12 Additional Differentiation Topics

After an uncomfortable trip in a vehicle, passengers sometimes describe the ride as "jerky." But what is jerkiness, exactly? What does it mean for, say, an engineer designing a new transportation system?

Travel in a straight line at a constant speed is called *uniform motion,* and there is nothing jerky about it. But if either the path or the speed changes, the ride may become jerky. Change in velocity over time is the derivative of velocity. Called acceleration, the change in velocity is the *second derivative* of position with respect to time—the derivative of the derivative of position. One of the important concepts covered in this chapter is that of a higher-order derivative, of which acceleration is an example.

But is acceleration responsible for jerkiness? The feeling of being jerked back and forth on a roller coaster is certainly related to acceleration. On the other hand, automotive magazines often praise a car for having *smooth* acceleration. So apparently acceleration has something to do with jerkiness but is not itself the cause.

The derivative of acceleration is the *third* derivative of position with respect to time. When this third derivative is large, the acceleration is changing rapidly. A roller coaster in a steady turn to the left is undergoing steady leftward acceleration. But when the coaster changes abruptly from a hard left turn to a hard right turn, the acceleration changes directions—and the riders experience a jerk. The third derivative of position is, in fact, so apt a measure of jerkiness that it is customarily called the *jerk,* just as the second derivative is called the acceleration.

Jerk has implications not only for passenger comfort in vehicles but also for equipment reliability. Engineers designing equipment for spacecraft, for instance, follow guidelines about the maximum jerk the equipment must be able to survive without damage to its internal components.

Objective

To develop a differentiation formula for $y = \ln u$, to apply the formula, and to use it to differentiate a logarithmic function to a base other than e.

12.1 Derivatives of Logarithmic Functions

So far, the only derivatives we have been able to calculate are those of functions that are constructed from power functions using multiplication by a constant, arithmetic operations, and composition. (As pointed out in Problem 65 of Section 11.6, we can calculate the derivative of a constant function c by writing $c = cx^0$; then

$$\frac{d}{dx}(c) = \frac{d}{dx}(cx^0) = c\frac{d}{dx}(x^0) = c \cdot 0x^{-1} = 0$$

Thus, we really have only one *basic* differention formula so far.) The logarithmic functions $\log_b x$ and the exponential functions b^x *cannot* be constructed from power functions using multiplication by a constant, arithmetic operations, and composition. It follows that we will need at least another truly *basic* differentiation formula.

In this section, we develop formulas for differentiating logarithmic functions. We begin with the derivative of $\ln x$, commenting further on the numbered steps at the end of the calculation.

$$\frac{d}{dx}(\ln x) \overset{(1)}{=} \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} \qquad \text{definition of derivative}$$

$$\overset{(2)}{=} \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} \qquad \text{since } \ln m - \ln n = \ln(m/n)$$

$$\overset{(3)}{=} \lim_{h\to 0}\left(\frac{1}{h}\ln\left(1+\frac{h}{x}\right)\right) \qquad \text{algebra}$$

$$\overset{(4)}{=} \lim_{h\to 0}\left(\frac{1}{x}\cdot\frac{x}{h}\ln\left(1+\frac{h}{x}\right)\right) \qquad \text{writing } \frac{1}{h} = \frac{1}{x}\cdot\frac{x}{h}$$

$$\overset{(5)}{=} \lim_{h\to 0}\left(\frac{1}{x}\ln\left(1+\frac{h}{x}\right)^{x/h}\right) \qquad \text{since } r\ln m = \ln m^r$$

$$\overset{(6)}{=} \frac{1}{x}\cdot\lim_{h\to 0}\left(\ln\left(1+\frac{h}{x}\right)^{x/h}\right) \qquad \text{by limit property 1 in Section 10.1}$$

$$\overset{(7)}{=} \frac{1}{x}\cdot\ln\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{x/h}\right) \qquad \ln \text{ is continuous}$$

$$\overset{(8)}{=} \frac{1}{x}\cdot\ln\left(\lim_{h/x\to 0}\left(1+\frac{h}{x}\right)^{x/h}\right) \qquad \text{for fixed } x > 0$$

$$\overset{(9)}{=} \frac{1}{x}\cdot\ln\left(\lim_{k\to 0}(1+k)^{1/k}\right) \qquad \text{setting } k = h/x$$

$$\overset{(10)}{=} \frac{1}{x}\cdot\ln(e) \qquad \text{as shown in Section 10.1}$$

$$\overset{(11)}{=} \frac{1}{x} \qquad \text{since } \ln e = 1$$

The calculation is long but following it step by step allows for review of many important ideas. Step (1) is the key definition introduced in Section 11.1. Steps (2), (5), and (11) involve properties found in 4.3. In step (3), labeled simply *algebra*, we use properties of fractions first given in 0.2. Step (4) is admittedly a *trick* that requires experience to discover. Note that necessarily $x \neq 0$ since x is in the domain of ln, which is $(0, \infty)$. To understand the justification for Step (6), we must observe that x, and hence $1/x$, is constant with respect to the limit variable h. We have already remarked in Section 10.3 that logarithmic functions are continuous and this is what allows us to interchange the processes of applying the ln function and taking a limit in (7). In (8) the point is that, for fixed

$x > 0$, h/x goes to 0 when h goes to 0 and conversely h goes to 0 when h/x goes to 0. Thus, we can regard h/x as a new limit variable, k, and this we do in step (9).

In conclusion, we have derived the following:

BASIC RULE 2 Derivative of $\ln x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \text{for } x > 0$$

Some care is required with this rule because while the left-hand side is defined only for $x > 0$, the right-hand side is defined for all $x \neq 0$. For $x < 0$, $\ln(-x)$ is defined and by the chain rule we have

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{-1}{-x} = \frac{1}{x} \quad \text{for } x < 0$$

We can combine the last two equations by using the absolute function to get

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \text{for } x \neq 0 \tag{1}$$

EXAMPLE 1 Differentiating Functions Involving $\ln x$

a. Differentiate $f(x) = 5\ln x$.

Solution: Here f is a constant (5) times a function ($\ln x$), so by Basic Rule 2, we have

$$f'(x) = 5\frac{d}{dx}(\ln x) = 5 \cdot \frac{1}{x} = \frac{5}{x} \quad \text{for } x > 0$$

b. Differentiate $y = \dfrac{\ln x}{x^2}$.

Solution: By the quotient rule and Basic Rule 2,

$$y' = \frac{x^2\frac{d}{dx}(\ln x) - (\ln x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2\left(\frac{1}{x}\right) - (\ln x)(2x)}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3} \quad \text{for } x > 0$$

Now Work Problem 1 ◁

The chain rule is used to develop the differentiation formula for $\ln|u|$.

We will now extend Equation (1) to cover a broader class of functions. Let $y = \ln|u|$, where u is a differentiable function of x. By the chain rule,

$$\frac{d}{dx}(\ln|u|) = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}(\ln|u|) \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx} \quad \text{for } u \neq 0$$

Thus,

$$\frac{d}{du}(\ln|u|) = \frac{1}{u} \cdot \frac{du}{dx} \quad \text{for } u \neq 0 \tag{2}$$

Of course, Equation (2) gives us $\dfrac{d}{du}(\ln u) = \dfrac{1}{u} \cdot \dfrac{du}{dx}$ for $u > 0$.

APPLY IT ▷

1. The supply of q units of a product at a price of p dollars per unit is given by $q(p) = 25 + 2\ln(3p^2 + 4)$. Find the rate of change of supply with respect to price, $\dfrac{dq}{dp}$.

EXAMPLE 2 Differentiating Functions Involving $\ln u$

a. Differentiate $y = \ln(x^2 + 1)$.

Solution: This function has the form $\ln u$ with $u = x^2 + 1$, and since $x^2 + 1 > 0$, for all x, $y = \ln(x^2 + 1)$ is defined for all x. Using Equation (2), we have

$$\frac{dy}{dx} = \frac{1}{x^2+1}\frac{d}{dx}(x^2+1) = \frac{1}{x^2+1}(2x) = \frac{2x}{x^2+1}$$

b. Differentiate $y = x^2 \ln(4x+2)$.

Solution: Using the product rule gives

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\ln(4x+2)) + (\ln(4x+2))\frac{d}{dx}(x^2)$$

By Equation (2) with $u = 4x+2$,

$$\frac{dy}{dx} = x^2\left(\frac{1}{4x+2}\right)(4) + (\ln(4x+2))(2x)$$

$$= \frac{2x^2}{2x+1} + 2x\ln(4x+2) \quad \text{for } 4x+2>0$$

Since $4x+2>0$ exactly when $x > -1/2$, we have

$$\frac{d}{dx}(x^2\ln(4x+2)) = \frac{2x^2}{2x+1} + 2x\ln(4x+2) \quad \text{for } x > -1/2$$

c. Differentiate $y = \ln|\ln|x||$.

Solution: This has the form $y = \ln|u|$ with $u = \ln|x|$. Using Equation (2), we obtain

$$y' = \frac{1}{\ln|x|}\frac{d}{dx}(\ln|x|) = \frac{1}{\ln|x|}\left(\frac{1}{x}\right) = \frac{1}{x\ln|x|} \quad \text{for } x, u \neq 0$$

Since $\ln|x| = 0$ when $x = -1, 1$, we have

$$\frac{d}{dx}(\ln|\ln|x||) = \frac{1}{x\ln|x|} \quad \text{for } x \neq -1, 0, 1$$

Now Work Problem 9 ◁

Frequently, we can reduce the work involved in differentiating the logarithm of a product, quotient, or power by using properties of logarithms to rewrite the logarithm *before* differentiating. The next example will illustrate.

EXAMPLE 3 Rewriting Logarithmic Functions before Differentiating

a. Find $\frac{dy}{dx}$ if $y = \ln(2x+5)^3$.

Solution: Here we have the logarithm of a power. First we simplify the right side by using properties of logarithms. Then we differentiate. We have

$$y = \ln(2x+5)^3 = 3\ln(2x+5) \quad \text{for } 2x+5>0$$

$$\frac{dy}{dx} = 3\left(\frac{1}{2x+5}\right)(2) = \frac{6}{2x+5} \quad \text{for } x > -5/2$$

Comparing both methods, we note that the easier one is to simplify first and then differentiate.

Alternatively, if the simplification were not performed first, we would write

$$\frac{dy}{dx} = \frac{1}{(2x+5)^3}\frac{d}{dx}((2x+5)^3)$$

$$= \frac{1}{(2x+5)^3}(3)(2x+5)^2(2) = \frac{6}{2x+5}$$

b. Find $f'(p)$ if $f(p) = \ln((p+1)^2(p+2)^3(p+3)^4)$.

Solution: We simplify the right side and then differentiate:

$$f(p) = 2\ln(p+1) + 3\ln(p+2) + 4\ln(p+3)$$

$$f'(p) = 2\left(\frac{1}{p+1}\right)(1) + 3\left(\frac{1}{p+2}\right)(1) + 4\left(\frac{1}{p+3}\right)(1)$$

$$= \frac{2}{p+1} + \frac{3}{p+2} + \frac{4}{p+3}$$

Now Work Problem 5 ◁

EXAMPLE 4 **Differentiating Functions Involving Logarithms**

a. Find $f'(w)$ if $f(w) = \ln\sqrt{\dfrac{1+w^2}{w^2-1}}$.

