit the absolute-value bars:

$$\int \frac{2x}{x^2+5}\,dx = \ln(x^2+5) + C$$

Now Work Problem 31 ◁

EXAMPLE 7 An Integral Involving $\frac{1}{u}\,du$

Find $\displaystyle\int \frac{(2x^3+3x)\,dx}{x^4+3x^2+7}$.

Solution: If $u = x^4 + 3x^2 + 7$, then $du = (4x^3 + 6x)\,dx$, which is two times the numerator giving $(2x^3 + 3x)\,dx = \frac{du}{2}$. To apply Equation (3), we write

$$\begin{aligned}
\int \frac{2x^3+3x}{x^4+3x^2+7}\,dx &= \frac{1}{2}\int \frac{1}{u}\,du \\
&= \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|x^4+3x^2+7| + C && \text{Rewrite } u \text{ in terms of } x. \\
&= \frac{1}{2}\ln(x^4+3x^2+7) + C && x^4+3x^2+7>0 \quad \text{for all } x
\end{aligned}$$

Now Work Problem 51 ◁

EXAMPLE 8 An Integral Involving Two Forms

Find $\displaystyle\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw$.

Solution:

$$\begin{aligned}
\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw &= \int (1-w)^{-2}dw + \int \frac{1}{w-1}dw \\
&= -1\int (1-w)^{-2}[-dw] + \int \frac{1}{w-1}dw
\end{aligned}$$

The first integral has the form $\int u^{-2}\,du$, and the second has the form $\int \frac{1}{v}\,dv$. Thus,

$$\begin{aligned}
\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw &= -\frac{(1-w)^{-1}}{-1} + \ln|w-1| + C \\
&= \frac{1}{1-w} + \ln|w-1| + C
\end{aligned}$$

◁

PROBLEMS 14.4

In Problems 1–80, find the indefinite integrals.

1. $\int (x+5)^7\,dx$
2. $\int 15(x+2)^4\,dx$
3. $\int 2x(x^2+3)^5\,dx$
4. $\int (4x+3)(2x^2+3x+1)\,dx$
5. $\int (3y^2+6y)(y^3+3y^2+1)^{2/3}\,dy$
6. $\int (15t^2-6t+1)(5t^3-3t^2+t)^{17}\,dt$
7. $\int \frac{5}{(3x-1)^3}\,dx$
8. $\int \frac{4x}{(2x^2-7)^{10}}\,dx$
9. $\int \sqrt{7x+3}\,dx$
10. $\int \frac{1}{\sqrt{x-5}}\,dx$
11. $\int (7x-6)^4\,dx$
12. $\int x^2(3x^3+7)^3\,dx$
13. $\int u(5u^2-9)^{14}\,du$
14. $\int x\sqrt{3+5x^2}\,dx$
15. $\int 4x^4(27+x^5)^{1/3}\,dx$
16. $\int (4-5x)^9\,dx$
17. $\int 3e^{3x}\,dx$
18. $\int 5e^{3t+7}\,dt$
19. $\int (3t+1)e^{3t^2+2t+1}\,dt$
20. $\int -3w^2e^{-w^3}\,dw$
21. $\int xe^{7x^2}\,dx$
22. $\int x^3e^{4x^4}\,dx$
23. $\int 4e^{-3x}\,dx$
24. $\int 24x^5e^{-2x^6+7}\,dx$
25. $\int \frac{1}{x+5}\,dx$
26. $\int \frac{12x^2+4x+2}{x+x^2+2x^3}\,dx$
27. $\int \frac{3x^2+4x^3}{x^3+x^4}\,dx$
28. $\int \frac{6x^2-6x}{1-3x^2+2x^3}\,dx$
29. $\int \frac{8z}{(z^2-5)^7}\,dz$
30. $\int \frac{3}{(5v-1)^4}\,dv$

