Objective

To develop the idea of a tangent line to a curve, to define the slope of a curve, and to define a derivative and give it a geometric interpretation. To compute derivatives by using the limit definition.

11.1 The Derivative

The main problem of differential calculus deals with finding the slope of the *tangent line* at a point on a curve. In high school geometry a tangent line, or *tangent*, to a circle is often defined as a line that meets the circle at exactly one point (Figure 11.1). However, this idea of a tangent is not very useful for other kinds of curves. For example, in Figure 11.2(a), the lines L_1 and L_2 intersect the curve at exactly one point P. Although we would not think of L_2 as the tangent at this point, it seems natural that L_1 is. In Figure 11.2(b) we intuitively would consider L_3 to be the tangent at point P, even though L_3 intersects the curve at other points.

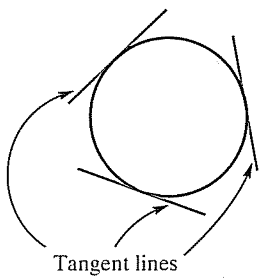
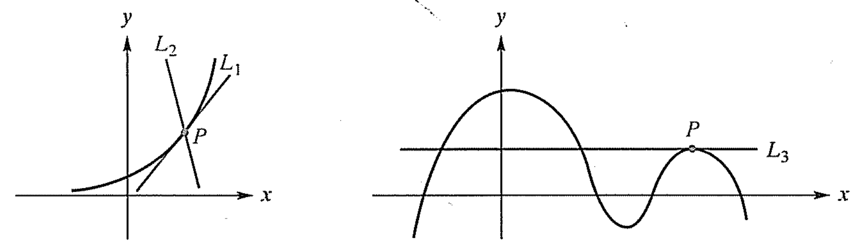


FIGURE 11.1 Tangent lines to a circle.



L_1 is a tangent line at P, but L_2 is not.

(a)

L_3 is a tangent line at P.

(b)

FIGURE 11.2 Tangent line at a point.

From these examples, we see that the idea of a tangent as simply a line that intersects a curve at only one point is inadequate. To obtain a suitable definition of tangent line, we use the limit concept and the geometric notion of a *secant line*. A **secant line** is a line that intersects a curve at two or more points.

Look at the graph of the function $y = f(x)$ in Figure 11.3. We wish to define the tangent line at point P. If Q is a different point on the curve, the line PQ is a secant line. If Q moves along the curve and approaches P from the right (see Figure 11.4), typical secant lines are PQ', PQ'', and so on. As Q approaches P from the left, typical secant lines are PQ_1, PQ_2, and so on. *In both cases, the secant lines approach the same limiting position.* This common limiting position of the secant lines is defined to be the **tangent line** to the curve at P. This definition seems reasonable and applies to curves in general, not just circles.

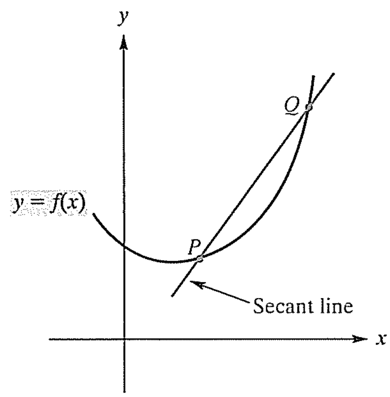


FIGURE 11.3 Secant line PQ.

A curve does not necessarily have a tangent line at each of its points. For example, the curve $y = |x|$ does not have a tangent at (0,0). As can be seen in Figure 11.5, a secant line through (0,0) and a nearby point to its right on the curve must always be the line

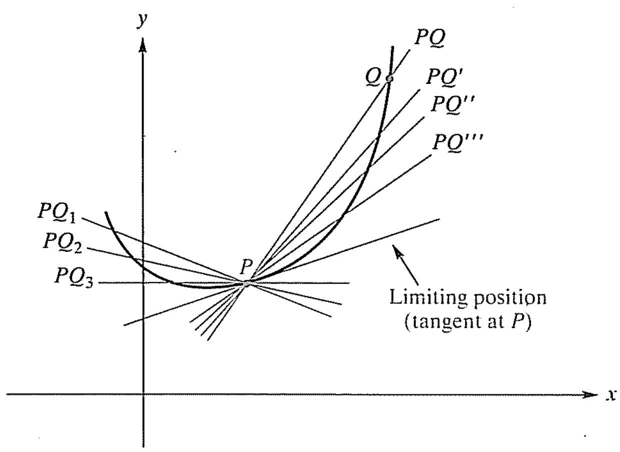


FIGURE 11.4 The tangent line is a limiting position of secant lines.

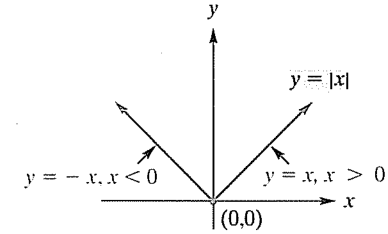


FIGURE 11.5 No tangent line to graph of $y = |x|$ at (0,0).

$y = x$. Thus the limiting position of such secant lines is also the line $y = x$. However, a secant line through (0,0) and a nearby point to its left on the curve must always be the line $y = -x$. Hence, the limiting position of such secant lines is also the line $y = -x$. Since there is no common limiting position, there is no tangent line at (0,0).

Now that we have a suitable definition of a tangent to a curve at a point, we can define the *slope of a curve* at a point.

Definition

The **slope of a curve** at a point P is the slope, if it exists, of the tangent line at P.

Since the tangent at P is a limiting position of secant lines PQ, we consider the slope of the tangent to be the limiting value of the slopes of the secant lines as Q approaches P. For example, let us consider the curve $f(x) = x^2$ and the slopes of some secant lines PQ, where $P = (1, 1)$. For the point $Q = (2.5, 6.25)$, the slope of PQ (see Figure 11.6) is

$$m_{PQ} = \frac{\text{rise}}{\text{run}} = \frac{6.25 - 1}{2.5 - 1} = 3.5$$

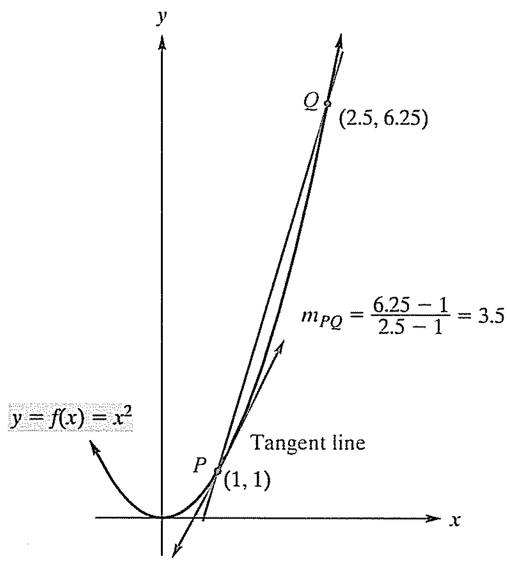


FIGURE 11.6 Secant line to $f(x) = x^2$ through (1, 1) and (2.5, 6.25).