Solution: We simplify by using properties of logarithms and then differentiate:

$$f(w) = \frac{1}{2}(\ln(1+w^2) - \ln(w^2-1))$$

$$f'(w) = \frac{1}{2}\left(\frac{1}{1+w^2}(2w) - \frac{1}{w^2-1}(2w)\right)$$

$$= \frac{w}{1+w^2} - \frac{w}{w^2-1} = -\frac{2w}{w^4-1}$$

b. Find $f'(x)$ if $f(x) = \ln^3(2x+5)$.

Solution: The exponent 3 refers to the cubing of $\ln(2x+5)$. That is,

$$f(x) = \ln^3(2x+5) = [\ln(2x+5)]^3$$

CAUTION!

Do not confuse $\ln^3(2x+5)$ with $\ln(2x+5)^3$, which occurred in Example 3(a). It is advisable to write $\ln^3(2x+5)$ explicitly as $[\ln(2x+5)]^3$ and avoid $\ln^3(2x+5)$.

By the power rule,

$$f'(x) = 3(\ln(2x+5))^2\frac{d}{dx}(\ln(2x+5))$$

$$= 3(\ln(2x+5))^2\left(\frac{1}{2x+5}(2)\right)$$

$$= \frac{6}{2x+5}(\ln(2x+5))^2$$

Now Work Problem 39 ◁

Derivatives of Logarithmic Functions to the Base b

To differentiate a logarithmic function to a base different from e, we can first convert the logarithm to natural logarithms via the change-of-base formula and then differentiate the resulting expression. For example, consider $y = \log_b u$, where u is a differentiable function of x. By the change-of-base formula,

$$y = \log_b u = \frac{\ln u}{\ln b} \quad \text{for } u > 0$$

Differentiating, we have

CAUTION!

Note that $\ln b$ is just a constant!

$$\frac{d}{dx}(\log_b u) = \frac{d}{dx}\left(\frac{\ln u}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}(\ln u) = \frac{1}{\ln b}\cdot\frac{1}{u}\frac{du}{dx}$$

Summarizing,

$$\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)u}\cdot\frac{du}{dx} \quad \text{for } u > 0$$

Rather than memorize this rule, we suggest that you remember the procedure used to obtain it.

Procedure to Differentiate $\log_b u$

Convert $\log_b u$ to natural logarithms to obtain $\dfrac{\ln u}{\ln b}$, and then differentiate.

EXAMPLE 5 **Differentiating a Logarithmic Function to the Base 2**

Differentiate $y = \log_2 x$.

Solution: Following the foregoing procedure, we have

$$\frac{d}{dx}(\log_2 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 2}\right) = \frac{1}{\ln 2}\frac{d}{dx}(\ln x) = \frac{1}{(\ln 2)x}$$

It is worth mentioning that we can write our answer in terms of the original base. Because

$$\frac{1}{\ln b} = \frac{1}{\frac{\log_b b}{\log_b e}} = \frac{\log_b e}{1} = \log_b e$$

we can express $\frac{1}{(\ln 2)x}$ as $\frac{\log_2 e}{x}$. More generally, $\frac{d}{dx}(\log_b u) = \frac{\log_b e}{u} \cdot \frac{du}{dx}$.

Now Work Problem 15 ◁

APPLY IT ▷

2. The intensity of an earthquake is measured on the Richter scale. The reading is given by $R = \log \frac{I}{I_0}$, where I is the intensity and I_0 is a standard minimum intensity. If $I_0 = 1$, find $\frac{dR}{dI}$, the rate of change of the Richter-scale reading with respect to the intensity.

EXAMPLE 6 **Differentiating a Logarithmic Function to the Base 10**

If $y = \log(2x + 1)$, find the rate of change of y with respect to x.

Solution: The rate of change is dy/dx, and the base involved is 10. Therefore, we have

$$\frac{dy}{dx} = \frac{d}{dx}(\log(2x+1)) = \frac{d}{dx}\left(\frac{\ln(2x+1)}{\ln 10}\right)$$
$$= \frac{1}{\ln 10} \cdot \frac{1}{2x+1}(2) = \frac{2}{\ln 10(2x+1)}$$

◁

PROBLEMS 12.1

In Problems 1–44, differentiate the functions. If possible, first use properties of logarithms to simplify the given function.

1. $y = a \ln x$ **2.** $y = \frac{5 \ln x}{9}$ **3.** $y = \ln(3x - 7)$

4. $y = \ln(5x - 6)$ **5.** $y = \ln x^2$

6. $y = \ln(5x^3 + 3x^2 + 2x + 1)$ **7.** $y = \ln(1 - x^2)$

8. $y = \ln(-x^2 + 6x)$ **9.** $f(X) = \ln(4X^6 + 2X^3)$

10. $f(r) = \ln(2r^4 - 3r^2 + 2r + 1)$

11. $f(t) = t \ln t - t$ **12.** $y = x^2 \ln x$

13. $y = x^3 \ln(2x + 5)$ **14.** $y = (ax + b)^3 \ln(ax + b)$

15. $y = \log_3(8x - 1)$ **16.** $f(w) = \log(w^2 + 2w + 1)$

17. $y = x^2 + \log_2(x^2 + 4)$ **18.** $y = x^2 \log_2 x$

19. $f(z) = \frac{\ln z}{z}$ **20.** $y = \frac{x^2}{\ln x}$

21. $y = \frac{x^4 + 3x^2 + x}{\ln x}$ **22.** $y = \ln x^{100}$

23. $y = \ln(x^2 + 4x + 5)^3$ **24.** $y = 6 \ln \sqrt[3]{x}$

25. $y = 9 \ln \sqrt{1 + x^2}$ **26.** $f(t) = \ln\left(\frac{t^4}{1 + 6t + t^2}\right)$

27. $f(l) = \ln\left(\frac{1 + l}{1 - l}\right)$ **28.** $y = \ln\left(\frac{2x + 3}{3x - 4}\right)$

29. $y = \ln \sqrt[4]{\frac{1 + x^2}{1 - x^2}}$ **30.** $y = \ln \sqrt[3]{\frac{x^3 - 1}{x^3 + 1}}$

31. $y = \ln[(ax^2 + bx + c)^p(hx^2 + kx + l)^q]$

32. $y = \ln[(5x + 2)^4(8x - 3)^6]$ **33.** $y = 13 \ln\left(x^2 \sqrt[3]{5x + 2}\right)$

34. $y = 6 \ln \frac{x}{\sqrt{2x + 1}}$ **35.** $y = (x^2 + 1)\ln(2x + 1)$

36. $y = (ax^2 + bx + c)\ln(hx^2 + kx + l)$

37. $y = \ln x^3 + \ln^3 x$ **38.** $y = x^{\ln 2}$

39. $y = \ln^4(ax)$ **40.** $y = \ln^2(2x + 11)$

41. $y = \ln \sqrt{f(x)}$ **42.** $y = \ln\left(x^3 \sqrt[4]{2x + 1}\right)$

43. $y = \sqrt{4 + 3 \ln x}$ **44.** $y = \ln\left(x + \sqrt{1 + x^2}\right)$

45. Find an equation of the tangent line to the curve

$$y = \ln(x^2 - 3x - 3)$$

when $x = 4$.

46. Find an equation of the tangent line to the curve

$$y = x \ln x - x$$

at the point where $x = 1$.

47. Find the slope of the curve $y = \frac{x}{\ln x}$ when $x = 3$.

48. Marginal Revenue Find the marginal-revenue function if the demand function is $p = 25/\ln(q + 2)$.

49. Marginal Cost A total-cost function is given by

$$c = 25 \ln(q + 1) + 12$$

Find the marginal cost when $q = 6$.

50. Marginal Cost A manufacturer's average-cost function, in dollars, is given by

$$\bar{c} = \frac{500}{\ln(q+20)}$$

Find the marginal cost (rounded to two decimal places) when $q = 50$.

51. Supply Change The supply of q units of a product at a price of p dollars per unit is given by $q(p) = 27 + 11\ln(2p+1)$. Find the rate of change of supply with respect to price, $\frac{dq}{dp}$.

52. Sound Perception The loudness of sound L, measured in decibels, perceived by the human ear depends upon intensity levels I according to $L = 10\log\frac{I}{I_0}$, where I_0 is the standard threshold of audibility. If $I_0 = 17$, find $\frac{dL}{dI}$, the rate of change of the loudness with respect to the intensity.

53. Biology In a certain experiment with bacteria, it is observed that the relative activeness of a given bacteria colony is described by

$$A = 6\ln\left(\frac{T}{a-T} - a\right)$$

where a is a constant and T is the surrounding temperature. Find the rate of change of A with respect to T.

54. Show that the relative rate of change of $y = f(x)$ with respect to x is equal to the derivative of $y = \ln f(x)$.

55. Show that $\frac{d}{dx}(\log_b u) = \frac{1}{u}(\log_b e)\frac{du}{dx}$.

In Problems 56 and 57, use differentiation rules to find $f'(x)$. Then use your graphing calculator to find all roots of $f'(x) = 0$. Round your answers to two decimal places.

56. $f(x) = x^3 \ln x$

57. $f(x) = \frac{\ln(x^2)}{x^2}$

Objective

To develop a differentiation formula for $y = e^u$, to apply the formula, and to use it to differentiate an exponential function with a base other than e.

12.2 Derivatives of Exponential Functions

As we pointed out in Section 12.1, the exponential functions cannot be constructed from power functions using multiplication by a constant, arithmetic operations, and composition. However, the functions b^x, for $b > 0$ and $b \neq 1$, are inverse to the functions $\log_b(x)$, and if an invertible function f is differentiable, it is fairly easy to see that its inverse is differentiable. The key idea is that the graph of the inverse of a function is obtained by reflecting the graph of the original function in the line $y = x$. This reflection process preserves smoothness so that if the graph of an invertible function is smooth, then so is the graph of its inverse. Differentiating $f(f^{-1}(x)) = x$, we have

$$\frac{d}{dx}(f(f^{-1}(x))) = \frac{d}{dx}(x)$$

$$f'(f^{-1}(x))\frac{d}{dx}(f^{-1}(x)) = 1 \qquad \text{Chain Rule}$$

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

Thus we have

COMBINING RULE 6 Inverse Function Rule

If f is an invertible, differentiable function, then f^{-1} is differentiable and

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

As with the chain rule, Leibniz notation is well suited for inverse functions. Indeed, if $y = f^{-1}(x)$, then $\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x))$ and since $f(y) = x$, $f'(y) = \frac{dx}{dy}$. When we substitute these in Combining Rule 6, we get

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)} = \frac{1}{\frac{dx}{dy}}$$

so that Combining Rule 6 can be rewritten as

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad (1)$$

In the immediate case of interest, with $y = e^x$ so that $x = \ln y$ and $dx/dy = 1/y = 1/e^x$, we have

$$\frac{d}{dx}(e^x) = \frac{1}{\frac{1}{e^x}} = e^x$$

which we record as

$$\frac{d}{dx}(e^x) = e^x \tag{2}$$

CAUTION!