31. $\int \frac{4}{x}\,dx$

32. $\int \frac{3}{1+2y}\,dy$

33. $\int \frac{s^2}{s^3+5}\,ds$

34. $\int \frac{32x^3}{4x^4+9}\,dx$

35. $\int \frac{5}{4-2x}\,dx$

36. $\int \frac{7t}{5t^2-6}\,dt$

37. $\int \sqrt{5x}\,dx$

38. $\int \frac{1}{(3x)^6}\,dx$

39. $\int \frac{x}{\sqrt{ax^2+b}}\,dx$

40. $\int \frac{9}{1-3x}\,dx$

41. $\int 2y^3e^{y^4+1}\,dy$

42. $\int 2\sqrt{2x-1}\,dx$

43. $\int v^2e^{-2v^3+1}\,dv$

44. $\int \frac{x^2+x+1}{\sqrt[3]{x^3+\frac{3}{2}x^2+3x}}\,dx$

45. $\int (e^{-5x}+2e^x)\,dx$

46. $\int 4\sqrt[3]{y+1}\,dy$

47. $\int (8x+10)(7-2x^2-5x)^3\,dx$

48. $\int 2ye^{3y^2}\,dy$

49. $\int \frac{6x^2+8}{x^3+4x}\,dx$

50. $\int (e^x+2e^{-3x}-e^{5x})\,dx$

51. $\int \frac{16s-4}{3-2s+4s^2}\,ds$

52. $\int (6t^2+4t)(t^3+t^2+1)^6\,dt$

53. $\int x(2x^2+1)^{-1}\,dx$

54. $\int (45w^4+18w^2+12)(3w^5+2w^3+4)^{-4}\,dw$

55. $\int -(x^2-2x^5)(x^3-x^6)^{-10}\,dx$

56. $\int \frac{3}{5}(v-2)e^{2-4v+v^2}\,dv$

57. $\int (2x^3+x)(x^4+x^2)\,dx$

58. $\int (e^{3.1})^2\,dx$

59. $\int \frac{9+18x}{(5-x-x^2)^4}\,dx$

60. $\int (e^x-e^{-x})^2\,dx$

61. $\int x(2x+1)e^{4x^3+3x^2-4}\,dx$

62. $\int (u^3-ue^{6-3u^2})\,du$

63. $\int x\sqrt{(8-5x^2)^3}\,dx$

64. $\int e^{ax}\,dx$

65. $\int \left(\sqrt{2x}-\frac{1}{\sqrt{2x}}\right)dx$

66. $\int 3\frac{x^4}{e^{x^5}}\,dx$

67. $\int (x^2+1)^2\,dx$

68. $\int \left[x(x^2-16)^2-\frac{1}{2x+5}\right]dx$

69. $\int \left(\frac{x}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx$

70. $\int \left[\frac{3}{x-1}+\frac{1}{(x-1)^2}\right]dx$

71. $\int \left[\frac{2}{4x+1}-(4x^2-8x^5)(x^3-x^6)^{-8}\right]dx$

72. $\int (r^3+5)^2\,dr$

73. $\int \left[\sqrt{3x+1}-\frac{x}{x^2+3}\right]dx$

74. $\int \left(\frac{x}{7x^2+2}-\frac{x^2}{(x^3+2)^4}\right)dx$

75. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

76. $\int (e^5-3^e)\,dx$

77. $\int \frac{1+e^{2x}}{4e^x}\,dx$

78. $\int \frac{2}{t^2}\sqrt{\frac{1}{t}+9}\,dt$

79. $\int \frac{4x+3}{2x^2+3x}\ln(2x^2+3x)\,dx$

80. $\int \sqrt[3]{x}e^{\sqrt[3]{8x^4}}\,dx$

In Problems 81–84, find y subject to the given conditions.

81. $y'=(3-2x)^2;\quad y(0)=1$

82. $y'=\frac{x}{x^2+6};\quad y(1)=0$

83. $y''=\frac{1}{x^2};\quad y'(-2)=3, y(1)=2$

84. $y''=(x+1)^{1/2};\quad y'(8)=19, y(24)=\frac{2572}{3}$

85. Real Estate The rate of change of the value of a house that cost \$350,000 to build can be modeled by $\frac{dV}{dt}=8e^{0.05t}$, where t is the time in years since the house was built and V is the value (in thousands of dollars) of the house. Find $V(t)$.

86. Life Span If the rate of change of the expected life span l at birth of people born in the United States can be modeled by $\frac{dl}{dt}=\frac{12}{2t+50}$, where t is the number of years after 1940 and the expected life span was 63 years in 1940, find the expected life span for people born in 1998.

87. Oxygen in Capillary In a discussion of the diffusion of oxygen from capillaries,[5] concentric cylinders of radius r are used as a model for a capillary. The concentration C of oxygen in the capillary is given by

$$C=\int\left(\frac{Rr}{2K}+\frac{B_1}{r}\right)dr$$

where R is the constant rate at which oxygen diffuses from the capillary, and K and B_1 are constants. Find C. (Write the constant of integration as B_2.)

88. Find $f(2)$ if $f\left(\frac{1}{3}\right)=2$ and $f'(x)=e^{3x+2}-3x$.

Objective

To discuss techniques of handling more challenging integration problems, namely, by algebraic manipulation and by fitting the integrand to a familiar form. To integrate an exponential function with a base different from *e* and to find the consumption function, given the marginal propensity to consume.

14.5 Techniques of Integration

We turn now to some more difficult integration problems.

When integrating fractions, sometimes a preliminary division is needed to get familiar integration forms, as the next example shows.

[5] W. Simon, *Mathematical Techniques for Physiology and Medicine* (New York: Academic Press, Inc., 1972).

EXAMPLE 1 Preliminary Division before Integration

a. Find $\int \frac{x^3+x}{x^2}\,dx$.

Solution: A familiar integration form is not apparent. However, we can break up the integrand into two fractions by dividing each term in the numerator by the denominator. We then have

Here we split up the integrand.

$$\int \frac{x^3+x}{x^2}\,dx = \int \left(\frac{x^3}{x^2}+\frac{x}{x^2}\right) dx = \int \left(x+\frac{1}{x}\right) dx$$

$$= \frac{x^2}{2} + \ln|x| + C$$

b. Find $\int \frac{2x^3+3x^2+x+1}{2x+1}\,dx$.

Solution: Here the integrand is a quotient of polynomials in which the degree of the numerator is greater than or equal to that of the denominator. In such a situation we first use long division. Recall that if f and g are polynomials, with the degree of f greater than or equal to the degree of g, then long division allows us to find (uniquely) polynomials q and r, where either r is the zero polynomial or the degree of r is strictly less than the degree of g, satisfying

$$\frac{f}{g} = q + \frac{r}{g}$$

Using an obvious, abbreviated notation, we see that

$$\int \frac{f}{g} = \int \left(q+\frac{r}{g}\right) = \int q + \int \frac{r}{g}$$

Since integrating a polynomial is easy, we see that integrating rational functions reduces to the task of integrating *proper rational functions*—those for which the degree of the numerator is strictly less than the degree of the denominator. In this case we obtain

Here we used long division to rewrite the integrand.

$$\int \frac{2x^3+3x^2+x+1}{2x+1}\,dx = \int \left(x^2+x+\frac{1}{2x+1}\right) dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+\int \frac{1}{2x+1}\,dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+\frac{1}{2}\int \frac{1}{2x+1}\,d(2x+1)$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+\frac{1}{2}\ln|2x+1|+C$$

Now Work Problem 1 ◁

EXAMPLE 2 Indefinite Integrals

a. Find $\int \frac{1}{\sqrt{x}(\sqrt{x}-2)^3}\,dx$.

Solution: We can write this integral as $\int \frac{(\sqrt{x}-2)^{-3}}{\sqrt{x}}\,dx$. Let us try the power rule for integration with $u = \sqrt{x}-2$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, so that $\frac{dx}{\sqrt{x}} = 2\,du$, and

Here the integral is fit to the form to which the power rule for integration applies.

$$\int \frac{(\sqrt{x}-2)^{-3}}{\sqrt{x}}\,dx = \int (\sqrt{x}-2)^{-3}\left[\frac{dx}{\sqrt{x}}\right]$$

$$= 2\int u^{-3}\,du = 2\left(\frac{u^{-2}}{-2}\right)+C$$

$$= -\frac{1}{u^2}+C = -\frac{1}{(\sqrt{x}-2)^2}+C$$

b. Find $\int \frac{1}{x \ln x}\, dx$.