Table 11.1 includes other points Q on the curve, as well as the corresponding slopes of PQ. Notice that as Q approaches P, the slopes of the secant lines seem to approach 2. Thus, we expect the slope of the indicated tangent line at (1, 1) to be 2. This will be confirmed later, in Example 1. But first, we wish to generalize our procedure.

Table 11.1 Slopes of Secant Lines to the Curve $f(x) = x^2$ at $P = (1, 1)$

Q	Slope of PQ
(2.5, 6.25)	$(6.25 - 1)/(2.5 - 1) = 3.5$
(2, 4)	$(4 - 1)/(2 - 1) = 3$
(1.5, 2.25)	$(2.25 - 1)/(1.5 - 1) = 2.5$
(1.25, 1.5625)	$(1.5625 - 1)/(1.25 - 1) = 2.25$
(1.1, 1.21)	$(1.21 - 1)/(1.1 - 1) = 2.1$
(1.01, 1.0201)	$(1.0201 - 1)/(1.01 - 1) = 2.01$

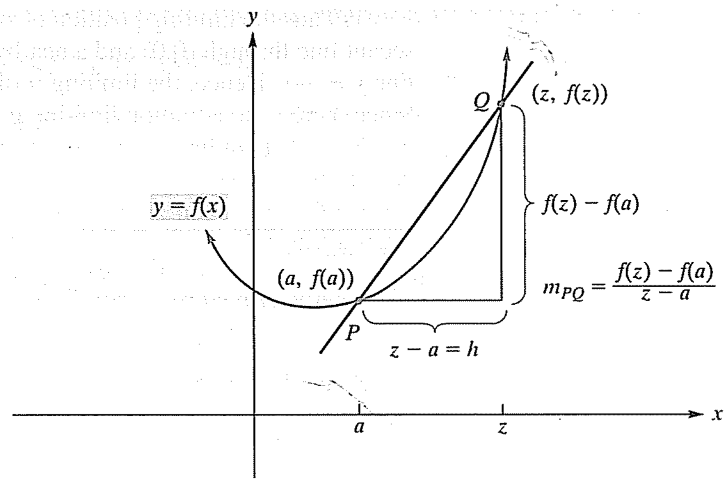


FIGURE 11.7 Secant line through P and Q.

For the curve $y = f(x)$ in Figure 11.7, we will find an expression for the slope at the point $P = (a,f(a))$. If $Q = (z,f(z))$, the slope of the secant line PQ is

$$m_{PQ} = \frac{f(z) - f(a)}{z - a}$$

If the difference $z - a$ is called h, then we can write z as $a + h$. Here we must have $h \neq 0$, for if $h = 0$, then $z = a$, and no secant line exists. Accordingly,

$$m_{PQ} = \frac{f(z) - f(a)}{z - a} = \frac{f(a + h) - f(a)}{h}$$

Which of these two forms for m_{PQ} is most convenient depends on the nature of the function f. As Q moves along the curve toward P, z approaches a. This means that h approaches zero. The limiting value of the slopes of the secant lines—which is the slope of the tangent line at $(a,f(a))$—is

$$m_{\tan} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \qquad (1)$$

Again, which of these two forms is most convenient—which limit is easiest to determine—depends on the nature of the function f. In Example 1, we will use this limit to confirm our previous expectation that the slope of the tangent line to the curve $f(x) = x^2$ at (1, 1) is 2.

EXAMPLE 1 Finding the Slope of a Tangent Line

Find the slope of the tangent line to the curve $y = f(x) = x^2$ at the point (1, 1).

Solution: The slope is the limit in Equation (1) with $f(x) = x^2$ and $a = 1$:

$$\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{(1 + h)^2 - (1)^2}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2 + h)}{h} = \lim_{h \to 0} (2 + h) = 2$$

Therefore, the tangent line to $y = x^2$ at (1, 1) has slope 2. (Refer to Figure 11.6.)

Now Work Problem 1 ◁

We can generalize Equation (1) so that it applies to any point $(x,f(x))$ on a curve. Replacing a by x gives a function, called the *derivative* of f, whose input is x and whose output is the slope of the tangent line to the curve at $(x,f(x))$, provided that the tangent line *exists* and *has* a slope. (If the tangent line exists but is *vertical*, then it has no slope.) We thus have the following definition, which forms the basis of differential calculus:

Definition

The **derivative** of a function f is the function denoted f' (read "f prime") and defined by

$$f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \tag{2}$$

provided that this limit exists. If $f'(a)$ can be found [while perhaps not all $f'(x)$ can be found] f is said to be ***differentiable*** at a, and $f'(a)$ is called the derivative of f at a or the derivative of f with respect to x at a. The process of finding the derivative is called ***differentiation.***

In the definition of the derivative, the expression

$$\frac{f(z)-f(x)}{z-x} = \frac{f(x+h)-f(x)}{h}$$

where $z = x+h$, is called a **difference quotient.** Thus $f'(x)$ is the limit of a difference quotient.

EXAMPLE 2 Using the Definition to Find the Derivative

If $f(x) = x^2$, find the derivative of f.

Calculating a derivative via the definition requires precision. Typically, the difference quotient requires considerable manipulation before the limit step is taken. This requires that each written step be preceded by "$\lim_{h\to 0}$" to acknowledge that the limit step is still pending. Observe that after the limit step is taken, h will no longer be present.

Solution: Applying the definition of a derivative gives

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh+h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x+h) = 2x \end{aligned}$$

Observe that, in taking the limit, we treated x as a constant, because it was h, not x, that was changing. Also, note that $f'(x) = 2x$ defines a function of x, which we can interpret as giving the slope of the tangent line to the graph of f at $(x,f(x))$. For example, if $x = 1$, then the slope is $f'(1) = 2 \cdot 1 = 2$, which confirms the result in Example 1.

Now Work Problem 3 ◁

CAUTION!

The notation $\frac{dy}{dx}$, which is called *Leibniz notation*, should **not** be thought of as a fraction, although it looks like one. It is a single symbol for a derivative. We have not yet attached any meaning to individual symbols such as dy and dx.

Besides the notation $f'(x)$, other common ways to denote the derivative of $y = f(x)$ at x are

$\frac{dy}{dx}$	pronounced "dee y, dee x" or "dee y by dee x"
$\frac{d}{dx}(f(x))$	"dee $f(x)$, dee x" or "dee by dee x of $f(x)$"
y'	"y prime"
$D_x y$	"dee x of y"
$D_x(f(x))$	"dee x of $f(x)$"

Because the derivative gives the slope of the tangent line, $f'(a)$ is the slope of the line tangent to the graph of $y = f(x)$ at $(a,f(a))$.

Two other notations for the derivative of f at a are

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{and} \quad y'(a)$$

EXAMPLE 3 Finding an Equation of a Tangent Line

If $f(x) = 2x^2 + 2x + 3$, find an equation of the tangent line to the graph of f at $(1, 7)$.

Solution:

Strategy We will first determine the slope of the tangent line by computing the derivative and evaluating it at $x = 1$. Using this result and the point $(1, 7)$ in a point-slope form gives an equation of the tangent line.

In Example 3 it is *not* correct to say that, since the derivative is $4x + 2$, the tangent line at $(1, 7)$ is $y - 7 = (4x + 2)(x - 1)$. (This is not even the equation of a line.) The derivative must be **evaluated** at the point of tangency to determine the slope of the tangent line.