The power rule does not apply to e^x and other exponential functions, b^x. The power rule applies to power functions, x^a. Note the location of the variable.

For u a differentiable function of x, an application of the Chain Rule gives

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx} \tag{3}$$

EXAMPLE 1 **Differentiating Functions Involving e^x**

a. Find $\frac{d}{dx}(3e^x)$. Since 3 is a constant factor,

$$\begin{aligned}\frac{d}{dx}(3e^x) &= 3\frac{d}{dx}(e^x)\\ &= 3e^x \qquad \text{by Equation (2)}\end{aligned}$$

If a quotient can be easily rewritten as a product, then we can use the somewhat simpler product rule rather than the quotient rule.

b. If $y = \frac{x}{e^x}$, find $\frac{dy}{dx}$.

Solution: We could use first the quotient rule and then Equation (2), but it is a little easier first to rewrite the function as $y = xe^{-x}$ and use the product rule and Equation (3):

$$\frac{dy}{dx} = e^{-x}\frac{d}{dx}(x) + x\frac{d}{dx}(e^{-x}) = e^{-x}(1) + x(e^{-x})(-1) = e^{-x}(1-x) = \frac{1-x}{e^x}$$

c. If $y = e^2 + e^x + \ln 3$, find y'.

Solution: Since e^2 and $\ln 3$ are constants, $y' = 0 + e^x + 0 = e^x$.

Now Work Problem 1 ◁

APPLY IT ▷

3. When an object is moved from one environment to another, the change in temperature of the object is given by $T = Ce^{kt}$, where C is the temperature difference between the two environments, t is the time in the new environment, and k is a constant. Find the rate of change of temperature with respect to time.

EXAMPLE 2 **Differentiating Functions Involving e^u**

a. Find $\frac{d}{dx}\left(e^{x^3+3x}\right)$.

Solution: The function has the form e^u with $u = x^3 + 3x$. From Equation (2),

$$\begin{aligned}\frac{d}{dx}\left(e^{x^3+3x}\right) &= e^{x^3+3x}\frac{d}{dx}(x^3+3x) = e^{x^3+3x}(3x^2+3)\\ &= 3(x^2+1)e^{x^3+3x}\end{aligned}$$

$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$. Don't forget the $\frac{du}{dx}$.

b. Find $\frac{d}{dx}(e^{x+1}\ln(x^2+1))$.

Solution: By the product rule,

$$\begin{aligned}\frac{d}{dx}(e^{x+1}\ln(x^2+1)) &= e^{x+1}\frac{d}{dx}(\ln(x^2+1)) + (\ln(x^2+1))\frac{d}{dx}(e^{x+1})\\ &= e^{x+1}\left(\frac{1}{x^2+1}\right)(2x) + (\ln(x^2+1))e^{x+1}(1)\\ &= e^{x+1}\left(\frac{2x}{x^2+1} + \ln(x^2+1)\right)\end{aligned}$$

Now Work Problem 3 ◁

EXAMPLE 3 **The Normal-Distribution Density Function**

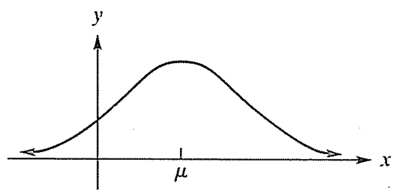

FIGURE 12.1 The normal-distribution density function.

An important function used in the social sciences is the **normal-distribution density function**

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)((x-\mu)/\sigma)^2}$$

where σ (a Greek letter read "sigma") and μ (a Greek letter read "mu") are constants. The graph of this function, called the normal curve, is bell shaped. (See Figure 12.1.) Determine the rate of change of y with respect to x when $x = \mu + \sigma$.

Solution: The rate of change of y with respect to x is dy/dx. We note that the factor $\dfrac{1}{\sigma\sqrt{2\pi}}$ is a constant and the second factor has the form e^u, where

$$u = -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

Thus,

$$\frac{dy}{dx} = \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-(1/2)((x-\mu)/\sigma)^2}\right)\left(-\frac{1}{2}(2)\left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\right)$$

Evaluating dy/dx when $x = \mu + \sigma$, we obtain

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=\mu+\sigma} &= \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-(1/2)((\mu+\sigma-\mu)/\sigma)^2}\right)\left(-\frac{\mu+\sigma-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\\ &= \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-(1/2)}\right)\left(-\frac{1}{\sigma}\right)\\ &= \frac{-e^{-(1/2)}}{\sigma^2\sqrt{2\pi}} = \frac{-1}{\sigma^2\sqrt{2\pi e}}\end{aligned}$$

◁

Differentiating Exponential Functions to the Base b

Now that we are familiar with the derivative of e^u, we consider the derivative of the more general exponential function b^u. Because $b = e^{\ln b}$, we can express b^u as an exponential function with the base e, a form we can differentiate. We have

$$\begin{aligned}\frac{d}{dx}(b^u) &= \frac{d}{dx}((e^{\ln b})^u) = \frac{d}{dx}(e^{(\ln b)u})\\ &= e^{(\ln b)u}\frac{d}{dx}((\ln b)u)\\ &= e^{(\ln b)u}(\ln b)\left(\frac{du}{dx}\right)\\ &= b^u(\ln b)\frac{du}{dx} \qquad \text{since } e^{(\ln b)u} = b^u\end{aligned}$$

Summarizing,

$$\frac{d}{dx}(b^u) = b^u(\ln b)\frac{du}{dx} \tag{4}$$

Note that if $b = e$, then the factor $\ln b$ in Equation (4) is 1. Thus, if exponential functions to the base e are used, we have a simpler differentiation formula with which to work. This is the reason natural exponential functions are used extensively in calculus. Rather than memorizing Equation (4), we advocate remembering the procedure for obtaining it.

Procedure to Differentiate b^u

Convert b^u to a natural exponential function by using the property that $b = e^{\ln b}$, and then differentiate.

The next example will illustrate this procedure.

EXAMPLE 4 Differentiating an Exponential Function with Base 4

Find $\frac{d}{dx}(4^x)$.

Solution: Using the preceding procedure, we have

$$\begin{aligned}\frac{d}{dx}(4^x) &= \frac{d}{dx}((e^{\ln 4})^x) \\ &= \frac{d}{dx}\left(e^{(\ln 4)x}\right) && \text{form}: \frac{d}{dx}(e^u) \\ &= e^{(\ln 4)x}(\ln 4) && \text{by Equation (2)} \\ &= 4^x(\ln 4)\end{aligned}$$

Verify the result by using Equation (4) directly.

Now Work Problem 15 ◁

EXAMPLE 5 Differentiating Different Forms

Find $\frac{d}{dx}\left(e^2 + x^e + 2^{\sqrt{x}}\right)$.

Solution: Here we must differentiate three different forms; do not confuse them! The first (e^2) is a constant base to a constant power, so it is a constant itself. Thus, its derivative is zero. The second (x^e) is a variable base to a constant power, so the power rule applies. The third ($2^{\sqrt{x}}$) is a constant base to a variable power, so we must differentiate an exponential function. Taken all together, we have

$$\begin{aligned}\frac{d}{dx}\left(e^2 + x^e + 2^{\sqrt{x}}\right) &= 0 + ex^{e-1} + \frac{d}{dx}\left[e^{(\ln 2)\sqrt{x}}\right] \\ &= ex^{e-1} + \left[e^{(\ln 2)\sqrt{x}}\right](\ln 2)\left(\frac{1}{2\sqrt{x}}\right) \\ &= ex^{e-1} + \frac{2^{\sqrt{x}}\ln 2}{2\sqrt{x}}\end{aligned}$$

Now Work Problem 17 ◁

EXAMPLE 6 Differentiating Power Functions Again

We have often used the rule $d/dx(x^a) = ax^{a-1}$, but we have only *proved* it for a a positive integer and a few other special cases. At least for $x > 0$, we can now improve our understanding of power functions, using Equation (2).

For $x > 0$, we can write $x^a = e^{a\ln x}$. So we have

$$\frac{d}{dx}(x^a) = \frac{d}{dx}e^{a\ln x} = e^{a\ln x}\frac{d}{dx}(a\ln x) = x^a(ax^{-1}) = ax^{a-1}$$

Now Work Problem 19 ◁

PROBLEMS 12.2

In Problems 1–28, differentiate the functions.

1. $y = 5e^x$

2. $y = \frac{ae^x}{b}$

3. $y = e^{2x^2+3}$

4. $y = e^{2x^2+5}$

5. $y = e^{9-5x}$

6. $f(q) = e^{-q^3+6q-1}$

7. $f(r) = e^{4r^3+5r^2+2r+6}$

8. $y = e^{x^2+6x^3+1}$

9. $y = xe^x$

10. $y = 3x^4e^{-x}$

11. $y = x^2e^{-x^2}$

12. $y = xe^{ax}$

13. $y = \frac{e^x + e^{-x}}{3}$

14. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

15. $y = 5^{2x^3}$

16. $y = 2^x x^2$

17. $f(w) = \frac{e^{aw}}{w^2 + w + 1}$

18. $y = e^{x-\sqrt{x}}$

19. $y = e^{1+\sqrt{x}}$

20. $y = (e^{2x} + 1)^3$

21. $y = x^5 - 5^x$

22. $f(z) = e^{1/z}$

23. $y = \frac{e^x - 1}{e^x + 1}$

24. $y = e^{2x}(x + 6)$

25. $y = \ln e^x$

26. $y = e^{-x} \ln x$

27. $y = x^x$

28. $y = \ln e^{4x+1}$

29. If $f(x) = ee^xe^{x^2}$, find $f'(-1)$.

30. If $f(x) = 5^{x^2 \ln x}$, find $f'(1)$.

31. Find an equation of the tangent line to the curve $y = e^x$ when $x = -2$.

32. Find an equation of the tangent line to the curve $y = e^x$ at the point $(1, e)$. Show that this tangent line passes through $(0, 0)$ and show that it is the only tangent line to $y = e^x$ that passes through $(0, 0)$.

For each of the demand equations in Problems 33 and 34, find the rate of change of price p with respect to quantity q. What is the rate of change for the indicated value of q?

33. $p = 15e^{-0.001q}$; $q = 500$

34. $p = 9e^{-5q/750}$; $q = 300$

In Problems 35 and 36, $\bar{c}$ is the average cost of producing q units of a product. Find the marginal-cost function and the marginal cost for the given values of q.