Here the integral fits the familiar form $\int \frac{1}{u}\, du$.

Solution: If $u = \ln x$, then $du = \frac{1}{x}\, dx$, and

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{\ln x}\left(\frac{1}{x}\, dx\right) = \int \frac{1}{u}\, du$$
$$= \ln|u| + C = \ln|\ln x| + C$$

c. Find $\int \frac{5}{w(\ln w)^{3/2}}\, dw$.

Here the integral is fit to the form to which the power rule for integration applies.

Solution: If $u = \ln w$, then $du = \frac{1}{w} dw$. Applying the power rule for integration, we have

$$\int \frac{5}{w(\ln w)^{3/2}} dw = 5\int (\ln w)^{-3/2}\left[\frac{1}{w} dw\right]$$
$$= 5\int u^{-3/2}\, du = 5 \cdot \frac{u^{-1/2}}{-\frac{1}{2}} + C$$
$$= \frac{-10}{u^{1/2}} + C = -\frac{10}{(\ln w)^{1/2}} + C$$

Now Work Problem 23 ◁

Integrating b^u

In Section 14.4, we integrated an exponential function to the base e:

$$\int e^u\, du = e^u + C$$

Now let us consider the integral of an exponential function with an arbitrary base, b.

$$\int b^u\, du$$

To find this integral, we first convert to base e using

$$b^u = e^{(\ln b)u} \tag{1}$$

(as we did in many differentiation examples too). Example 3 will illustrate.

EXAMPLE 3 An Integral Involving b^u

Find $\int 2^{3-x}\, dx$.

Solution:

Strategy We want to integrate an exponential function to the base 2. To do this, we will first convert from base 2 to base e by using Equation (1).

$$\int 2^{3-x}\, dx = \int e^{(\ln 2)(3-x)}\, dx$$

The integrand of the second integral is of the form e^u, where $u = (\ln 2)(3 - x)$. Since $du = -\ln 2\, dx$, we can solve for dx and write

$$\int e^{(\ln 2)(3-x)}\, dx = -\frac{1}{\ln 2}\int e^u\, du$$
$$= -\frac{1}{\ln 2}e^u + C = -\frac{1}{\ln 2}e^{(\ln 2)(3-x)} + C = -\frac{1}{\ln 2}2^{3-x} + C$$

Thus,

$$\int 2^{3-x}\,dx = -\frac{1}{\ln 2}2^{3-x} + C$$

Notice that we expressed our answer in terms of an exponential function to the base 2, the base of the original integrand.

Now Work Problem 27 ◁

Generalizing the procedure described in Example 3, we can obtain a formula for integrating b^u:

$$\begin{aligned}\int b^u\,du &= \int e^{(\ln b)u}\,du \\ &= \frac{1}{\ln b}\int e^{(\ln b)u}\,d((\ln b)u) && \ln b \text{ is a constant} \\ &= \frac{1}{\ln b}e^{(\ln b)u} + C \\ &= \frac{1}{\ln b}b^u + C\end{aligned}$$

Hence, we have

$$\int b^u\,du = \frac{1}{\ln b}b^u + C$$

Applying this formula to the integral in Example 3 gives

$$\begin{aligned}&\int 2^{3-x}\,dx && b = 2, u = 3 - x \\ &= -\int 2^{3-x}\,d(3-x) && -d(3-x) = dx \\ &= -\frac{1}{\ln 2}2^{3-x} + C\end{aligned}$$

which is the same result that we obtained before.

Application of Integration

We will now consider an application of integration that relates a consumption function to the marginal propensity to consume.

EXAMPLE 4 Finding a Consumption Function from Marginal Propensity to Consume

For a certain country, the marginal propensity to consume is given by

$$\frac{dC}{dI} = \frac{3}{4} - \frac{1}{2\sqrt{3I}}$$

where consumption C is a function of national income I. Here I is expressed in large denominations of money. Determine the consumption function for the country if it is known that consumption is 10 ($C = 10$) when $I = 12$.

Solution: Since the marginal propensity to consume is the derivative of C, we have

$$\begin{aligned}C = C(I) &= \int\left(\frac{3}{4} - \frac{1}{2\sqrt{3I}}\right)dI = \int \frac{3}{4}dI - \frac{1}{2}\int (3I)^{-1/2}\,dI \\ &= \frac{3}{4}I - \frac{1}{2}\int (3I)^{-1/2}\,dI\end{aligned}$$

If we let $u = 3I$, then $du = 3\,dI = d(3I)$, and

$$C = \frac{3}{4}I - \left(\frac{1}{2}\right)\frac{1}{3}\int (3I)^{-1/2}\,d(3I)$$

$$= \frac{3}{4}I - \frac{1}{6}\frac{(3I)^{1/2}}{\frac{1}{2}} + K$$

$$C = \frac{3}{4}I - \frac{\sqrt{3I}}{3} + K$$

This is an example of an initial-value problem.

When $I = 12$, $C = 10$, so

$$10 = \frac{3}{4}(12) - \frac{\sqrt{3(12)}}{3} + K$$

$$10 = 9 - 2 + K$$

Thus, $K = 3$, and the consumption function is

$$C = \frac{3}{4}I - \frac{\sqrt{3I}}{3} + 3$$

Now Work Problem 61 ◁

PROBLEMS 14.5

In Problems 1–56, determine the indefinite integrals.