We have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(2(x+h)^2 + 2(x+h) + 3) - (2x^2 + 2x + 3)}{h} \\ &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 2x + 2h + 3 - 2x^2 - 2x - 3}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 + 2h}{h} = \lim_{h \to 0} (4x + 2h + 2) \end{aligned}$$

So

$$f'(x) = 4x + 2$$

and

$$f'(1) = 4(1) + 2 = 6$$

Thus, the tangent line to the graph at (1, 7) has slope 6. A point-slope form of this tangent is

$$y - 7 = 6(x - 1)$$

which in slope-intercept form is

$$y = 6x + 1$$

Now Work Problem 25 ◁

EXAMPLE 4 Finding the Slope of a Curve at a Point

Find the slope of the curve $y = 2x + 3$ at the point where $x = 6$.

Solution: The slope of the curve is the slope of the tangent line. Letting $y = f(x) = 2x + 3$, we have

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(2(x+h) + 3) - (2x + 3)}{h} \\ &= \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2 \end{aligned}$$

Since $dy/dx = 2$, the slope when $x = 6$, or in fact at any point, is 2. Note that the curve is a straight line and thus has the same slope at each point.

Now Work Problem 19 ◁

EXAMPLE 5 A Function with a Vertical Tangent Line

Find $\dfrac{d}{dx}(\sqrt{x})$.

Solution: Letting $f(x) = \sqrt{x}$, we have

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Rationalizing numerators or denominators of fractions is often helpful in calculating limits.

As $h \to 0$, both the numerator and denominator approach zero. This can be avoided by rationalizing the *numerator*:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

Therefore,

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

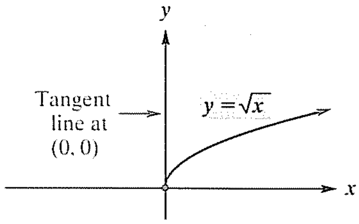


FIGURE 11.8 Vertical tangent line at (0, 0).

Note that the original function, $\sqrt{x}$, is defined for $x \geq 0$, but its derivative, $1/(2\sqrt{x})$, is defined only when $x > 0$. The reason for this is clear from the graph of $y = \sqrt{x}$ in Figure 11.8. When $x = 0$, the tangent is a vertical line, so its slope is not defined.

Now Work Problem 17 ◁

In Example 5 we saw that the function $y = \sqrt{x}$ is not differentiable when $x = 0$, because the tangent line is vertical at that point. It is worthwhile to mention that $y = |x|$ also is not differentiable when $x = 0$, but for a different reason: There is *no* tangent line at all at that point. (Refer to Figure 11.5.) Both examples show that the domain of f' may be strictly contained in the domain of f.

To indicate a derivative, Leibniz notation is often useful because it makes it convenient to emphasize the independent and dependent variables involved. For example, if the variable p is a function of the variable q, we speak of the derivative of p with respect to q, written dp/dq.

Variables other than x and y are often more natural in applied problems. Time denoted by t, quantity by q, and price by p are obvious examples. Example 6 illustrates.

EXAMPLE 6 Finding the Derivative of p with Respect to q

If $p = f(q) = \dfrac{1}{2q}$, find $\dfrac{dp}{dq}$.

Solution: We will do this problem first using the $h \to 0$ limit (the only one we have used so far) and then using $r \to q$ to illustrate the other variant of the limit.

$$\frac{dp}{dq} = \frac{d}{dq}\left(\frac{1}{2q}\right) = \lim_{h\to 0} \frac{f(q+h)-f(q)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2(q+h)} - \frac{1}{2q}}{h} = \lim_{h\to 0} \frac{\frac{q-(q+h)}{2q(q+h)}}{h}$$

$$= \lim_{h\to 0} \frac{q-(q+h)}{h(2q(q+h))} = \lim_{h\to 0} \frac{-h}{h(2q(q+h))}$$

$$= \lim_{h\to 0} \frac{-1}{2q(q+h)} = -\frac{1}{2q^2}$$

We also have

$$\frac{dp}{dq} = \lim_{r\to q} \frac{f(r)-f(q)}{r-q}$$

$$= \lim_{r\to q} \frac{\frac{1}{2r} - \frac{1}{2q}}{r-q} = \lim_{r\to q} \frac{\frac{q-r}{2rq}}{r-q}$$

$$= \lim_{r\to q} \frac{-1}{2rq} = \frac{-1}{2q^2}$$

APPLY IT ▷

1. If a ball is thrown upward at a speed of 40 ft/s from a height of 6 feet, its height H in feet after t seconds is given by $H = 6 + 40t - 16t^2$. Find $\dfrac{dH}{dt}$.

We leave it you to decide which form leads to the simpler limit calculation in this case.

Note that when $q = 0$ the function is not defined, so the derivative is also not even defined when $q = 0$.

Now Work Problem 15 ◁

Keep in mind that the derivative of $y = f(x)$ at x is nothing more than a limit, namely

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

equivalently

$$\lim_{z\to x} \frac{f(z)-f(x)}{z-x}$$

whose use we have just illustrated. Although we can interpret the derivative as a function that gives the slope of the tangent line to the curve $y = f(x)$ at the point $(x, f(x))$, this interpretation is simply a geometric convenience that assists our understanding. The preceding limit may exist, aside from any geometric considerations at all. As we will see later, there are other useful interpretations of the derivative.

In Section 11.4, we will make technical use of the following relationship between differentiability and continuity. However, it is of fundamental importance and needs to be understood from the outset.

If f is differentiable at a, then f is continuous at a.

To establish this result, we will assume that f is differentiable at a. Then $f'(a)$ exists, and

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

Consider the numerator $f(a+h) - f(a)$ as $h \to 0$. We have

$$\begin{aligned}\lim_{h\to 0}(f(a+h)-f(a)) &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\cdot h\right)\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot \lim_{h\to 0} h\\ &= f'(a)\cdot 0 = 0\end{aligned}$$

Thus, $\lim_{h\to 0}(f(a+h)-f(a)) = 0$. This means that $f(a+h) - f(a)$ approaches 0 as $h \to 0$. Consequently,

$$\lim_{h\to 0} f(a+h) = f(a)$$

As stated in Section 10.3, this condition means that f is continuous at a. The foregoing, then, proves that f is continuous at a when f is differentiable there. More simply, we say that **differentiability at a point implies continuity at that point.**

If a function is not continuous at a point, then it cannot have a derivative there. For example, the function in Figure 11.9 is discontinuous at a. The curve has no tangent at that point, so the function is not differentiable there.

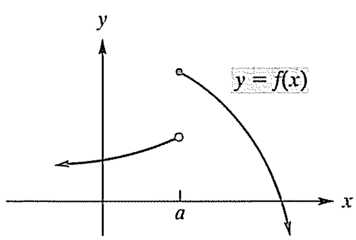


FIGURE 11.9 f is not continuous at a, so f is not differentiable at a.

EXAMPLE 7 Continuity and Differentiability

a. Let $f(x) = x^2$. The derivative, $2x$, is defined for all values of x, so $f(x) = x^2$ must be continuous for all values of x.

b. The function $f(p) = \dfrac{1}{2p}$ is not continuous at $p = 0$ because f is not defined there. Thus, the derivative does not exist at $p = 0$.