35. $\bar{c} = \frac{7000e^{q/700}}{q}$; $q = 350, q = 700$

36. $\bar{c} = \frac{850}{q} + 4000\frac{e^{(2q+6)/800}}{q}$; $q = 97, q = 197$

37. If $w = e^{x^2}$ and $x = \frac{t+1}{t-1}$, find $\frac{dw}{dt}$ when $t = 2$.

38. If $f'(x) = x^3$ and $u = e^x$, show that

$$\frac{d}{dx}[f(u)] = e^{4x}$$

39. Determine the value of the positive constant c if

$$\left.\frac{d}{dx}(c^x - x^c)\right|_{x=1} = 0$$

40. Calculate the relative rate of change of

$$f(x) = 10^{-x} + \ln(8 + x) + 0.01e^{x-2}$$

when $x = 2$. Round your answer to four decimal places.

41. Production Run For a firm, the daily output on the tth day of a production run is given by

$$q = 500(1 - e^{-0.2t})$$

Find the rate of change of output q with respect to t on the tenth day.

42. Normal-Density Function For the normal-density function

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

find $f'(-1)$.

43. Population The population, in millions, of the greater Seattle area t years from 1970 is estimated by $P = 1.92e^{0.0176t}$. Show that $dP/dt = kP$, where k is a constant. This means that the rate of change of population at any time is proportional to the population at that time.

44. Market Penetration In a discussion of diffusion of a new process into a market, Hurter and Rubenstein[1] refer to an equation of the form

$$Y = k\alpha^{\beta^t}$$

where Y is the cumulative level of diffusion of the new process at time t and k, α, and β are positive constants. Verify their claim that

$$\frac{dY}{dt} = k\alpha^{\beta^t}(\beta^t \ln \alpha) \ln \beta$$

45. Finance After t years, the value S of a principal of P dollars invested at the annual rate of r compounded continuously is given by $S = Pe^{rt}$. Show that the relative rate of change of S with respect to t is r.

46. Predator–Prey Relationship In an article concerning predators and prey, Holling[2] refers to an equation of the form

$$y = K(1 - e^{-ax})$$

where x is the prey density, y is the number of prey attacked, and K and a are constants. Verify his statement that

$$\frac{dy}{dx} = a(K - y)$$

47. Earthquakes According to Richter,[3] the number of earthquakes of magnitude M or greater per unit of time is given by $N = 10^A 10^{-bM}$, where A and b are constants. Find dN/dM.

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[1] A. P. Hurter, Jr., A. H. Rubenstein, et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change*, 11 (1978), 197–221.

[2] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist*, XCI, no. 7 (1959), 385–98.

[3] C. F. Richter, *Elementary Seismology* (San Francisco: W. H. Freeman and Company, Publishers, 1958).

48. Psychology Short-term retention was studied by Peterson and Peterson.[4] The two researchers analyzed a procedure in which an experimenter verbally gave a subject a three-letter consonant syllable, such as CHJ, followed by a three-digit number, such as 309. The subject then repeated the number and counted backward by 3's, such as 309, 306, 303, ... After a period of time, the subject was signaled by a light to recite the three-letter consonant syllable. The time between the experimenter's completion of the last consonant to the onset of the light was called the *recall interval*. The time between the onset of the light and the completion of a response was referred to as *latency*. After many trials, it was determined that, for a recall interval of t seconds, the approximate proportion of correct recalls with latency below 2.83 seconds was

$$p = 0.89[0.01 + 0.99(0.85)^t]$$

(a) Find dp/dt and interpret your result.
(b) Evaluate dp/dt when $t = 2$. Round your answer to two decimal places.

49. Medicine Suppose a tracer, such as a colored dye, is injected instantly into the heart at time $t = 0$ and mixes uniformly with blood inside the heart. Let the initial concentration of the tracer in the heart be C_0, and assume that the heart has constant volume V. Also assume that, as fresh blood flows into the heart, the diluted mixture of blood and tracer flows out at the constant positive rate r. Then the concentration of the tracer in the heart at time t is given by

$$C(t) = C_0 e^{-(r/V)t}$$

Show that $dC/dt = (-r/V)C(t)$.

50. Medicine In Problem 49, suppose the tracer is injected at a constant rate R. Then the concentration at time t is

$$C(t) = \frac{R}{r}\left[1 - e^{-(r/V)t}\right]$$

(a) Find $C(0)$.
(b) Show that $\frac{dC}{dt} = \frac{R}{V} - \frac{r}{V}C(t)$.

51. Schizophrenia Several models have been used to analyze the length of stay in a hospital. For a particular group of schizophrenics, one such model is[5]

$$f(t) = 1 - e^{-0.008t}$$

where $f(t)$ is the proportion of the group that was discharged at the end of t days of hospitalization. Find the rate of discharge (the proportion discharged per day) at the end of 100 days. Round your answer to four decimal places.

52. Savings and Consumption A country's savings S (in billions of dollars) is related to its national income I (in billions of dollars) by the equation

$$S = \ln \frac{3}{2 + e^{-I}}$$

(a) Find the marginal propensity to consume as a function of income.
(b) To the nearest million dollars, what is the national income when the marginal propensity to save is $\frac{1}{7}$?

In Problems 53 and 54, use differentiation rules to find $f'(x)$. Then use your graphing calculator to find all real zeros of $f'(x)$. Round your answers to two decimal places.

53. $f(x) = e^{2x^3+x^2-3x}$ **54.** $f(x) = x + e^{-x}$

Objective

To give a mathematical analysis of the economic concept of elasticity.

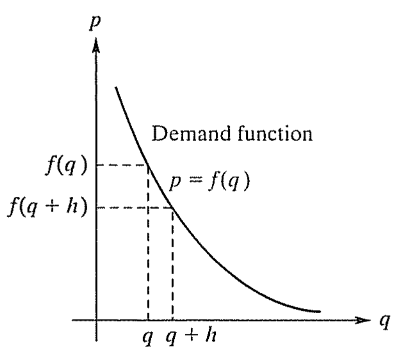

FIGURE 12.2 Change in demand.

12.3 Elasticity of Demand

Elasticity of demand is a means by which economists measure how a change in the price of a product will affect the quantity demanded. That is, it refers to consumer response to price changes. Loosely speaking, elasticity of demand is the ratio of the resulting percentage change in quantity demanded to a given percentage change in price:

$$\frac{\text{percentage change in quantity}}{\text{percentage change in price}}$$

For example, if, for a price increase of 5%, quantity demanded were to decrease by 2%, we would loosely say that elasticity of demand is $-2/5$.

To be more general, suppose $p = f(q)$ is the demand function for a product. Consumers will demand q units at a price of $f(q)$ per unit and will demand $q + h$ units at a price of $f(q + h)$ per unit (Figure 12.2). The *percentage* change in quantity demanded from q to $q + h$ is

$$\frac{(q+h) - q}{q} \cdot 100\% = \frac{h}{q} \cdot 100\%$$

The corresponding percentage change in price per unit is

$$\frac{f(q+h) - f(q)}{f(q)} \cdot 100\%$$

[4] L. R. Peterson and M. J. Peterson, "Short-Term Retention of Individual Verbal Items," *Journal of Experimental Psychology*, 58 (1959), 193–98.

[5] W. W. Eaton and G. A. Whitmore, "Length of Stay as a Stochastic Process: A General Approach and Application to Hospitalization for Schizophrenia," *Journal of Mathematical Sociology*, 5 (1977), 273–92.

The ratio of these percentage changes is

$$\frac{\frac{h}{q}\cdot 100\%}{\frac{f(q+h)-f(q)}{f(q)}\cdot 100\%} = \frac{h}{q}\cdot\frac{f(q)}{f(q+h)-f(q)}$$

$$= \frac{f(q)}{q}\cdot\frac{h}{f(q+h)-f(q)}$$

$$= \frac{\frac{f(q)}{q}}{\frac{f(q+h)-f(q)}{h}} \qquad (1)$$

If f is differentiable, then as $h \to 0$, the limit of $[f(q+h)-f(q)]/h$ is $f'(q) = dp/dq$. Thus, the limit of (1) is

$$\frac{\frac{f(q)}{q}}{f'(q)} = \frac{\frac{p}{q}}{\frac{dp}{dq}} \qquad \text{since } p = f(q)$$

which is called the *point elasticity of demand.*

Definition

If $p = f(q)$ is a differentiable demand function, the ***point elasticity of demand,*** denoted by the Greek letter η (eta), at (q, p) is given by

$$\eta = \eta(q) = \frac{\frac{p}{q}}{\frac{dp}{dq}}$$

CAUTION!

Since p is a function of q, dp/dq is a function of q and thus the ratio that defines η is a function of q. That is why we write $\eta = \eta(q)$.

To illustrate, let us find the point elasticity of demand for the demand function $p = 1200 - q^2$. We have

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{1200-q^2}{q}}{-2q} = -\frac{1200-q^2}{2q^2} = -\left(\frac{600}{q^2} - \frac{1}{2}\right) \qquad (2)$$

For example, if $q = 10$, then $\eta = -\left((600/10^2) - \frac{1}{2}\right) = -5\frac{1}{2}$. Since

$$\eta \approx \frac{\%\text{ change in demand}}{\%\text{ change in price}}$$

we have

$$(\%\text{ change in price})(\eta) \approx \%\text{ change in demand}$$

Thus, if price were increased by 1% when $q = 10$, then quantity demanded would change by approximately

$$(1\%)\left(-5\frac{1}{2}\right) = -5\frac{1}{2}\%$$

That is, demand would decrease $5\frac{1}{2}\%$. Similarly, decreasing price by $\frac{1}{2}\%$ when $q = 10$ results in a change in demand of approximately

$$\left(-\frac{1}{2}\%\right)\left(-5\frac{1}{2}\right) = 2\frac{3}{4}\%$$

Hence, demand increases by $2\frac{3}{4}\%$.

Note that when elasticity is evaluated, no units are attached to it—it is nothing more than a real number. In fact, the 100%'s arising from the word *percentage* cancel, so that elasticity is really an approximation of the ratio

$$\frac{\text{relative change in quantity}}{\text{relative change in price}}$$

and each of the relative changes is no more than a real number. For usual behavior of demand, an increase (decrease) in price corresponds to a decrease (increase) in quantity. This means that if price is plotted as a function of quantity then the graph will have a negative slope at each point. Thus, dp/dq will typically be negative, and since p and q are positive, η will typically be negative too. Some economists disregard the minus sign; in the preceding situation, they would consider the elasticity to be $5\frac{1}{2}$. We will not adopt this practice.

There are three categories of elasticity:

1. When $|\eta| > 1$, demand is *elastic*.
2. When $|\eta| = 1$, demand has *unit elasticity*.
3. When $|\eta| < 1$, demand is *inelastic*.

For example, in Equation (2), since $|\eta| = 5\frac{1}{2}$ when $q = 10$, demand is elastic. If $q = 20$, then $|\eta| = \left|-\left[(600/20^2) - \frac{1}{2}\right]\right| = 1$ so demand has unit elasticity. If $q = 25$, then $|\eta| = \left|-\frac{23}{50}\right|$, and demand is inelastic.