1. $\int \frac{2x^6 + 8x^4 - 4x}{2x^2}\,dx$

2. $\int \frac{9x^2 + 5}{3x}\,dx$

3. $\int (3x^2 + 2)\sqrt{2x^3 + 4x + 1}\,dx$

4. $\int \frac{x}{\sqrt[4]{x^2 + 1}}\,dx$

5. $\int \frac{3}{\sqrt{4 - 5x}}\,dx$

6. $\int \frac{2xe^{x^2}\,dx}{e^{x^2} - 2}$

7. $\int 4^{7x}\,dx$

8. $\int 5^t\,dt$

9. $\int 2x(7 - e^{x^2/4})\,dx$

10. $\int \frac{e^x + 1}{e^x}\,dx$

11. $\int \frac{6x^2 - 11x + 5}{3x - 1}\,dx$

12. $\int \frac{(3x + 2)(x - 4)}{x - 3}\,dx$

13. $\int \frac{5e^{2x}}{7e^{2x} + 4}\,dx$

14. $\int 6(e^{4-3x})^2\,dx$

15. $\int \frac{5e^{13/x}}{x^2}\,dx$

16. $\int \frac{2x^4 - 6x^3 + x - 2}{x - 2}\,dx$

17. $\int \frac{5x^3}{x^2 + 9}\,dx$

18. $\int \frac{5 - 4x^2}{3 + 2x}\,dx$

19. $\int \frac{(\sqrt{x} + 2)^2}{3\sqrt{x}}\,dx$

20. $\int \frac{5e^s}{1 + 3e^s}\,ds$

21. $\int \frac{5(x^{1/3} + 2)^4}{\sqrt[3]{x^2}}\,dx$

22. $\int \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}}\,dx$

23. $\int \frac{\ln x}{x}\,dx$

24. $\int \sqrt{t}(3 - t\sqrt{t})^{0.6}\,dt$

25. $\int \frac{r\sqrt{\ln(r^2 + 1)}}{r^2 + 1}\,dr$

26. $\int \frac{9x^5 - 6x^4 - ex^3}{7x^2}\,dx$

27. $\int \frac{3^{\ln x}}{x}\,dx$

28. $\int \frac{4}{x\ln(2x^2)}\,dx$

29. $\int x^2\sqrt{e^{x^3+1}}\,dx$

30. $\int \frac{ax + b}{cx + d}\,dx \quad c \neq 0$

31. $\int \frac{8}{(x + 3)\ln(x + 3)}\,dx$

32. $\int (e^{e^2} + x^e - 2x)\,dx$

33. $\int \frac{x^3 + x^2 - x - 3}{x^2 - 3}\,dx$

34. $\int \frac{4x\ln\sqrt{1 + x^2}}{1 + x^2}\,dx$

35. $\int \frac{12x^3\sqrt{\ln(x^4 + 1)^3}}{x^4 + 1}\,dx$

36. $\int 3(x^2 + 2)^{-1/2}xe^{\sqrt{x^2+2}}\,dx$

37. $\int \left(\frac{x^3 - 1}{\sqrt{x^4 - 4x}} - \ln 7\right)dx$

38. $\int \frac{x - x^{-2}}{x^2 + 2x^{-1}}\,dx$

39. $\int \frac{2x^4 - 8x^3 - 6x^2 + 4}{x^3}\,dx$

40. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

41. $\int \frac{x}{x + 1}\,dx$

42. $\int \frac{2x}{(x^2 + 1)\ln(x^2 + 1)}\,dx$

43. $\int \frac{xe^{x^2}}{\sqrt{e^{x^2} + 2}}\,dx$

44. $\int \frac{5}{(3x + 1)[1 + \ln(3x + 1)]^2}\,dx$

45. $\int \frac{(e^{-x} + 5)^3}{e^x}\,dx$

46. $\int \left[\frac{1}{8x + 1} - \frac{1}{e^x(8 + e^{-x})^2}\right]dx$

47. $\int (x^3 + ex)\sqrt{x^2 + e}\,dx$

48. $\int 3^{x\ln x}(1 + \ln x)\,dx$ [*Hint:* $\frac{d}{dx}(x\ln x) = 1 + \ln x$]

49. $\int \sqrt{x}\sqrt{(8x)^{3/2} + 3}\,dx$

50. $\int \frac{7}{x(\ln x)^{\pi}}\,dx$

51. $\int \frac{\sqrt{s}}{e^{\sqrt{s^3}}}\,ds$

52. $\int \frac{\ln^3 x}{3x}\,dx$

53. $\int e^{\ln(x^2+1)}\,dx$

54. $\int dx$

55. $\displaystyle\int \frac{\ln\left(\frac{e^x}{x}\right)}{x}\,dx$

56. $\displaystyle\int e^{f(x)+\ln(f'(x))}\,dx$ assuming $f' > 0$

In Problems 57 and 58, dr/dq is a marginal-revenue function. Find the demand function.

57. $\dfrac{dr}{dq} = \dfrac{200}{(q+2)^2}$

58. $\dfrac{dr}{dq} = \dfrac{900}{(2q+3)^3}$

In Problems 59 and 60, dc/dq is a marginal-cost function. Find the total-cost function if fixed costs in each case are 2000.

59. $\dfrac{dc}{dq} = \dfrac{20}{q+5}$

60. $\dfrac{dc}{dq} = 4e^{0.005q}$

In Problems 61–63, dC/dI represents the marginal propensity to consume. Find the consumption function subject to the given condition.

61. $\dfrac{dC}{dI} = \dfrac{1}{\sqrt{I}}$; $C(9) = 8$

62. $\dfrac{dC}{dI} = \dfrac{1}{2} - \dfrac{1}{2\sqrt{2I}}$; $C(2) = \dfrac{3}{4}$

63. $\dfrac{dC}{dI} = \dfrac{3}{4} - \dfrac{1}{6\sqrt{I}}$; $C(25) = 23$

64. Cost Function The marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = 10 - \frac{100}{q+10}$$

where c is the total cost in dollars when q units are produced. When 100 units are produced, the average cost is \$50 per unit. To the nearest dollar, determine the manufacturer's fixed cost.

65. Cost Function Suppose the marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = \frac{100q^2 - 3998q + 60}{q^2 - 40q + 1}$$

where c is the total cost in dollars when q units are produced.

(a) Determine the marginal cost when 40 units are produced.
(b) If fixed costs are \$10,000, find the total cost of producing 40 units.
(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 42 units.

66. Cost Function The marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = \frac{9}{10}\sqrt{q}\sqrt{0.04q^{3/4} + 4}$$

where c is the total cost in dollars when q units are produced. Fixed costs are \$360.

(a) Determine the marginal cost when 25 units are produced.
(b) Find the total cost of producing 25 units.
(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 23 units.