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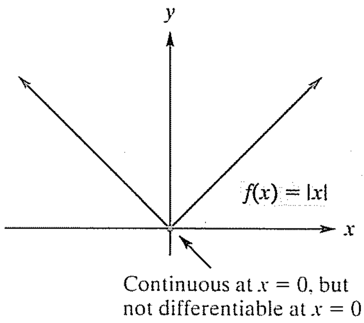

FIGURE 11.10 Continuity does not imply differentiability.

The converse of the statement that differentiability implies continuity is *false*. That is, continuity does not imply differentiability. In Example 8, we give a function that is continuous at a point, but not differentiable there.

EXAMPLE 8 Continuity Does Not Imply Differentiability

The function $y = f(x) = |x|$ is continuous at $x = 0$. (See Figure 11.10.) As we mentioned earlier, there is no tangent line at $x = 0$. Thus, the derivative does not exist there. This shows that continuity does *not* imply differentiability.

◁

Finally, we remark that while differentiability of f at a implies continuity of f at a, the derivative function, f', is not necessarily continuous at a. Unfortunately, the classic example is constructed from a function not considered in this book.

PROBLEMS 11.1

In Problems 1 and 2, a function f and a point P on its graph are given.

(a) *Find the slope of the secant line PQ for each point $Q = (x, f(x))$ whose x-value is given in the table. Round your answers to four decimal places.*
(b) *Use your results from part (a) to estimate the slope of the tangent line at P.*

1. $f(x) = x^3 + 3, P = (-2, -5)$

x-value of Q	-3	-2.5	-2.2	-2.1	-2.01	-2.001
m_{PQ}						

2. $f(x) = e^x, P = (0, 1)$

x-value of Q	1	0.5	0.2	0.1	0.01	0.001
m_{PQ}						

In Problems 3–18, use the definition of the derivative to find each of the following.

3. $f'(x)$ if $f(x) = x$

4. $f'(x)$ if $f(x) = 4x - 1$

5. $\dfrac{dy}{dx}$ if $y = 3x + 5$

6. $\dfrac{dy}{dx}$ if $y = -5x$

7. $\dfrac{d}{dx}(3 - 2x)$

8. $\dfrac{d}{dx}\left(1 - \dfrac{x}{2}\right)$

9. $f'(x)$ if $f(x) = 3$

10. $f'(x)$ if $f(x) = 7.01$

11. $\dfrac{d}{dx}(x^2 + 4x - 8)$

12. y' if $y = x^2 + 3x + 2$

13. $\dfrac{dp}{dq}$ if $p = 3q^2 + 2q + 1$

14. $\dfrac{d}{dx}(x^2 - x - 3)$

15. y' if $y = \dfrac{6}{x}$

16. $\dfrac{dC}{dq}$ if $C = 7 + 2q - 3q^2$

17. $f'(x)$ if $f(x) = \sqrt{2x}$

18. $H'(x)$ if $H(x) = \dfrac{3}{x - 2}$

19. Find the slope of the curve $y = x^2 + 4$ at the point $(-2, 8)$.

20. Find the slope of the curve $y = 1 - x^2$ at the point $(1, 0)$.

21. Find the slope of the curve $y = 4x^2 - 5$ when $x = 0$.

22. Find the slope of the curve $y = \sqrt{2x}$ when $x = 18$.

In Problems 23–28, find an equation of the tangent line to the curve at the given point.

23. $y = x + 4$; $(3, 7)$

24. $y = 3x^2 - 4$; $(1, -1)$

25. $y = x^2 + 2x + 3$; $(1, 6)$

26. $y = (x - 7)^2$; $(6, 1)$

27. $y = \dfrac{4}{x + 1}$; $(3, 1)$

28. $y = \dfrac{5}{1 - 3x}$; $(2, -1)$

29. Banking Equations may involve derivatives of functions. In an article on interest rate deregulation, Christofi and Agapos[1] solve the equation

$$r = \left(\frac{\eta}{1 + \eta}\right)\left(r_L - \frac{dC}{dD}\right)$$

for η (the Greek letter "eta"). Here r is the deposit rate paid by commercial banks, r_L is the rate earned by commercial banks, C is the administrative cost of transforming deposits into return-earning assets, D is the savings deposits level, and η is the deposit elasticity with respect to the deposit rate. Find η.

In Problems 30 and 31, use the numerical derivative feature of your graphing calculator to estimate the derivatives of the functions at the indicated values. Round your answers to three decimal places.

30. $f(x) = \sqrt{2x^2 + 3x}$; $x = 1, x = 2$

31. $f(x) = e^x(4x - 7)$; $x = 0, x = 1.5$

In Problems 32 and 33, use the "limit of a difference quotient" approach to estimate $f'(x)$ at the indicated values of x. Round your answers to three decimal places.

32. $f(x) = x \ln x - x$; $x = 1, x = 10$

33. $f(x) = \dfrac{x^2 + 4x + 2}{x^3 - 3}$; $x = 2, x = -4$

[1] A. Christofi and A. Agapos, "Interest Rate Deregulation: An Empirical Justification," *Review of Business and Economic Research*, XX, no. 1 (1984), 39–49.

34. Find an equation of the tangent line to the curve $f(x) = x^2 + x$ at the point $(-2, 2)$. Graph both the curve and the tangent line. Notice that the tangent line is a good approximation to the curve near the point of tangency.

35. The derivative of $f(x) = x^3 - x + 2$ is $f'(x) = 3x^2 - 1$. Graph both the function f and its derivative f'. Observe that there are two points on the graph of f where the tangent line is horizontal. For the x-values of these points, what are the corresponding values of $f'(x)$? Why are these results expected? Observe the intervals where $f'(x)$ is positive. Notice that tangent lines to the graph of f have positive slopes over these intervals. Observe the interval where $f'(x)$ is negative. Notice that tangent lines to the graph of f have negative slopes over this interval.

In Problems 36 and 37, verify the identity $(z - x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) = z^n - x^n$ *for the indicated values of n and calculate the derivative using the* $z \to x$ *form of the definition of the derivative in Equation (2).*

36. $n = 4, n = 3, n = 2;\quad f'(x)$ if $f(x) = 2x^4 + x^3 - 3x^2$

37. $n = 5, n = 3;\quad f'(x)$ if $f(x) = 4x^5 - 3x^3$

Objective

To develop the basic rules for differentiating constant functions and power functions and the combining rules for differentiating a constant multiple of a function and a sum of two functions.

11.2 Rules for Differentiation

Differentiating a function by direct use of the definition of derivative can be tedious. However, if a function is constructed from simpler functions, then the derivative of the more complicated function can be constructed from the derivatives of the simpler functions. Ultimately, we need to know only the derivatives of a few basic functions and ways to assemble derivatives of constructed functions from the derivatives of their components. For example, if functions f and g have derivatives f' and g', respectively, then $f + g$ has a derivative given by $(f + g)' = f' + g'$. However, some *rules* are less intuitive. For example, if $f \cdot g$ denotes the function whose value at x is given by $(f \cdot g)(x) = f(x) \cdot g(x)$, then $(f \cdot g)' = f' \cdot g + f \cdot g'$. In this chapter we study most such combining rules and some basic rules for calculating derivatives of certain basic functions.