Loosely speaking, for a given percentage change in price, there is a greater percentage change in quantity demanded if demand is elastic, a smaller percentage change if demand is inelastic, and an equal percentage change if demand has unit elasticity. To better understand elasticity, it is helpful to think of typical examples. Demand for an essential utilty such as electricity tends to be inelastic through a wide range of prices. If electricity prices are increased by 10%, consumers can be expected to reduce their consumption somewhat, but a full 10% decrease may not be possible if most of their electricity usage is for essentials of life such as heating and food preparation. On the other hand, demand for luxury goods tends to be quite elastic. A 10% increase in the price of jewelry, for example, may result in a 50% decrease in demand.

EXAMPLE 1 **Finding Point Elasticity of Demand**

Determine the point elasticity of the demand equation

$$p = \frac{k}{q}, \quad \text{where } k > 0 \text{ and } q > 0$$

Solution: From the definition, we have

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{k}{q^2}}{\frac{-k}{q^2}} = -1$$

Thus, the demand has unit elasticity for all $q > 0$. The graph of $p = k/q$ is called an *equilateral hyperbola* and is often found in economics texts in discussions of elasticity. (See Figure 2.11 for a graph of such a curve.)

Now Work Problem 1 ◁

If we are given $p = f(q)$ for our demand equation, as in our discussion thus far, then it is usually straightforward to calculate $dp/dq = f'(q)$. However, if instead we are given q as a function of p, then we will have $q = f^{-1}(p)$ and, from Section 12.2,

$$\frac{dp}{dq} = \frac{1}{\frac{dq}{dp}}$$

It follows that

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp} \tag{3}$$

provides another useful expression for η. Notice too that if $q = g(p)$, then

$$\eta = \eta(p) = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{g(p)} \cdot g'(p) = p \cdot \frac{g'(p)}{g(p)}$$

and thus

$$\text{elasticity} = \text{price} \cdot \text{relative rate of change of quantity as a function of price} \tag{4}$$

EXAMPLE 2 Finding Point Elasticity of Demand

Determine the point elasticity of the demand equation

$$q = p^2 - 40p + 400, \quad \text{where } q > 0$$

Solution: Here we have q given as a function of p and it is easy to see that $dq/dp = 2p - 40$. Thus,

$$\eta(p) = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{q(p)}(2p - 40)$$

For example, if $p = 15$, then $q = q(15) = 25$; hence, $\eta(15) = (15(-10))/25 = -6$, so demand is elastic for $p = 15$.

Now Work Problem 13 ◁

Here we analyze elasticity for linear demand.

Point elasticity for a *linear* demand equation is quite interesting. Suppose the equation has the form

$$p = mq + b, \quad \text{where } m < 0 \text{ and } b > 0$$

(See Figure 12.3.) We assume that $q > 0$; thus, $p < b$. The point elasticity of demand is

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{p}{q}}{m} = \frac{p}{mq} = \frac{p}{p - b}$$

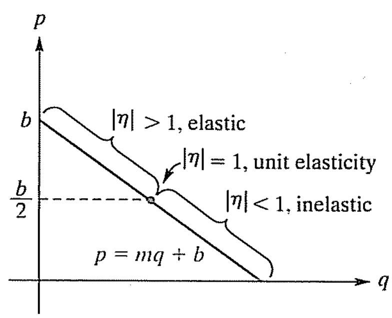

FIGURE 12.3 Elasticity for linear demand.

By considering $d\eta/dp$, we will show that η is a decreasing function of p. By the quotient rule,

$$\frac{d\eta}{dp} = \frac{(p - b) - p}{(p - b)^2} = -\frac{b}{(p - b)^2}$$

Since $b > 0$ and $(p - b)^2 > 0$, it follows that $d\eta/dp < 0$, meaning that the graph of $\eta = \eta(p)$ has a negative slope. This means that as price p increases, elasticity η decreases. However, p ranges between 0 and b, and at the midpoint of this range, $b/2$,

$$\eta = \eta(b) = \frac{\frac{b}{2}}{\frac{b}{2} - b} = \frac{\frac{b}{2}}{-\frac{b}{2}} = -1$$

Therefore, if $p < b/2$, then $\eta > -1$; if $p > b/2$, then $\eta < -1$. Because we typically have $\eta < 0$, we can state these facts another way: When $p < b/2, |\eta| < 1$, and demand is inelastic; when $p = b/2, |\eta| = 1$, and demand has unit elasticity; when $p > b/2, |\eta| > 1$ and demand is elastic. This shows that the slope of a demand curve is not a measure of elasticity. The slope of the line in Figure 12.3 is m everywhere, but elasticity varies with the point on the line. Of course, this is in accord with Equation (4).

Elasticity and Revenue

Here we analyze the relationship between elasticity and the rate of change of revenue.

Turning to a different situation, we can relate how elasticity of demand affects changes in revenue (marginal revenue). If $p = f(q)$ is a manufacturer's demand function, the total revenue is given by

$$r = pq$$

To find the marginal revenue, dr/dq, we differentiate r by using the product rule:

$$\frac{dr}{dq} = p + q\frac{dp}{dq}. \tag{5}$$

Factoring the right side of Equation (5), we have

$$\frac{dr}{dq} = p\left(1 + \frac{q}{p}\frac{dp}{dq}\right)$$

But

$$\frac{q}{p}\frac{dp}{dq} = \frac{\frac{dp}{dq}}{\frac{p}{q}} = \frac{1}{\eta}$$

Thus,

$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right) \tag{6}$$

If demand is elastic, then $\eta < -1$, so $1 + \frac{1}{\eta} > 0$. If demand is inelastic, then $\eta > -1$, so $1 + \frac{1}{\eta} < 0$. We can assume that $p > 0$. From Equation (6) we can conclude that $dr/dq > 0$ on intervals for which demand is elastic. As we will soon see, a function is increasing on intervals for which its derivative is positive and a function is decreasing on intervals for which its derivative is negative. Hence, total revenue r is increasing on intervals for which demand is elastic and total revenue is decreasing on intervals for which demand is inelastic.

Thus, we conclude from the preceding argument that as more units are sold, a manufacturer's total revenue increases if demand is elastic, but decreases if demand is inelastic. That is, if demand is elastic, a lower price will increase revenue. This means that a lower price will cause a large enough increase in demand to actually increase revenue. If demand is inelastic, a lower price will decrease revenue. For unit elasticity, a lower price leaves total revenue unchanged.

If we solve the demand equation to obtain the form $q = g(p)$, rather than $p = f(q)$, then a similar analysis gives

$$\frac{dr}{dp} = q(1 + \eta) \tag{7}$$

and the conclusions of the last paragraph follow even more directly.

PROBLEMS 12.3

In Problems 1–14, find the point elasticity of the demand equations for the indicated values of q or p, and determine whether demand is elastic, is inelastic, or has unit elasticity.

1. $p = 40 - 2q$; $q = 5$

2. $p = 10 - 0.04q$; $q = 100$

3. $p = \frac{3000}{q}$; $q = 300$

4. $p = \frac{500}{q^2}$; $q = 52$

5. $p = \frac{500}{q+2}$; $q = 104$

6. $p = \frac{800}{2q+1}$; $q = 24$

7. $p = 150 - e^{q/100}$; $q = 100$

8. $p = 250e^{-q/50}$; $q = 50$

9. $q = 1200 - 150p$; $p = 4$

10. $q = 100 - p$; $p = 50$

11. $q = \sqrt{500 - p}$; $p = 400$

12. $q = \sqrt{2500 - p^2}$; $p = 20$

13. $q = (p - 50)^2$; $p = 10$

14. $q = p^2 - 50p + 850$; $p = 20$

15. For the linear demand equation $p = 13 - 0.05q$, verify that demand is elastic when $p = 10$, is inelastic when $p = 3$, and has unit elasticity when $p = 6.50$.

16. For what value (or values) of q do the following demand equations have unit elasticity?

(a) $p = 36 - 0.25q$
(b) $p = 300 - q^2$

17. The demand equation for a product is

$$q = 500 - 40p + p^2$$

where p is the price per unit (in dollars) and q is the quantity of units demanded (in thousands). Find the point elasticity of demand when $p = 15$. If this price of 15 is increased by $\frac{1}{2}\%$, what is the approximate change in demand?

18. The demand equation for a certain product is

$$q = \sqrt{3000 - p^2}$$

where p is in dollars. Find the point elasticity of demand when $p = 40$, and use this value to compute the percentage change in demand if the price of \$40 is increased by 7%.

19. For the demand equation $p = 500 - 2q$, verify that demand is elastic and total revenue is increasing for $0 < q < 125$. Verify that demand is inelastic and total revenue is decreasing for $125 < q < 250$.

20. Verify that $\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right)$ if $p = 50 - 3q$.

21. Repeat Problem 20 for $p = \frac{1000}{q^2}$.

22. Suppose $p = mq + b$ is a linear demand equation, where $m \neq 0$ and $b > 0$.
(a) Show that $\lim_{p \to b^-} \eta = -\infty$.
(b) Show that $\eta = 0$ when $p = 0$.

23. The demand equation for a manufacturer's product has the form

$$q = a\sqrt{b - cp^2}$$

where a, b, and c are positive constants.
(a) Show that elasticity does not depend on a.
(b) Determine the interval of prices for which demand is elastic.
(c) For which price is there unit elasticity?

24. Given the demand equation $q^2(1 + p)^2 = p$, determine the point elasticity of demand when $p = 9$.

25. The demand equation for a product is

$$q = \frac{60}{p} + \ln(65 - p^3)$$

(a) Determine the point elasticity of demand when $p = 4$, and classify the demand as elastic, inelastic, or of unit elasticity at this price level.
(b) If the price is lowered by 2% (from \$4.00 to \$3.92), use the answer to part (a) to estimate the corresponding percentage change in quantity sold.
(c) Will the changes in part (b) result in an increase or decrease in revenue? Explain.

26. The demand equation for a manufacturer's product is

$$p = 50(151 - q)^{0.02\sqrt{q+19}}$$

(a) Find the value of dp/dq when 150 units are demanded.
(b) Using the result in part (a), determine the point elasticity of demand when 150 units are demanded. At this level, is demand elastic, inelastic, or of unit elasticity?
(c) Use the result in part (b) to approximate the price per unit if demand decreases from 150 to 140 units.
(d) If the current demand is 150 units, should the manufacturer increase or decrease price in order to increase revenue? (Justify your answer.)

27. A manufacturer of aluminum doors currently is able to sell 500 doors per week at a price of \$80 each. If the price were lowered to \$75 each, an additional 50 doors per week could be sold. Estimate the current elasticity of demand for the doors, and also estimate the current value of the manufacturer's marginal-revenue function.

28. Given the demand equation

$$p = 2000 - q^2$$

where $5 \leq q \leq 40$, for what value of q is $|\eta|$ a maximum? For what value is it a minimum?

29. Repeat Problem 28 for

$$p = \frac{200}{q + 5}$$

such that $5 \leq q \leq 95$.

Objective

To discuss the notion of a function defined implicitly and to determine derivatives by means of implicit differentiation.

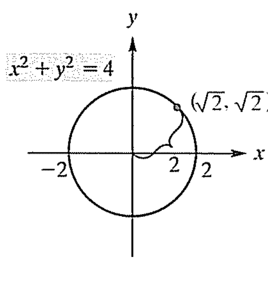

FIGURE 12.4 The circle $x^2 + y^2 = 4$.