67. Value of Land It is estimated that t years from now the value V (in dollars) of an acre of land near the ghost town of Cherokee, California, will be increasing at the rate of $\dfrac{8t^3}{\sqrt{0.2t^4 + 8000}}$ dollars per year. If the land is currently worth \$500 per acre, how much will it be worth in 10 years? Express your answer to the nearest dollar.

68. Revenue Function The marginal-revenue function for a manufacturer's product is of the form

$$\frac{dr}{dq} = \frac{a}{e^q + b}$$

for constants a and b, where r is the total revenue received (in dollars) when q units are produced and sold. Find the demand function, and express it in the form $p = f(q)$. (*Hint:* Rewrite dr/dq by multiplying both numerator and denominator by e^{-q}.)

69. Savings A certain country's marginal propensity to save is given by

$$\frac{dS}{dI} = \frac{5}{(I+2)^2}$$

where S and I represent total national savings and income, respectively, and are measured in billions of dollars. If total national consumption is \$7.5 billion when total national income is \$8 billion, for what value(s) of I is total national savings equal to zero?

70. Consumption Function A certain country's marginal propensity to save is given by

$$\frac{dS}{dI} = \frac{2}{5} - \frac{1.6}{\sqrt[3]{2I^2}}$$

where S and I represent total national savings and income, respectively, and are measured in billions of dollars.

(a) Determine the marginal propensity to consume when total national income is \$16 billion.
(b) Determine the consumption function, given that savings are \$10 billion when total national income is \$54 billion.
(c) Use the result in part (b) to show that consumption is $\$\frac{82}{5} = 16.4$ billion when total national income is \$16 billion (a deficit situation).
(d) Use differentials and the results in parts (a) and (c) to approximate consumption when total national income is \$18 billion.

14.6 The Definite Integral

Objective

To motivate, by means of the concept of area, the definite integral as a limit of a special sum; to evaluate simple definite integrals by using a limiting process.

Figure 14.2 shows the region R bounded by the lines $y = f(x) = 2x$, $y = 0$ (the x-axis), and $x = 1$. The region is simply a right triangle. If b and h are the lengths of the base and the height, respectively, then, from geometry, the area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(1)(2) = 1$ square unit. (Henceforth, we will treat areas as pure numbers and write *square unit* only if it seems necessary for emphasis.) We will now find this area by another method, which, as we will see later, applies to more complex regions. This method involves the summation of areas of rectangles.

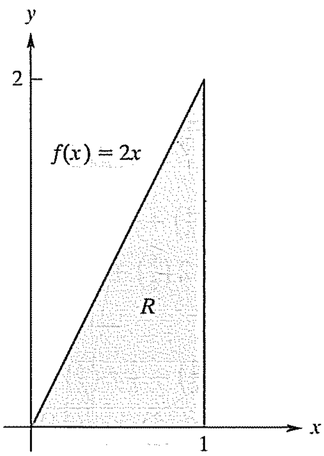


FIGURE 14.2 Region bounded by $f(x) = 2x$, $y = 0$, and $x = 1$.

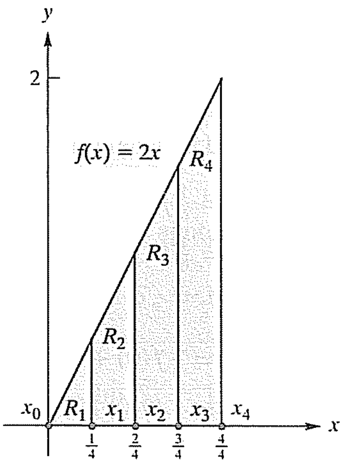


FIGURE 14.3 Four subregions of R.

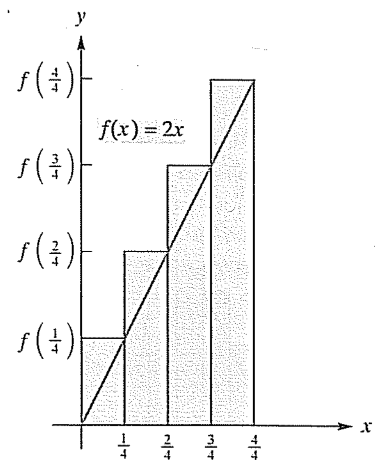


FIGURE 14.4 Four circumscribed rectangles.

Let us divide the interval [0, 1] on the x-axis into four subintervals of equal length by means of the equally spaced points $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4}$, and $x_4 = \frac{4}{4} = 1$. (See Figure 14.3.) Each subinterval has length $\Delta x = \frac{1}{4}$. These subintervals determine four subregions of R: R_1, R_2, R_3, and R_4, as indicated.

With each subregion, we can associate a *circumscribed* rectangle (Figure 14.4)—that is, a rectangle whose base is the corresponding subinterval and whose height is the *maximum* value of $f(x)$ on that subinterval. Since f is an increasing function, the maximum value of $f(x)$ on each subinterval occurs when x is the right-hand endpoint. Thus, the areas of the circumscribed rectangles associated with regions R_1, R_2, R_3, and R_4 are $\frac{1}{4}f(\frac{1}{4})$, $\frac{1}{4}f(\frac{2}{4})$, $\frac{1}{4}f(\frac{3}{4})$, and $\frac{1}{4}f(\frac{4}{4})$, respectively. The area of each rectangle is an approximation to the area of its corresponding subregion. Hence, the sum of the areas of these rectangles, denoted by $\overline{S}_4$ (read "S upper bar sub 4" or "the fourth upper sum"), approximates the area A of the triangle. We have

$$\begin{aligned}\overline{S}_4 &= \tfrac{1}{4}f\left(\tfrac{1}{4}\right) + \tfrac{1}{4}f\left(\tfrac{2}{4}\right) + \tfrac{1}{4}f\left(\tfrac{3}{4}\right) + \tfrac{1}{4}f\left(\tfrac{4}{4}\right)\\ &= \tfrac{1}{4}\left(2\left(\tfrac{1}{4}\right) + 2\left(\tfrac{2}{4}\right) + 2\left(\tfrac{3}{4}\right) + 2\left(\tfrac{4}{4}\right)\right) = \tfrac{5}{4}\end{aligned}$$

You can verify that $\overline{S}_4 = \sum_{i=1}^{4} f(x_i)\Delta x$. The fact that $\overline{S}_4$ is greater than the actual area of the triangle might have been expected, since $\overline{S}_4$ includes areas of shaded regions that are not in the triangle. (See Figure 14.4.)