We begin by showing that the derivative of a constant function is zero. Recall that the graph of the constant function $f(x) = c$ is a horizontal line (see Figure 11.11), which has a slope of zero at each point. This means that $f'(x) = 0$ regardless of x. As a formal proof of this result, we apply the definition of the derivative to $f(x) = c$:

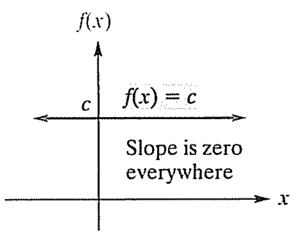

FIGURE 11.11 The slope of a constant function is 0.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h}$$

$$= \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

Thus, we have our first rule:

BASIC RULE 0 Derivative of a Constant

If c is a constant, then

$$\frac{d}{dx}(c) = 0$$

That is, the derivative of a constant function is zero.

EXAMPLE 1 Derivatives of Constant Functions

a. $\dfrac{d}{dx}(3) = 0$ because 3 is a constant function.

b. If $g(x) = \sqrt{5}$, then $g'(x) = 0$ because g is a constant function. For example, the derivative of g when $x = 4$ is $g'(4) = 0$.

c. If $s(t) = (1{,}938{,}623)^{807.4}$, then $ds/dt = 0$.

Now Work Problem 1 ◁

The next rule gives a formula for the derivative of "x raised to a constant power"—that is, the derivative of $f(x) = x^a$, where a is an arbitrary real number. A function of this form is called a **power function.** For example, $f(x) = x^2$ is a power function. While the rule we record is valid for all real a, we will establish it only in the case where

a is a positive integer, n. The rule is so central to differential calculus that it warrants a detailed calculation—if only in the case where a is a positive integer, n. Whether we use the $h \to 0$ form of the definition of derivative or the $z \to x$ form, the calculation of $\frac{dx^n}{dx}$ is instructive and provides good practice with summation notation, whose use is more essential in later chapters. We provide a calculation for each possibility. We must either expand $(x+h)^n$, to use the $h \to 0$ form of Equation (2) from Section 11.1, or factor $z^n - x^n$, to use the $z \to x$ form.

For the first of these we recall the *binomial theorem* of Section 9.2:

$$(x+h)^n = \sum_{i=0}^{n} {}_nC_i x^{n-i} h^i$$

where the ${}_nC_i$ are the binomial coefficients, whose precise descriptions, except for ${}_nC_0 = 1$ and ${}_nC_1 = n$, are not necessary here (but are given in Section 8.2). For the second we have

$$(z-x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) = z^n - x^n$$

which is easily verified by carrying out the multiplication using the rules for manipulating summations given in Section 1.5. In fact, we have

$$\begin{aligned}(z-x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) &= z\sum_{i=0}^{n-1} x^i z^{n-1-i} - x\sum_{i=0}^{n-1} x^i z^{n-1-i}\\ &= \sum_{i=0}^{n-1} x^i z^{n-i} - \sum_{i=0}^{n-1} x^{i+1} z^{n-1-i}\\ &= \left(z^n + \sum_{i=1}^{n-1} x^i z^{n-i}\right) - \left(\sum_{i=0}^{n-2} x^{i+1} z^{n-1-i} + x^n\right)\\ &= z^n - x^n\end{aligned}$$

where the reader should check that the two summations in the second to last line really do cancel as shown.

BASIC RULE 1 Derivative of x^a

If a is any real number, then

$$\frac{d}{dx}(x^a) = ax^{a-1}$$

That is, the derivative of a constant power of x is the exponent times x raised to a power one less than the given power.

CAUTION!

There is a lot more to calculus than this rule.

For n a positive integer, if $f(x) = x^n$, the definition of the derivative gives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

By our previous discussion on expanding $(x+h)^n$,

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{\sum_{i=0}^{n} {}_nC_i x^{n-i} h^i - x^n}{h}\\ &\overset{(1)}{=} \lim_{h\to 0} \frac{\sum_{i=1}^{n} {}_nC_i x^{n-i} h^i}{h}\end{aligned}$$

$$\overset{(2)}{=} \lim_{h \to 0} \frac{h \sum_{i=1}^{n} {}_nC_i x^{n-i} h^{i-1}}{h}$$

$$\overset{(3)}{=} \lim_{h \to 0} \sum_{i=1}^{n} {}_nC_i x^{n-i} h^{i-1}$$

$$\overset{(4)}{=} \lim_{h \to 0} \left(nx^{n-1} + \sum_{i=2}^{n} {}_nC_i x^{n-i} h^{i-1} \right)$$

$$\overset{(5)}{=} nx^{n-1}$$

where we justify the further steps as follows:

(1) The $i = 0$ term in the summation is ${}_nC_0 x^n h^0 = x^n$ so it cancels with the separate, last, term: $-x^n$.

(2) We are able to extract a common factor of h from each term in the sum.

(3) This is the crucial step. The expressions separated by the equal sign are limits as $h \to 0$ of functions of h that are equal for $h \neq 0$.

(4) The $i = 1$ term in the summation is ${}_nC_1 x^{n-1} h^0 = nx^{n-1}$. It is the only one that does not contain a factor of h, and we separated it from the other terms.

(5) Finally, in determining the limit we made use of the fact that the isolated term is independent of h; while all the others contain h as a factor and so have limit 0 as $h \to 0$.

Now, using the $z \to x$ limit for the definition of the derivative and $f(x) = x^n$, we have

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{h \to 0} \frac{z^n - x^n}{z - x}$$

By our previous discussion on factoring $z^n - x^n$, we have

$$f'(x) = \lim_{z \to x} \frac{(z - x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i} \right)}{z - x}$$

$$\overset{(1)}{=} \lim_{z \to x} \sum_{i=0}^{n-1} x^i z^{n-1-i}$$

$$\overset{(2)}{=} \sum_{i=0}^{n-1} x^i x^{n-1-i}$$

$$\overset{(3)}{=} \sum_{i=0}^{n-1} x^{n-1}$$

$$\overset{(4)}{=} nx^{n-1}$$

where this time we justify the further steps as follows:

(1) Here the crucial step comes first. The expressions separated by the equal sign are limits as $z \to x$ of functions of z that are equal for $z \neq x$.

(2) The limit is given by evaluation because the expression is a polynomial in the variable z.

(3) An obvious rule for exponents is used.

(4) Each term in the sum is x^{n-1}, independent of i, and there are n such terms.

EXAMPLE 2 **Derivatives of Powers of x**

a. By Basic Rule 1, $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$.

b. If $F(x) = x = x^1$, then $F'(x) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$. Thus, the derivative of x with respect to x is 1.

c. If $f(x) = x^{-10}$, then $f'(x) = -10x^{-10-1} = -10x^{-11}$.

Now Work Problem 3 ◁

When we apply a differentiation rule to a function, sometimes the function must first be rewritten so that it has the proper form for that rule. For example, to differentiate $f(x) = \frac{1}{x^{10}}$ we would first rewrite f as $f(x) = x^{-10}$ and then proceed as in Example 2(c).