12.4 Implicit Differentiation

Implicit differentiation is a technique for differentiating functions that are not given in the usual form $y = f(x)$ [nor in the form $x = g(y)$]. To introduce this technique, we will find the slope of a tangent line to a circle. Let us take the circle of radius 2 whose center is at the origin (Figure 12.4). Its equation is

$$x^2 + y^2 = 4$$

$$x^2 + y^2 - 4 = 0 \qquad (1)$$

The point $(\sqrt{2}, \sqrt{2})$ lies on the circle. To find the slope at this point, we need to find dy/dx there. Until now, we have always had y given explicitly (directly) in terms of x before determining y'; that is, our equation was always in the form $y = f(x)$ [or in the form $x = g(y)$]. In Equation (1), this is not so. We say that Equation (1) has the form $F(x, y) = 0$, where $F(x, y)$ denotes a function of two variables as introduced in Section 2.8. The obvious thing to do is solve Equation (1) for y in terms of x:

$$x^2 + y^2 - 4 = 0$$

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2} \qquad (2)$$

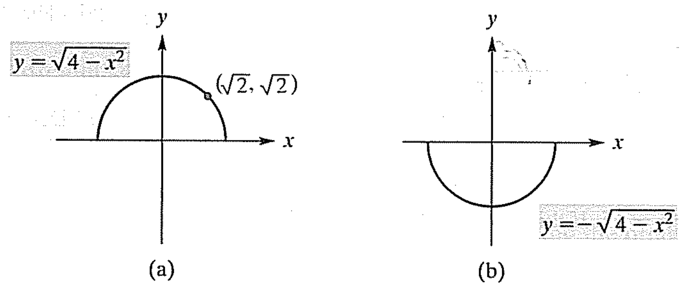

FIGURE 12.5 $x^2 + y^2 = 4$ gives rise to two different functions.

A problem now occurs: Equation (2) may give two values of y for a value of x. It does not define y explicitly as a function of x. We can, however, suppose that Equation (1) defines y as one of two different functions of x,

$$y = +\sqrt{4-x^2} \quad \text{and} \quad y = -\sqrt{4-x^2}$$

whose graphs are given in Figure 12.5. Since the point $(\sqrt{2}, \sqrt{2})$ lies on the graph of $y = \sqrt{4-x^2}$, we should differentiate that function:

$$\begin{aligned} y &= \sqrt{4-x^2} \\ \frac{dy}{dx} &= \frac{1}{2}(4-x^2)^{-1/2}(-2x) \\ &= -\frac{x}{\sqrt{4-x^2}} \end{aligned}$$

So

$$\left.\frac{dy}{dx}\right|_{x=\sqrt{2}} = -\frac{\sqrt{2}}{\sqrt{4-2}} = -1$$

Thus, the slope of the circle $x^2 + y^2 - 4 = 0$ at the point $(\sqrt{2}, \sqrt{2})$ is -1.

Let us summarize the difficulties we had. First, y was not originally given explicitly in terms of x. Second, after we tried to find such a relation, we ended up with more than one function of x. In fact, depending on the equation given, it may be very complicated or even impossible to find an explicit expression for y. For example, it would be difficult to solve $ye^x + \ln(x+y) = 0$ for y. We will now consider a method that avoids such difficulties.

An equation of the form $F(x, y) = 0$, such as we had originally, is said to express y *implicitly* as a function of x. The word *implicitly* is used, since y is not given explicitly as a function of x. However, it is assumed or *implied* that the equation defines y as at least one differentiable function of x. Thus, we assume that Equation (1), $x^2 + y^2 - 4 = 0$, defines some differentiable function of x, say, $y = f(x)$. Next, we treat y as a function of x and differentiate both sides of Equation (1) with respect to x. Finally, we solve the result for dy/dx. Applying this procedure, we obtain

$$\frac{d}{dx}(x^2 + y^2 - 4) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(4) = \frac{d}{dx}(0) \tag{3}$$

We know that $\frac{d}{dx}(x^2) = 2x$ and that both $\frac{d}{dx}(4)$ and $\frac{d}{dx}(0)$ are 0. But $\frac{d}{dx}(y^2)$ is **not** $2y$, because we are differentiating with respect to x, not y. That is, y is not the independent variable. Since y is assumed to be a function of x, y^2 has the form u^n, where y plays the role of u. Just as the power rule states that $\frac{d}{dx}(u^2) = 2u\frac{du}{dx}$, we have $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$. Hence, Equation (3) becomes

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for dy/dx gives

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{for } y \neq 0 \tag{4}$$

Notice that the expression for dy/dx involves the variable y as well as x. This means that to find dy/dx at a point, both coordinates of the point must be substituted into dy/dx. Thus,

$$\left.\frac{dy}{dx}\right|_{(\sqrt{2},\sqrt{2})} = -\frac{\sqrt{2}}{\sqrt{2}} = -1$$

as before. This method of finding dy/dx is called **implicit differentiation.** We note that Equation (4) is not defined when $y = 0$. Geometrically, this is clear, since the tangent line to the circle at either (2, 0) or (−2, 0) is vertical, and the slope is not defined.

Here are the steps to follow when differentiating implicitly:

Implicit Differentiation Procedure

For an equation that we assume defines y implicitly as a differentiable function of x, the derivative $\frac{dy}{dx}$ can be found as follows:

1. Differentiate both sides of the equation with respect to x.

2. Collect all terms involving $\frac{dy}{dx}$ on one side of the equation, and collect all other terms on the other side.

3. Factor $\frac{dy}{dx}$ from the side involving the $\frac{dy}{dx}$ terms.

4. Solve for $\frac{dy}{dx}$, noting any restrictions.

EXAMPLE 1 Implicit Differentiation

Find $\frac{dy}{dx}$ by implicit differentiation if $y + y^3 - x = 7$.

Solution: Here y is not given as an explicit function of x [that is, not in the form $y = f(x)$]. Thus, we assume that y is an implicit (differentiable) function of x and apply the preceding four-step procedure:

1. Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(y + y^3 - x) = \frac{d}{dx}(7)$$

$$\frac{d}{dx}(y) + \frac{d}{dx}(y^3) - \frac{d}{dx}(x) = \frac{d}{dx}(7)$$

CAUTION!

The derivative of y^3 with respect to x is $3y^2\frac{dy}{dx}$, not $3y^2$.

Now, $\frac{d}{dx}(y)$ can be written $\frac{dy}{dx}$, and $\frac{d}{dx}(x) = 1$. By the power rule,

$$\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

Hence, we obtain

$$\frac{dy}{dx} + 3y^2\frac{dy}{dx} - 1 = 0$$

2. Collecting all $\frac{dy}{dx}$ terms on the left side and all other terms on the right side gives

$$\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 1$$

3. Factoring $\frac{dy}{dx}$ from the left side, we have

$$\frac{dy}{dx}(1+3y^2)=1$$

In an implicit-differentiation problem, we are able to find the derivative of a function without knowing the function.

4. We solve for $\frac{dy}{dx}$ by dividing both sides by $1+3y^2$:

$$\frac{dy}{dx}=\frac{1}{1+3y^2}$$

Because step 4 of the process often involves division by an expression involving the variables, the answer obtained must often be restricted to exclude those values of the variables that would make the denominator zero. Here the denominator is always greater than or equal to 1, so there is no restriction.

Now Work Problem 3 ◁

APPLY IT ▷

4. Suppose that P, the proportion of people affected by a certain disease, is described by $\ln\left(\frac{P}{1-P}\right)=0.5t$, where t is the time in months. Find $\frac{dP}{dt}$, the rate at which P grows with respect to time.

EXAMPLE 2 Implicit Differentiation

Find $\frac{dy}{dx}$ if $x^3+4xy^2-27=y^4$.

Solution: Since y is not given explicitly in terms of x, we will use the method of implicit differentiation:

1. Assuming that y is a function of x and differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x^3+4xy^2-27)=\frac{d}{dx}(y^4)$$

$$\frac{d}{dx}(x^3)+4\frac{d}{dx}(xy^2)-\frac{d}{dx}(27)=\frac{d}{dx}(y^4)$$

To find $\frac{d}{dx}(xy^2)$, we use the product rule:

$$3x^2+4\left[x\frac{d}{dx}(y^2)+y^2\frac{d}{dx}(x)\right]-0=4y^3\frac{dy}{dx}$$

$$3x^2+4\left[x\left(2y\frac{dy}{dx}\right)+y^2(1)\right]=4y^3\frac{dy}{dx}$$

$$3x^2+8xy\frac{dy}{dx}+4y^2=4y^3\frac{dy}{dx}$$

2. Collecting $\frac{dy}{dx}$ terms on the left side and other terms on the right gives

$$8xy\frac{dy}{dx}-4y^3\frac{dy}{dx}=-3x^2-4y^2$$

3. Factoring $\frac{dy}{dx}$ from the left side yields

$$\frac{dy}{dx}(8xy-4y^3)=-3x^2-4y^2$$

4. Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx}=\frac{-3x^2-4y^2}{8xy-4y^3}=\frac{3x^2+4y^2}{4y^3-8xy}$$

which gives the value of dy/dx at points (x, y) for which $4y^3-8xy\neq 0$.

Now Work Problem 11 ◁

APPLY IT ▷

5. The volume V enclosed by a spherical balloon of radius r is given by the equation $V = \frac{4}{3}\pi r^3$. If the radius is increasing at a rate of 5 inches/minute (that is, $\frac{dr}{dt} = 5$), then find $\left.\frac{dV}{dt}\right|_{r=12}$, the rate of increase of the volume, when the radius is 12 inches.

EXAMPLE 3 Implicit Differentiation

Find the slope of the curve $x^3 = (y - x^2)^2$ at (1, 2).

Solution: The slope at (1, 2) is the value of dy/dx at that point. Finding dy/dx by implicit differentiation, we have

$$\frac{d}{dx}(x^3) = \frac{d}{dx}[(y - x^2)^2]$$

$$3x^2 = 2(y - x^2)\left(\frac{dy}{dx} - 2x\right)$$

$$3x^2 = 2\left(y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} + 2x^3\right)$$

$$3x^2 = 2y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} + 4x^3$$

$$3x^2 + 4xy - 4x^3 = 2y\frac{dy}{dx} - 2x^2\frac{dy}{dx}$$

$$3x^2 + 4xy - 4x^3 = 2\frac{dy}{dx}(y - x^2)$$

$$\frac{dy}{dx} = \frac{3x^2 + 4xy - 4x^3}{2(y - x^2)} \qquad \text{for } y - x^2 \neq 0$$

For the point $(1, 2)$, $y - x^2 = 2 - 1^2 = 1 \neq 0$. Thus, the slope of the curve at (1, 2) is

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{3(1)^2 + 4(1)(2) - 4(1)^3}{2(2 - (1)^2)} = \frac{7}{2}$$

Now Work Problem 25 ◁

APPLY IT ▷

6. A 10-foot ladder is placed against a vertical wall. Suppose the bottom of the ladder slides away from the wall at a constant rate of 3 ft/s. (That is, $\frac{dx}{dt} = 3$.) How fast is the top of the ladder sliding down the wall when the top of the ladder is 8 feet from the ground (that is, when $y = 8$)? (That is, what is $\frac{dy}{dt}$?) (Use the Pythagorean theorem for right triangles, $x^2 + y^2 = z^2$, where x and y are the legs of the triangle and z is the hypotenuse.)