On the other hand, with each subregion we can also associate an *inscribed* rectangle (Figure 14.5)—that is, a rectangle whose base is the corresponding subinterval, but whose height is the *minimum* value of $f(x)$ on that subinterval. Since f is an increasing function, the minimum value of $f(x)$ on each subinterval will occur when x is the left-hand endpoint. Thus, the areas of the four inscribed rectangles associated with R_1, R_2, R_3, and R_4 are $\frac{1}{4}f(0)$, $\frac{1}{4}f(\frac{1}{4})$, $\frac{1}{4}f(\frac{2}{4})$, and $\frac{1}{4}f(\frac{3}{4})$, respectively. Their sum, denoted $\underline{S}_4$ (read "S lower bar sub 4" or "the fourth lower sum"), is also an approximation to the area A of the triangle. We have

$$\begin{aligned}\underline{S}_4 &= \tfrac{1}{4}f(0) + \tfrac{1}{4}f\left(\tfrac{1}{4}\right) + \tfrac{1}{4}f\left(\tfrac{2}{4}\right) + \tfrac{1}{4}f\left(\tfrac{3}{4}\right)\\ &= \tfrac{1}{4}\left(2(0) + 2\left(\tfrac{1}{4}\right) + 2\left(\tfrac{2}{4}\right) + 2\left(\tfrac{3}{4}\right)\right) = \tfrac{3}{4}\end{aligned}$$

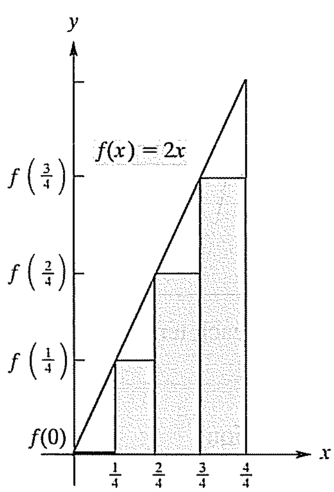


FIGURE 14.5 Four inscribed rectangles.

Using summation notation, we can write $\underline{S}_4 = \sum_{i=0}^{3} f(x_i)\Delta x$. Note that $\underline{S}_4$ is less than the area of the triangle, because the rectangles do not account for the portion of the triangle that is not shaded in Figure 14.5.

Since

$$\frac{3}{4} = \underline{S}_4 \leq A \leq \overline{S}_4 = \frac{5}{4}$$

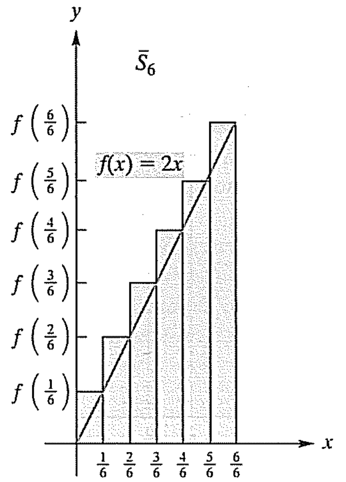


FIGURE 14.6 Six circumscribed rectangles.

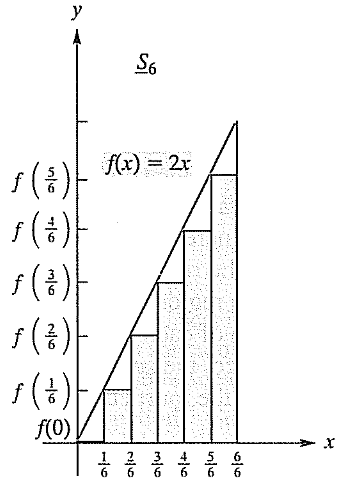


FIGURE 14.7 Six inscribed rectangles.

TO REVIEW summation notation, refer to Section 1.5.

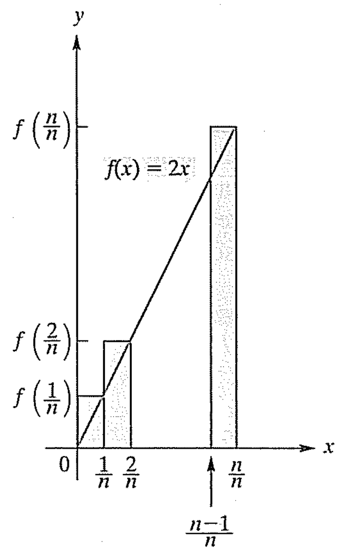


FIGURE 14.8 n circumscribed rectangles.

we say that $\underline{S}_4$ is an approximation to A from *below* and $\overline{S}_4$ is an approximation to A from *above*.

If [0, 1] is divided into more subintervals, we expect that better approximations to A will occur. To test this, let us use six subintervals of equal length $\Delta x = \frac{1}{6}$. Then $\overline{S}_6$, the total area of six circumscribed rectangles (see Figure 14.6), and $\underline{S}_6$, the total area of six inscribed rectangles (see Figure 14.7), are

$$\begin{aligned}\overline{S}_6 &= \tfrac{1}{6}f\left(\tfrac{1}{6}\right)+\tfrac{1}{6}f\left(\tfrac{2}{6}\right)+\tfrac{1}{6}f\left(\tfrac{3}{6}\right)+\tfrac{1}{6}f\left(\tfrac{4}{6}\right)+\tfrac{1}{6}f\left(\tfrac{5}{6}\right)+\tfrac{1}{6}f\left(\tfrac{6}{6}\right)\\ &= \tfrac{1}{6}\left(2\left(\tfrac{1}{6}\right)+2\left(\tfrac{2}{6}\right)+2\left(\tfrac{3}{6}\right)+2\left(\tfrac{4}{6}\right)+2\left(\tfrac{5}{6}\right)+2\left(\tfrac{6}{6}\right)\right)=\tfrac{7}{6}\end{aligned}$$

and

$$\begin{aligned}\underline{S}_6 &= \tfrac{1}{6}f(0)+\tfrac{1}{6}f\left(\tfrac{1}{6}\right)+\tfrac{1}{6}f\left(\tfrac{2}{6}\right)+\tfrac{1}{6}f\left(\tfrac{3}{6}\right)+\tfrac{1}{6}f\left(\tfrac{4}{6}\right)+\tfrac{1}{6}f\left(\tfrac{5}{6}\right)\\ &= \tfrac{1}{6}\left(2(0)+2\left(\tfrac{1}{6}\right)+2\left(\tfrac{2}{6}\right)+2\left(\tfrac{3}{6}\right)+2\left(\tfrac{4}{6}\right)+2\left(\tfrac{5}{6}\right)\right)=\tfrac{5}{6}\end{aligned}$$