EXAMPLE 3 **Rewriting Functions in the Form x^a**

a. To differentiate $y = \sqrt{x}$, we rewrite $\sqrt{x}$ as $x^{1/2}$ so that it has the form x^n. Thus,

$$\frac{dy}{dx} = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

which agrees with our limit calculation in Example 5 of Section 11.1.

CAUTION!

In Example 3(b), do not rewrite $\frac{1}{x\sqrt{x}}$ as $\frac{1}{x^{3/2}}$ and then merely differentiate the denominator.

b. Let $h(x) = \frac{1}{x\sqrt{x}}$. To apply Basic Rule 1, we must rewrite $h(x)$ as $h(x) = x^{-3/2}$ so that it has the form x^n. We have

$$h'(x) = \frac{d}{dx}(x^{-3/2}) = -\frac{3}{2}x^{(-3/2)-1} = -\frac{3}{2}x^{-5/2}$$

Now Work Problem 39 ◁

Now that we can say immediately that the derivative of x^3 is $3x^2$, the question arises as to what we could say about the derivative of a *multiple* of x^3, such as $5x^3$. Our next rule will handle this situation of differentiating a constant times a function.

COMBINING RULE 1 **Constant Factor Rule**

If f is a differentiable function and c is a constant, then $cf(x)$ is differentiable, and

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

That is, the derivative of a constant times a function is the constant times the derivative of the function.

Proof. If $g(x) = cf(x)$, applying the definition of the derivative of g gives

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= \lim_{h\to 0}\left(c\cdot\frac{f(x+h)-f(x)}{h}\right) = c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is $f'(x)$; so $g'(x) = cf'(x)$.

EXAMPLE 4 Differentiating a Constant Times a Function

Differentiate the following functions.

a. $g(x) = 5x^3$

Solution: Here g is a constant (5) times a function (x^3). So

$$\frac{d}{dx}(5x^3) = 5\frac{d}{dx}(x^3) \qquad \text{Combining Rule 1}$$

$$= 5(3x^{3-1}) = 15x^2 \qquad \text{Basic Rule 1}$$

b. $f(q) = \dfrac{13q}{5}$

Solution:

Strategy We first rewrite f as a constant times a function and then apply Basic Rule 1.

Because $\dfrac{13q}{5} = \dfrac{13}{5}q$, f is the constant $\dfrac{13}{5}$ times the function q. Thus,

$$f'(q) = \frac{13}{5}\frac{d}{dq}(q) \qquad \text{Combining Rule 1}$$

$$= \frac{13}{5}\cdot 1 = \frac{13}{5} \qquad \text{Basic Rule 1}$$

c. $y = \dfrac{0.25}{\sqrt[5]{x^2}}$

Solution: We can express y as a constant times a function:

$$y = 0.25 \cdot \frac{1}{\sqrt[5]{x^2}} = 0.25x^{-2/5}$$

Hence,

$$y' = 0.25\frac{d}{dx}(x^{-2/5}) \qquad \text{Combining Rule 1}$$

$$= 0.25\left(-\frac{2}{5}x^{-7/5}\right) = -0.1x^{-7/5} \qquad \text{Basic Rule 1}$$

Now Work Problem 7 ◁

CAUTION!

In differentiating $f(x) = (4x)^3$, Basic Rule 1 cannot be applied directly. It applies to a power of the variable x, *not* to a power of an expression involving x, such as $4x$. To apply our rules, write $f(x) = (4x)^3 = 4^3x^3 = 64x^3$. Thus,

$$f'(x) = 64\frac{d}{dx}(x^3) = 64(3x^2) = 192x^2.$$

The next rule involves derivatives of sums and differences of functions.

COMBINING RULE 2 Sum or Difference Rule

If f and g are differentiable functions, then $f + g$ and $f - g$ are differentiable, and

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

and

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

That is, the derivative of the sum (difference) of two functions is the sum (difference) of their derivatives.

Proof. For the case of a sum, if $F(x) = f(x) + g(x)$, applying the definition of the derivative of F gives

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0}\frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h)-f(x)) + (g(x+h) - g(x))}{h} \qquad \text{regrouping}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right)$$

Because the limit of a sum is the sum of the limits,

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

But these two limits are $f'(x)$ and $g'(x)$. Thus,

$$F'(x) = f'(x) + g'(x)$$

The proof for the derivative of a difference of two functions is similar.

Combining Rule 2 can be extended to the derivative of any number of sums and differences of functions. For example,

$$\frac{d}{dx}[f(x) - g(x) + h(x) + k(x)] = f'(x) - g'(x) + h'(x) + k'(x)$$

APPLY IT ▷

2. If the revenue function for a certain product is $r(q) = 50q - 0.3q^2$, find the derivative of this function, also known as the marginal revenue.

EXAMPLE 5 Differentiating Sums and Differences of Functions

Differentiate the following functions.

a. $F(x) = 3x^5 + \sqrt{x}$

Solution: Here F is the sum of two functions, $3x^5$ and $\sqrt{x}$. Therefore,

$$F'(x) = \frac{d}{dx}(3x^5) + \frac{d}{dx}(x^{1/2}) \qquad \text{Combining Rule 2}$$

$$= 3\frac{d}{dx}(x^5) + \frac{d}{dx}(x^{1/2}) \qquad \text{Combining Rule 1}$$

$$= 3(5x^4) + \frac{1}{2}x^{-1/2} = 15x^4 + \frac{1}{2\sqrt{x}} \qquad \text{Basic Rule 1}$$

b. $f(z) = \dfrac{z^4}{4} - \dfrac{5}{z^{1/3}}$

Solution: To apply our rules, we will rewrite $f(z)$ in the form $f(z) = \frac{1}{4}z^4 - 5z^{-1/3}$. Since f is the difference of two functions,

$$f'(z) = \frac{d}{dz}\left(\frac{1}{4}z^4\right) - \frac{d}{dz}(5z^{-1/3}) \qquad \text{Combining Rule 2}$$

$$= \frac{1}{4}\frac{d}{dz}(z^4) - 5\frac{d}{dz}(z^{-1/3}) \qquad \text{Combining Rule 1}$$

$$= \frac{1}{4}(4z^3) - 5\left(-\frac{1}{3}z^{-4/3}\right) \qquad \text{Basic Rule 1}$$

$$= z^3 + \frac{5}{3}z^{-4/3}$$

c. $y = 6x^3 - 2x^2 + 7x - 8$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(6x^3) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(8)$$

$$= 6\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) - \frac{d}{dx}(8)$$

$$= 6(3x^2) - 2(2x) + 7(1) - 0$$

$$= 18x^2 - 4x + 7$$

Now Work Problem 47 ◁

In Examples 6 and 7, we need to rewrite the given function in a form to which our rules apply.

EXAMPLE 6 Finding a Derivative

Find the derivative of $f(x) = 2x(x^2 - 5x + 2)$ when $x = 2$.

Solution: We multiply and then differentiate each term:

$$\begin{aligned} f(x) &= 2x^3 - 10x^2 + 4x \\ f'(x) &= 2(3x^2) - 10(2x) + 4(1) \\ &= 6x^2 - 20x + 4 \\ f'(2) &= 6(2)^2 - 20(2) + 4 = -12 \end{aligned}$$

Now Work Problem 75 ◁

EXAMPLE 7 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the curve

$$y = \frac{3x^2 - 2}{x}$$

when $x = 1$.