EXAMPLE 4 Implicit Differentiation

If $q - p = \ln q + \ln p$, find dq/dp.

Solution: We assume that q is a function of p and differentiate both sides with respect to p:

$$\frac{d}{dp}(q) - \frac{d}{dp}(p) = \frac{d}{dp}(\ln q) + \frac{d}{dp}(\ln p)$$

$$\frac{dq}{dp} - 1 = \frac{1}{q}\frac{dq}{dp} + \frac{1}{p}$$

$$\frac{dq}{dp} - \frac{1}{q}\frac{dq}{dp} = \frac{1}{p} + 1$$

$$\frac{dq}{dp}\left(1 - \frac{1}{q}\right) = \frac{1}{p} + 1$$

$$\frac{dq}{dp}\left(\frac{q - 1}{q}\right) = \frac{1 + p}{p}$$

$$\frac{dq}{dp} = \frac{(1 + p)q}{p(q - 1)} \qquad \text{for } p(q - 1) \neq 0$$

Now Work Problem 19 ◁

PROBLEMS 12.4

In Problems 1–24, find dy/dx by implicit differentiation.

1. $x^2 + 4y^2 = 4$
2. $3x^2 + 6y^2 = 1$
3. $2y^3 - 7x^2 = 5$
4. $5y^2 - 2x^2 = 10$
5. $\sqrt[3]{x} + \sqrt[3]{y} = 3$
6. $x^{1/5} + y^{1/5} = 4$
7. $x^{3/4} + y^{3/4} = 5$
8. $y^3 = 4x$
9. $xy = 36$
10. $x^2 + xy - 2y^2 = 0$
11. $xy - y - 11x = 5$
12. $x^3 - y^3 = 3x^2y - 3xy^2$
13. $2x^3 + y^3 - 12xy = 0$
14. $5x^3 + 6xy + 7y^3 = 0$
15. $x = \sqrt{y} + \sqrt[4]{y}$
16. $x^3y^3 + x = 9$
17. $5x^3y^4 - x + y^2 = 25$
18. $y^2 + y = \ln x$
19. $\ln(xy) = e^{xy}$
20. $\ln(xy) + x = 4$
21. $xe^y + y = 13$
22. $4x^2 + 9y^2 = 16$
23. $(1 + e^{3x})^2 = 3 + \ln(x + y)$
24. $e^{x-y} = \ln(x - y)$

25. If $x + xy + y^2 = 7$, find dy/dx at (1, 2).

26. If $x\sqrt{y+1} = y\sqrt{x+1}$, find dy/dx at (3, 3).

27. Find the slope of the curve $4x^2 + 9y^2 = 1$ at the point $\left(0, \frac{1}{3}\right)$; at the point (x_0, y_0).

28. Find the slope of the curve $(x^2 + y^2)^2 = 4y^2$ at the point (0, 2).

29. Find equations of the tangent lines to the curve

$$x^3 + xy + y^3 = -1$$

at the points $(-1, -1)$, $(-1, 0)$, and $(-1, 1)$.

30. Repeat Problem 29 for the curve

$$y^2 + xy - x^2 = 5$$

at the point (4, 3).

For the demand equations in Problems 31–34, find the rate of change of q with respect to p.

31. $p = 100 - q^2$
32. $p = 400 - \sqrt{q}$
33. $p = \dfrac{20}{(q+5)^2}$
34. $p = \dfrac{3}{q^2 + 1}$

35. Radioactivity The relative activity I/I_0 of a radioactive element varies with elapsed time according to the equation

$$\ln\left(\frac{I}{I_0}\right) = -\lambda t$$

where λ (a Greek letter read "lambda") is the disintegration constant and I_0 is the initial intensity (a constant). Find the rate of change of the intensity I with respect to the elapsed time t.

36. Earthquakes The magnitude M of an earthquake and its energy E are related by the equation[6]

$$1.5M = \log\left(\frac{E}{2.5 \times 10^{11}}\right)$$

Here M is given in terms of Richter's preferred scale of 1958 and E is in ergs. Determine the rate of change of energy with respect to magnitude and the rate of change of magnitude with respect to energy.

37. Physical Scale The relationship among the speed (v), frequency (f), and wavelength (λ) of any wave is given by

$$v = f\lambda$$

Find $df/d\lambda$ by differentiating implicitly. (Treat v as a constant.) Then show that the same result is obtained if you first solve the equation for f and then differentiate with respect to λ.

38. Biology The equation $(P + a)(v + b) = k$ is called the "fundamental equation of muscle contraction."[7] Here P is the load imposed on the muscle, v is the velocity of the shortening of the muscle fibers, and a, b, and k are positive constants. Use implicit differentiation to show that, in terms of P,

$$\frac{dv}{dP} = -\frac{k}{(P+a)^2}$$

39. Marginal Propensity to Consume A country's savings S is defined implicitly in terms of its national income I by the equation

$$S^2 + \frac{1}{4}I^2 = SI + I$$

where both S and I are in billions of dollars. Find the marginal propensity to consume when $I = 16$ and $S = 12$.

40. Technological Substitution New products or technologies often tend to replace old ones. For example, today most commercial airlines use jet engines rather than prop engines. In discussing the forecasting of technological substitution, Hurter and Rubenstein[8] refer to the equation

$$\ln\frac{f(t)}{1 - f(t)} + \sigma\frac{1}{1 - f(t)} = C_1 + C_2 t$$

where $f(t)$ is the market share of the substitute over time t and C_1, C_2, and σ (a Greek letter read "sigma") are constants. Verify their claim that the rate of substitution is

$$f'(t) = \frac{C_2 f(t)[1 - f(t)]^2}{\sigma f(t) + [1 - f(t)]}$$

Objective

To describe the method of logarithmic differentiation and to show how to differentiate a function of the form u^v.

12.5 Logarithmic Differentiation

A technique called **logarithmic differentiation** often simplifies the differentiation of $y = f(x)$ when $f(x)$ involves products, quotients, or powers. The procedure is as

[6] K. E. Bullen, *An Introduction to the Theory of Seismology* (Cambridge, U.K.: Cambridge at the University Press, 1963).

[7] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).

[8] A. P. Hurter, Jr., A. H. Rubenstein et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change*, 11 (1978), 197–221.

follows:

Logarithmic Differentiation

To differentiate $y = f(x)$,

1. Take the natural logarithm of both sides. This results in
$$\ln y = \ln(f(x))$$
2. Simplify $\ln(f(x))$ by using properties of logarithms.
3. Differentiate both sides with respect to x.
4. Solve for $\dfrac{dy}{dx}$.
5. Express the answer in terms of x only. This requires substituting $f(x)$ for y.

There are a couple of points worth noting. First, irrespective of any simplification, the procedure produces

$$\frac{y'}{y} = \frac{d}{dx}(\ln(f(x))$$

so that

$$\frac{dy}{dx} = y\frac{d}{dx}(\ln(f(x))$$

is a formula that you can memorize, if you prefer. Second, the quantity $\dfrac{f'(x)}{f(x)}$, which results from differentiating $\ln(f(x))$, is what was called the *relative rate of change of* $f(x)$ in Section 11.3.

The next example illustrates the procedure.

EXAMPLE 1 Logarithmic Differentiation

Find y' if $y = \dfrac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}}$.

Solution: Differentiating this function in the usual way is messy because it involves the quotient, power, and product rules. Logarithmic differentiation makes the work less of a chore.

1. We take the natural logarithm of both sides:
$$\ln y = \ln \frac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}}$$
2. Simplifying by using properties of logarithms, we have
$$\begin{aligned}\ln y &= \ln(2x-5)^3 - \ln\left(x^2\sqrt[4]{x^2+1}\right)\\ &= 3\ln(2x-5) - (\ln x^2 + \ln(x^2+1)^{1/4})\\ &= 3\ln(2x-5) - 2\ln x - \frac{1}{4}\ln(x^2+1)\end{aligned}$$
3. Differentiating with respect to x gives
$$\begin{aligned}\frac{y'}{y} &= 3\left(\frac{1}{2x-5}\right)(2) - 2\left(\frac{1}{x}\right) - \frac{1}{4}\left(\frac{1}{x^2+1}\right)(2x)\\ &= \frac{6}{2x-5} - \frac{2}{x} - \frac{x}{2(x^2+1)}\end{aligned}$$
4. Solving for y' yields
$$y' = y\left(\frac{6}{2x-5} - \frac{2}{x} - \frac{x}{2(x^2+1)}\right)$$

CAUTION!

Since y is a function of x, differentiating $\ln y$ with respect to x gives $\dfrac{y'}{y}$.

5. Substituting the original expression for y gives y' in terms of x only:

$$y' = \frac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}}\left[\frac{6}{2x-5} - \frac{2}{x} - \frac{x}{2(x^2+1)}\right]$$

Now Work Problem 1 ◁

Logarithmic differentiation can also be used to differentiate a function of the form $y = u^v$, where both u and v are differentiable functions of x. Because neither the base nor the exponent is necessarily a constant, the differentiation techniques for u^n and a^u do not apply here.

EXAMPLE 2 Differentiating the Form u^v

Differentiate $y = x^x$ by using logarithmic differentiation.

Solution: This example is a good candidate for the *formula* approach to logarithmic differentiation.

$$y' = y\frac{d}{dx}(\ln x^x) = x^x\frac{d}{dx}(x\ln x) = x^x\left((1)(\ln x) + (x)\left(\frac{1}{x}\right)\right) = x^x(\ln x + 1)$$

It is worthwhile mentioning that an alternative technique for differentiating a function of the form $y = u^v$ is to convert it to an exponential function to the base e. To illustrate, for the function in this example, we have

$$y = x^x = (e^{\ln x})^x = e^{x\ln x}$$

$$y' = e^{x\ln x}\left(1\ln x + x\frac{1}{x}\right) = x^x(\ln x + 1)$$

Now Work Problem 15 ◁

EXAMPLE 3 Relative Rate of Change of a Product

Show that the relative rate of change of a product is the sum of the relative rates of change of its factors. Use this result to express the percentage rate of change in revenue in terms of the percentage rate of change in price.