Note that $\underline{S}_6 \le A \le \overline{S}_6$, and, with appropriate labeling, both $\overline{S}_6$ and $\underline{S}_6$ will be of the form $\Sigma f(x)\,\Delta x$. Clearly, using six subintervals gives better approximations to the area than does four subintervals, as expected.

More generally, if we divide [0, 1] into n subintervals of equal length Δx, then $\Delta x = 1/n$, and the endpoints of the subintervals are $x = 0, 1/n, 2/n, \ldots, (n-1)/n$, and $n/n = 1$. (See Figure 14.8.) The endpoints of the kth subinterval, for $k = 1, \ldots n$, are $(k-1)/n$ and k/n and the maximum value of f occurs at the right-hand endpoint k/n. It follows that the area of the kth circumscribed rectangle is $1/n \cdot f(k/n) = 1/n \cdot 2(k/n) = 2k/n^2$, for $k = 1, \ldots, n$. The total area of *all n circumscribed* rectangles is

$$\begin{aligned}\overline{S}_n &= \sum_{k=1}^{n} f(k/n)\Delta x = \sum_{k=1}^{n}\frac{2k}{n^2} && (1)\\ &= \frac{2}{n^2}\sum_{k=1}^{n}k && \text{by factoring } \frac{2}{n^2} \text{ from each term}\\ &= \frac{2}{n^2}\cdot\frac{n(n+1)}{2} && \text{from Section 1.5}\\ &= \frac{n+1}{n}\end{aligned}$$

(We recall that $\sum_{k=1}^{n} k = 1+2+\cdots+n$ is the sum of the first n positive integers and the formula used above was derived in Section 1.5 in anticipation of its application here.)

For *inscribed* rectangles, we note that the minimum value of f occurs at the left-hand endpoint, $(k-1)/n$, of $[(k-1)/n, k/n]$, so that the area of the kth inscribed rectangle is $1/n \cdot f(k-1/n) = 1/n \cdot 2((k-1)/n) = 2(k-1)/n^2$, for $k = 1, \ldots n$. The total area determined of *all n inscribed* rectangles (see Figure 14.9) is

$$\begin{aligned}\underline{S}_n &= \sum_{k=1}^{n} f((k-1)/n)\,\Delta x = \sum_{k=1}^{n}\frac{2(k-1)}{n^2} && (2)\\ &= \frac{2}{n^2}\sum_{k=1}^{n}k-1 && \text{by factoring } \frac{2}{n^2} \text{ from each term}\\ &= \frac{2}{n^2}\sum_{k=0}^{n-1}k && \text{adjusting the summation}\\ &= \frac{2}{n^2}\cdot\frac{(n-1)n}{2} && \text{adapted from Section 1.5}\\ &= \frac{n-1}{n}\end{aligned}$$

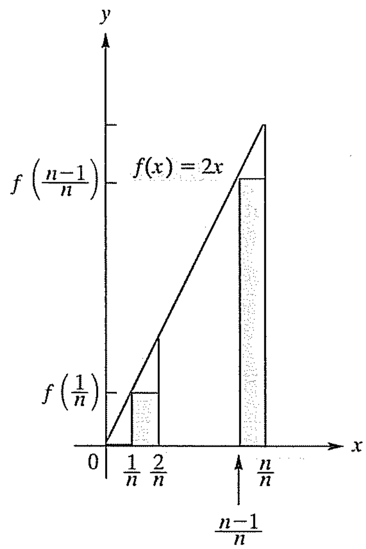


FIGURE 14.9 n inscribed rectangles.

From Equations (1) and (2), we again see that both $\overline{S}_n$ and $\underline{S}_n$ are sums of the form $\sum f(x)\Delta x$, namely, $\overline{S}_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\Delta x$ and $\underline{S}_n = \sum_{k=1}^{n} f\left(\frac{k-1}{n}\right)\Delta x$.

From the nature of $\overline{S}_n$ and $\underline{S}_n$, it seems reasonable—and it is indeed true—that

$$\underline{S}_n \le A \le \overline{S}_n$$

As n becomes larger, $\underline{S}_n$ and $\overline{S}_n$ become better approximations to A. In fact, let us take the limits of $\underline{S}_n$ and $\overline{S}_n$ as n approaches ∞ through positive integral values:

$$\lim_{n\to\infty} \underline{S}_n = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1$$

$$\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$$

Since $\overline{S}_n$ and $\underline{S}_n$ have the same limit, namely,

$$\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \underline{S}_n = 1 \tag{3}$$

and since

$$\underline{S}_n \le A \le \overline{S}_n$$

we will take this limit to be the area of the triangle. Thus $A = 1$, which agrees with our prior finding. It is important to understand that here we developed a *definition of the notion of area* that is applicable to many different regions.