Solution:

Strategy First we find $\frac{dy}{dx}$, which gives the slope of the tangent line at any point. Evaluating $\frac{dy}{dx}$ when $x = 1$ gives the slope of the required tangent line. We then determine the y-coordinate of the point on the curve when $x = 1$. Finally, we substitute the slope and both of the coordinates of the point in point-slope form to obtain an equation of the tangent line.

Rewriting y as a difference of two functions, we have

$$y = \frac{3x^2}{x} - \frac{2}{x} = 3x - 2x^{-1}$$

Thus,

$$\frac{dy}{dx} = 3(1) - 2((-1)x^{-2}) = 3 + \frac{2}{x^2}$$

The slope of the tangent line to the curve when $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 + \frac{2}{1^2} = 5$$

To find the y-coordinate of the point on the curve where $x = 1$, we evaluate $y = \frac{3x^2 - 2}{x}$ at $x = 1$. This gives

$$y = \frac{3(1)^2 - 2}{1} = 1$$

CAUTION!

To obtain the y-value of the point on the curve when $x = 1$, evaluate the *original* function at $x = 1$.

Hence, the point $(1, 1)$ lies on both the curve and the tangent line. Therefore, an equation of the tangent line is

$$y - 1 = 5(x - 1)$$

In slope-intercept form, we have

$$y = 5x - 4$$

Now Work Problem 81 ◁

PROBLEMS 11.2

In Problems 1–74, differentiate the functions.

1. $f(x) = \pi$
2. $f(x) = \left(\frac{6}{7}\right)^{2/3}$
3. $y = x^6$
4. $f(x) = x^{21}$
5. $y = x^{80}$
6. $y = x^{2.1}$
7. $f(x) = 9x^2$
8. $y = 4x^3$
9. $g(w) = 8w^7$
10. $v(x) = x^e$
11. $y = \frac{3}{5}x^6$
12. $f(p) = \sqrt{3}p^4$
13. $f(t) = \dfrac{t^7}{25}$
14. $y = \dfrac{x^7}{7}$
15. $f(x) = x + 3$
16. $f(x) = 5x - e$
17. $f(x) = 4x^2 - 2x + 3$
18. $F(x) = 5x^2 - 9x$
19. $g(p) = p^4 - 3p^3 - 1$
20. $f(t) = -13t^2 + 14t + 1$
21. $y = x^4 - \sqrt[3]{x}$
22. $y = -8x^4 + \ln 2$
23. $y = -13x^3 + 14x^2 - 2x + 3$
24. $V(r) = r^8 - 7r^6 + 3r^2 + 1$
25. $f(x) = 2(13 - x^4)$
26. $\psi(t) = e(t^7 - 5^3)$
27. $g(x) = \dfrac{13 - x^4}{3}$
28. $f(x) = \dfrac{5(x^4 - 6)}{2}$
29. $h(x) = 4x^4 + x^3 - \dfrac{9x^2}{2} + 8x$
30. $k(x) = -2x^2 + \dfrac{5}{3}x + 11$
31. $f(x) = \dfrac{5}{7}x^9 + \dfrac{3}{5}x^7$
32. $p(x) = \dfrac{x^7}{7} + \dfrac{2x}{3}$
33. $f(x) = x^{3/5}$
34. $f(x) = 2x^{-14/5}$
35. $y = x^{3/4} + 2x^{5/3}$
36. $y = 4x^2 - x^{-3/5}$
37. $y = 11\sqrt{x}$
38. $y = \sqrt{x^7}$
39. $f(r) = 6\sqrt[3]{r}$
40. $y = 4\sqrt[8]{x^2}$
41. $f(x) = x^{-6}$
42. $f(s) = 2s^{-3}$
43. $f(x) = x^{-3} + x^{-5} - 2x^{-6}$
44. $f(x) = 100x^{-3} + 10x^{1/2}$
45. $y = \dfrac{1}{x}$
46. $f(x) = \dfrac{3}{x^4}$
47. $y = \dfrac{8}{x^5}$
48. $y = \dfrac{1}{4x^5}$
49. $g(x) = \dfrac{4}{3x^3}$
50. $y = \dfrac{1}{x^2}$
51. $f(t) = \dfrac{3}{5t^3}$
52. $g(x) = \dfrac{7}{9x}$
53. $f(x) = \dfrac{x}{7} + \dfrac{7}{x}$
54. $\Phi(x) = \dfrac{x^3}{3} - \dfrac{3}{x^3}$
55. $f(x) = -9x^{1/3} + 5x^{-2/5}$
56. $f(z) = 5z^{3/4} - 6^2 - 8z^{1/4}$
57. $q(x) = \dfrac{1}{\sqrt[3]{8x^2}}$
58. $f(x) = \dfrac{3}{\sqrt[4]{x^3}}$
59. $y = \dfrac{2}{\sqrt{x}}$
60. $y = \dfrac{1}{2\sqrt{x}}$
61. $y = x^3\sqrt[3]{x}$
62. $f(x) = (2x^3)(4x^2)$
63. $f(x) = x(3x^2 - 10x + 7)$
64. $f(x) = x^3(3x^6 - 5x^2 + 4)$
65. $f(x) = x^3(3x)^2$
66. $s(x) = \sqrt{x}(\sqrt[5]{x} + 7x + 2)$
67. $v(x) = x^{-2/3}(x + 5)$
68. $f(x) = x^{3/5}(x^2 + 7x + 11)$
69. $f(q) = \dfrac{3q^2 + 4q - 2}{q}$
70. $f(w) = \dfrac{w - 5}{w^5}$
71. $f(x) = (x - 1)(x + 2)$
72. $f(x) = x^2(x - 2)(x + 4)$
73. $w(x) = \dfrac{x^2 + x^3}{x^2}$
74. $f(x) = \dfrac{7x^3 + x}{6\sqrt{x}}$

For each curve in Problems 75–78, find the slopes at the indicated points.

75. $y = 3x^2 + 4x - 8$; $(0, -8), (2, 12), (-3, 7)$
76. $y = 3 + 5x - 3x^3$; $(0, 3), (\frac{1}{2}, \frac{41}{8}), (2, -11)$
77. $y = 4$; when $x = -4, x = 7, x = 22$
78. $y = 3x - 4\sqrt{x}$; when $x = 4, x = 9, x = 25$

In Problems 79–82, find an equation of the tangent line to the curve at the indicated point.

79. $y = 4x^2 + 5x + 6$; $(1, 15)$
80. $y = \dfrac{1 - x^2}{5}$; $(4, -3)$
81. $y = \dfrac{1}{x^2}$; $(2, \frac{1}{4})$
82. $y = -\sqrt[3]{x}$; $(8, -2)$

83. Find an equation of the tangent line to the curve

$$y = 3 + x - 5x^2 + x^4$$

when $x = 0$.

84. Repeat Problem 83 for the curve

$$y = \frac{\sqrt{x}(2 - x^2)}{x}$$

when $x = 4$.

85. Find all points on the curve

$$y = \frac{5}{2}x^2 - x^3$$

where the tangent line is horizontal.