Solution: Recall that the relative rate of change of a function r is $\frac{r'}{r}$. We are to show that if $r = pq$, then $\frac{r'}{r} = \frac{p'}{p} + \frac{q'}{q}$. From $r = pq$ we have $\ln r = \ln p + \ln q$, which, when both sides are differentiated, gives

$$\frac{r'}{r} = \frac{p'}{p} + \frac{q'}{q}$$

as required. Multiplying both sides by 100% gives an expression for the percentage rate of change of r in terms of those of p and q:

$$\frac{r'}{r}100\% = \frac{p'}{p}100\% + \frac{q'}{q}100\%$$

If p is *price* per item and q is *quantity* sold, then $r = pq$ is total *revenue*. In this case we take differentiation to be with respect to p and note that now $\frac{q'}{q} = \eta\frac{p'}{p}$, where η is the elasticity of demand as in Section 12.3. It follows that in this case we have

$$\frac{r'}{r}100\% = (1+\eta)\frac{p'}{p}100\%$$

expressing the percentage rate of change in revenue in terms of the percentage rate of change in price. For example, if at a given price and quantity, $\eta = -5$, then a 1% increase in price will result in a $(1-5)\% = -4\%$ increase in revenue, which is to say a 4% *decrease* in revenue, while a 3% decrease in price—that is, a -3% *increase* in price—will result in a $(1-5)(-3)\% = 12\%$ increase in revenue. It is also clear that at points at which there is unit elasticity ($\eta = -1$), any percentage change in price produces no percentage change in revenue.

Now Work Problem 29 ◁

EXAMPLE 4 Differentiating the Form u^v

Find the derivative of $y = (1 + e^x)^{\ln x}$.

Solution: This has the form $y = u^v$, where $u = 1 + e^x$ and $v = \ln x$. Using logarithmic differentiation, we have

$$\ln y = \ln\left((1 + e^x)^{\ln x}\right)$$
$$\ln y = (\ln x)\ln(1 + e^x)$$
$$\frac{y'}{y} = \left(\frac{1}{x}\right)(\ln(1 + e^x)) + (\ln x)\left(\frac{1}{1 + e^x} \cdot e^x\right)$$
$$\frac{y'}{y} = \frac{\ln(1 + e^x)}{x} + \frac{e^x \ln x}{1 + e^x}$$
$$y' = y\left(\frac{\ln(1 + e^x)}{x} + \frac{e^x \ln x}{1 + e^x}\right)$$
$$y' = (1 + e^x)^{\ln x}\left(\frac{\ln(1 + e^x)}{x} + \frac{e^x \ln x}{1 + e^x}\right)$$

Now Work Problem 17 ◁

Alternatively, we can differentiate even a general function of the form $y = u(x)^{v(x)}$ with $u(x) > 0$ by using the equation

$$u^v = e^{v \ln u}$$

Indeed, if $y = u(x)^{v(x)} = e^{v(x)\ln u(x)}$ for $u(x) > 0$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{v(x)\ln u(x)}\right) = e^{v(x)\ln u(x)}\frac{d}{dx}(v(x)\ln u(x)) = u^v\left(v'(x)\ln u(x) + v(x)\frac{u'(x)}{u(x)}\right)$$

which could be summarized as

$$(u^v)' = u^v\left(v' \ln u + v\frac{u'}{u}\right)$$

As is often the case, there is no suggestion that the preceding formula should be memorized. The point here is that we have shown that *any* function of the form u^v can be differentiated using the equation $u^v = e^{v \ln u}$. The same result will be obtained from logarithmic differentiation:

$$\ln y = \ln(u^v)$$
$$\ln y = v \ln u$$
$$\frac{y'}{y} = v' \ln u + v\frac{u'}{u}$$
$$y' = y\left(v' \ln u + v\frac{u'}{u}\right)$$
$$(u^v)' = u^v\left(v' \ln u + v\frac{u'}{u}\right)$$

After completing this section, we understand how to differentiate each of the following forms:

$$y = \begin{cases} (f(x))^a & \text{(a)} \\ b^{f(x)} & \text{(b)} \\ (f(x))^{g(x)} & \text{(c)} \end{cases}$$

For type (a), use the power rule. For type (b), use the differentiation formula for exponential functions. [If $b \neq e$, first convert $b^{f(x)}$ to an e^u function.] For type (c), use logarithmic differentiation or first convert to an e^u function. Do not apply a rule in a situation where the rule does not apply. For example, the power rule does not apply to x^x.

PROBLEMS 12.5

In Problems 1–12, find y′ by using logarithmic differentiation.

1. $y = (x+1)^2(x-2)(x^2+3)$

2. $y = (3x+4)(8x-1)^2(3x^2+1)^4$

3. $y = (3x^3-1)^2(2x+5)^3$

4. $y = (2x^2+1)\sqrt{8x^2-1}$

5. $y = \sqrt{x+1}\sqrt{x-1}\sqrt{x^2+1}$

6. $y = (2x+1)\sqrt{x^3+2}\sqrt[3]{2x+5}$

7. $y = \dfrac{\sqrt{1-x^2}}{1-2x}$

8. $y = \sqrt{\dfrac{x^2+5}{x+9}}$

9. $y = \dfrac{(2x^2+2)^2}{(x+1)^2(3x+2)}$

10. $y = \dfrac{x^2(1+x^2)}{\sqrt{x^2+4}}$

11. $y = \sqrt{\dfrac{(x+3)(x-2)}{2x-1}}$

12. $y = \sqrt[3]{\dfrac{6(x^3+1)^2}{x^6e^{-4x}}}$

In Problems 13–20, find y′.

13. $y = x^{x^2+1}$

14. $y = (2x)^{\sqrt{x}}$

15. $y = x^{\sqrt{x}}$

16. $y = \left(\dfrac{3}{x^2}\right)^x$

17. $y = (3x+1)^{2x}$

18. $y = (x^2+1)^{x+1}$

19. $y = 4e^x x^{3x}$

20. $y = (\sqrt{x})^x$

21. If $y = (4x-3)^{2x+1}$, find dy/dx when $x = 1$.

22. If $y = (\ln x)^{\ln x}$, find dy/dx when $x = e$.

23. Find an equation of the tangent line to

$$y = (x+1)(x+2)^2(x+3)^2$$

at the point where $x = 0$.

24. Find an equation of the tangent line to the graph of

$$y = x^x$$

at the point where $x = 1$.

25. Find an equation of the tangent line to the graph of

$$y = x^x$$

at the point where $x = e$.

26. If $y = x^x$, find the relative rate of change of y with respect to x when $x = 1$.

27. If $y = (3x)^{-2x}$, find the value of x for which the *percentage* rate of change of y with respect to x is 60.

28. Suppose $f(x)$ is a positive differentiable function and g is a differentiable function and $y = (f(x))^{g(x)}$. Use logarithmic differentiation to show that

$$\frac{dy}{dx} = (f(x))^{g(x)}\left(f'(x)\frac{g(x)}{f(x)} + g'(x)\ln(f(x))\right)$$

29. The demand equation for a compact disc is

$$q = 500 - 40p + p^2$$

If the price of \$15 is increased by 1/2%, find the corresponding percentage change in revenue.

30. Repeat Problem 29 with the same information except for a 5% *decrease* in price.

Objective

To approximate real roots of an equation by using calculus. The method shown is suitable for calculators.

12.6 Newton's Method

It is easy to solve equations of the form $f(x) = 0$ when f is a linear or quadratic function. For example, we can solve $x^2+3x-2 = 0$ by the quadratic formula. However, if $f(x)$ has a degree greater than 2 (or is not a polynomial), it may be difficult, or even impossible, to find solutions (or roots) of $f(x) = 0$ by the methods to which you are accustomed. For this reason, we may settle for approximate solutions, which can be obtained in a variety of efficient ways. For example, a graphing calculator can be used to estimate the real roots of $f(x) = 0$. In this section, we will study how the derivative can be so used (provided that f is differentiable). The procedure we will develop, called *Newton's method,* is well suited to a calculator or computer.

Newton's method requires an initial estimate for a root of $f(x) = 0$. One way of obtaining this estimate is by making a rough sketch of the graph of $y = f(x)$ and estimating the root from the graph. A point on the graph where $y = 0$ is an x-intercept, and the x-value of this point is a root of $f(x) = 0$. Another way of locating a root is based on the following fact:

If f is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one real root between a and b.

Figure 12.6 depicts this situation. The x-intercept between a and b corresponds to a root of $f(x) = 0$, and we can use either a or b to approximate this root.

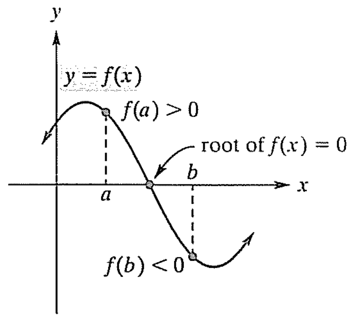

FIGURE 12.6 Root of $f(x) = 0$ between a and b, where $f(a)$ and $f(b)$ have opposite signs.

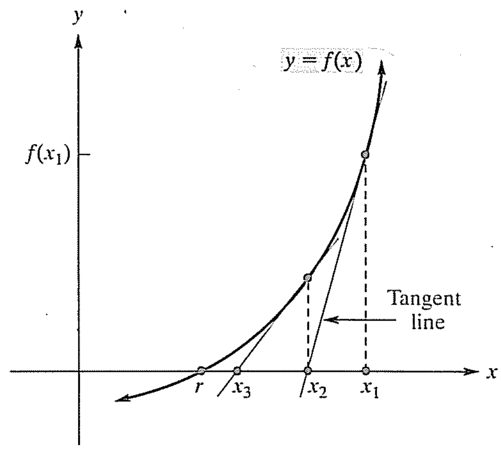

FIGURE 12.7 Improving approximation of root via tangent line.

Assuming that we have an estimated (but incorrect) value for a root, we turn to a way of getting a better approximation. In Figure 12.7, we see that $f(r) = 0$, so r is a root of the equation $f(x) = 0$. Suppose x_1 is an initial approximation to r (and one that is close to r). Observe that the tangent line to the curve at $(x_1, f(x_1))$ intersects the x-axis at the point $(x_2, 0)$, and x_2 is a better approximation to r than is x_1.

We can find x_2 from the equation of the tangent line. The slope of the tangent line is $f'(x_1)$, so a point-slope form for this line is

$$y - f(x_1) = f'(x_1)(x - x_1) \tag{1}$$

Since $(x_2, 0)$ is on the tangent line, its coordinates must satisfy Equation (1). This gives

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$-\frac{f(x_1)}{f'(x_1)} = x_2 - x_1 \qquad \text{if } f'(x_1) \neq 0$$

Thus,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2}$$

To get a better approximation to r, we again perform the procedure described, but this time we use x_2 as our starting point. This gives the approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \tag{3}$$

Repeating (or *iterating*) this computation over and over, we hope to obtain better approximations, in the sense that the sequence of values

$$x_1, x_2, x_3, \ldots$$

will approach r. In practice, we terminate the process when we have reached a desired degree of accuracy.

Analyzing Equations (2) and (3), we see how x_2 is obtained from x_1 and how x_3 is obtained from x_2. In general, x_{n+1} is obtained from x_n by means of the following general formula, called **Newton's method:**

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 1, 2, 3, \ldots \tag{4}$$

TO REVIEW recursively defined sequences, see Section 1.6.

A formula, like Equation (4), that indicates how one number in a sequence is obtained from the preceding one is called a **recursion formula,** or an *iteration equation.*