We call the common limit of $\overline{S}_n$ and $\underline{S}_n$, namely, 1, the *definite integral* of $f(x) = 2x$ on the interval from $x = 0$ to $x = 1$, and we denote this quantity by writing

$$\int_0^1 2x\,dx = 1 \tag{4}$$

The reason for using the term *definite integral* and the symbolism in Equation (4) will become apparent in the next section. The numbers 0 and 1 appearing with the integral sign $\int$ in Equation (4) are called the *limits of integration;* 0 is the *lower limit* and 1 is the *upper limit.*

In general, for a function f defined on the interval from $x = a$ to $x = b$, where $a < b$, we can form the sums $\overline{S}_n$ and $\underline{S}_n$, which are obtained by considering the maximum and minimum values, respectively, on each of n subintervals of equal length Δx.[6] We can now state the following:

The common limit of $\overline{S}_n$ and $\underline{S}_n$ as $n \to \infty$, if it exists, is called the **definite integral** of f over $[a, b]$ and is written

$$\int_a^b f(x)\,dx$$

The numbers a and b are called **limits of integration;** a is the **lower limit** and b is the **upper limit.** The symbol x is called the **variable of integration** and $f(x)$ is the **integrand.**

In terms of a limiting process, we have

$$\sum f(x)\,\Delta x \to \int_a^b f(x)\,dx$$

The definite integral is the limit of sums of the form $\sum f(x)\,\Delta x$. This definition will be useful in later sections.

Two points must be made about the definite integral. First, the definite integral is the limit of a sum of the form $\sum f(x)\,\Delta x$. In fact, we can think of the integral sign as an elongated "S," the first letter of "Summation." Second, for an arbitrary function f

[6]Here we assume that the maximum and minimum values exist.

defined on an interval, we may be able to calculate the sums $\overline{S}_n$ and $\underline{S}_n$ and determine their common limit if it exists. However, some terms in the sums may be negative if $f(x)$ is negative at points in the interval. These terms are not areas of rectangles (an area is never negative), so the common limit may not represent an area. Thus, **the definite integral is nothing more than a real number; it may or may not represent an area.**

As we saw in Equation (3), $\lim_{n\to\infty} \underline{S}_n$ is equal to $\lim_{n\to\infty} \overline{S}_n$. For an arbitrary function, this is not always true. However, for the functions that we will consider, these limits will be equal, and the definite integral will always exist. To save time, we will just use the **right-hand endpoint** of each subinterval in computing a sum. For the functions in this section, this sum will be denoted S_n.

APPLY IT ▷

10. A company has determined that its marginal-revenue function is given by $R'(x) = 600 - 0.5x$, where R is the revenue (in dollars) received when x units are sold. Find the total revenue received for selling 10 units by finding the area in the first quadrant bounded by $y = R'(x) = 600 - 0.5x$ and the lines $y = 0, x = 0$, and $x = 10$.

In general, over $[a, b]$, we have

$$\Delta x = \frac{b-a}{n}$$

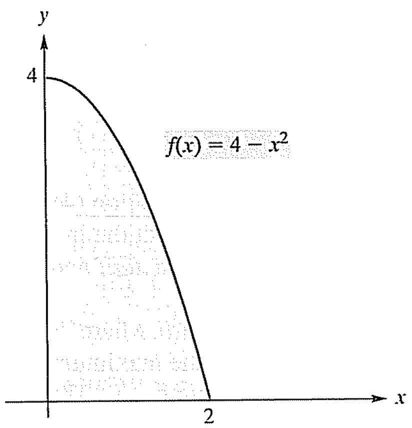


FIGURE 14.10 Region of Example 1.

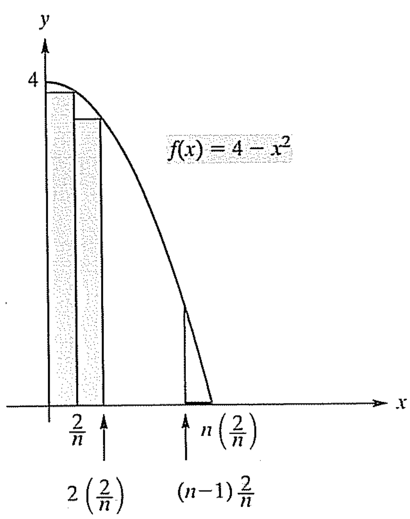


FIGURE 14.11 n subintervals and corresponding rectangles for Example 1.

EXAMPLE 1 Computing an Area by Using Right-Hand Endpoints

Find the area of the region in the first quadrant bounded by $f(x) = 4 - x^2$ and the lines $x = 0$ and $y = 0$.

Solution: A sketch of the region appears in Figure 14.10. The interval over which x varies in this region is seen to be $[0, 2]$, which we divide into n subintervals of equal length Δx. Since the length of $[0, 2]$ is 2, we take $\Delta x = 2/n$. The endpoints of the subintervals are $x = 0, 2/n, 2(2/n), \dots, (n-1)(2/n)$, and $n(2/n) = 2$, which are shown in Figure 14.11. The diagram also shows the corresponding rectangles obtained by using the right-hand endpoint of each subinterval. The area of the kth rectangle, for $k = 1, \dots n$, is the product of its width, $2/n$, and its height, $f(k(2/n)) = 4 - (2k/n)^2$, which is the function value at the right-hand endpoint of its base. Summing these areas, we get

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} f\left(k \cdot \left(\frac{2}{n}\right)\right) \Delta x = \sum_{k=1}^{n} \left(4 - \left(\frac{2k}{n}\right)^2\right) \frac{2}{n} \\
&= \sum_{k=1}^{n} \left(\frac{8}{n} - \frac{8k^2}{n^3}\right) = \sum_{k=1}^{n} \frac{8}{n} - \sum_{k=1}^{n} \frac{8k^2}{n^3} = \frac{8}{n} \sum_{k=1}^{n} 1 - \frac{8}{n^3} \sum_{k=1}^{n} k^2 \\
&= \frac{8}{n} n - \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= 8 - \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)
\end{aligned}$$

The second line of the preceding computations uses basic summation manipulations as discussed in Section 1.5. The third line uses two specific summation formulas, also from Section 1.5: The sum of n copies of 1 is n and the sum of the first n squares is $\frac{n(n+1)(2n+1)}{6}$.

Finally, we take the limit of the S_n as $n \to \infty$:

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \left(8 - \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)\right) \\
&= 8 - \frac{4}{3} \lim_{n\to\infty} \left(\frac{2n^2 + 3n + 1}{n^2}\right) \\
&= 8 - \frac{4}{3} \lim_{n\to\infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&= 8 - \frac{8}{3} = \frac{16}{3}
\end{aligned}$$

Hence, the area of the region is $\frac{16}{3}$.

Now Work Problem 7 ◁