86. Repeat Problem 85 for the curve

$$y = \frac{x^6}{6} - \frac{x^2}{2} + 1$$

87. Find all points on the curve

$$y = x^2 - 5x + 3$$

where the slope is 1.

88. Repeat Problem 87 for the curve

$$y = x^4 - 31x + 11$$

89. If $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, evaluate the expression

$$\frac{x - 1}{2x\sqrt{x}} - f'(x)$$

90. Economics Eswaran and Kotwal[2] consider agrarian economies in which there are two types of workers, permanent and casual. Permanent workers are employed on long-term contracts and may receive benefits such as holiday gifts and emergency aid. Casual workers are hired on a daily basis and perform routine and menial tasks such as weeding, harvesting, and threshing. The difference z in the present-value cost of hiring a permanent worker over that of hiring a casual worker is given by

$$z = (1 + b)w_p - bw_c$$

where w_p and w_c are wage rates for permanent labor and casual labor, respectively, b is a constant, and w_p is a function of w_c. Eswaran and Kotwal claim that

$$\frac{dz}{dw_c} = (1 + b)\left[\frac{dw_p}{dw_c} - \frac{b}{1+b}\right]$$

Verify this.

91. Find an equation of the tangent line to the graph of $y = x^3 - 2x + 1$ at the point $(1, 0)$. Graph both the function and the tangent line on the same screen.

92. Find an equation of the tangent line to the graph of $y = \sqrt[3]{x}$, at the point $(-8, -2)$. Graph both the function and the tangent line on the same screen. Notice that the line passes through $(-8, -2)$ and the line appears to be tangent to the curve.

Objective

To motivate the instantaneous rate of change of a function by means of velocity and to interpret the derivative as an instantaneous rate of change. To develop the "marginal" concept, which is frequently used in business and economics.

11.3 The Derivative as a Rate of Change

We have given a geometric interpretation of the derivative as being the slope of the tangent line to a curve at a point. Historically, an important application of the derivative involves the motion of an object traveling in a straight line. This gives us a convenient way to interpret the derivative as a *rate of change*.

To denote the change in a variable such as x, the symbol Δx (read "delta x") is commonly used. For example, if x changes from 1 to 3, then the change in x is $\Delta x = 3 - 1 = 2$. The new value of x(=3) is the old value plus the change, which is $1 + \Delta x$. Similarly, if t increases by Δt, the new value is $t + \Delta t$. We will use Δ-notation in the discussion that follows.

Suppose an object moves along the number line in Figure 11.12 according to the equation

$$s = f(t) = t^2$$

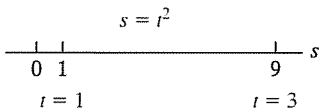

FIGURE 11.12 Motion along a number line.

where s is the position of the object at time t. This equation is called an **equation of motion,** and f is called a ***position function.*** Assume that t is in seconds and s is in meters. At $t = 1$ the position is $s = f(1) = 1^2 = 1$, and at $t = 3$ the position is $s = f(3) = 3^2 = 9$. Over this two-second time interval, the object has a change in position, or a *displacement,* of $9 - 1 = 8$ m, and the *average velocity* of the object is defined as

$$v_{\text{ave}} = \frac{\text{displacement}}{\text{length of time interval}} \tag{1}$$

$$= \frac{8}{2} = 4 \text{ m/s}$$

To say that the average velocity is 4 m/s from $t = 1$ to $t = 3$ means that, *on the average,* the position of the object changed by 4 m to the right each second during that time interval. Let us denote the changes in s-values and t-values by Δs and Δt, respectively. Then the average velocity is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = 4 \text{ m/s} \quad (\text{for the interval } t = 1 \text{ to } t = 3)$$

The ratio $\Delta s/\Delta t$ is also called the **average rate of change of s with respect to t** over the interval from $t = 1$ to $t = 3$.

Now, let the time interval be only 1 second long (that is, $\Delta t = 1$). Then, for the *shorter* interval from $t = 1$ to $t = 1 + \Delta t = 2$, we have $f(2) = 2^2 = 4$, so

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(2) - f(1)}{\Delta t} = \frac{4 - 1}{1} = 3 \text{ m/s}$$

[2]M. Eswaran and A. Kotwal, "A Theory of Two-Tier Labor Markets in Agrarian Economies," *The American Economic Review,* 75, no. 1 (1985), 162–77.

Table 11.2

Length of Time Interval Δt	Time Interval $t = 1$ to $t = 1 + \Delta t$	Average Velocity $\frac{\Delta s}{\Delta t} = \frac{f(1+\Delta t) - f(1)}{\Delta t}$
0.1	$t = 1$ to $t = 1.1$	2.1 m/s
0.07	$t = 1$ to $t = 1.07$	2.07 m/s
0.05	$t = 1$ to $t = 1.05$	2.05 m/s
0.03	$t = 1$ to $t = 1.03$	2.03 m/s
0.01	$t = 1$ to $t = 1.01$	2.01 m/s
0.001	$t = 1$ to $t = 1.001$	2.001 m/s

More generally, over the time interval from $t = 1$ to $t = 1 + \Delta t$, the object moves from position $f(1)$ to position $f(1 + \Delta t)$. Thus, its displacement is

$$\Delta s = f(1 + \Delta t) - f(1)$$

Since the time interval has length Δt, the object's average velocity is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(1 + \Delta t) - f(1)}{\Delta t}$$

If Δt were to become smaller and smaller, the average velocity over the interval from $t = 1$ to $t = 1 + \Delta t$ would be close to what we might call the *instantaneous velocity* at time $t = 1$; that is, the velocity at a *point* in time ($t = 1$) as opposed to the velocity over an *interval* of time. For some typical values of Δt between 0.1 and 0.001, we get the average velocities in Table 11.2, which the reader can verify.

The table suggests that as the length of the time interval approaches zero, the average velocity approaches the value 2 m/s. In other words, as Δt approaches 0, $\Delta s / \Delta t$ approaches 2 m/s. We define the limit of the average velocity as $\Delta t \to 0$ to be the **instantaneous velocity** (or simply the **velocity**), v, at time $t = 1$. This limit is also called the **instantaneous rate of change** of s with respect to t at $t = 1$:

$$v = \lim_{\Delta t \to 0} v_{\text{ave}} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(1 + \Delta t) - f(1)}{\Delta t}$$

If we think of Δt as h, then the limit on the right is simply the derivative of s with respect to t at $t = 1$. Thus, the instantaneous velocity of the object at $t = 1$ is just ds/dt at $t = 1$. Because $s = t^2$ and

$$\frac{ds}{dt} = 2t$$

the velocity at $t = 1$ is

$$v = \left.\frac{ds}{dt}\right|_{t=1} = 2(1) = 2 \text{ m/s}$$

which confirms our previous conclusion.

In summary, if $s = f(t)$ is the position function of an object moving in a straight line, then the average velocity of the object over the time interval $[t, t + \Delta t]$ is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

and the velocity at time t is given by

$$v = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{ds}{dt}$$

Selectively combining equations for v, we have

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

which provides motivation for the otherwise bizarre Leibniz notation. (After all, Δ is the [uppercase] Greek letter corresponding to d.)

EXAMPLE 1 Finding Average Velocity and Velocity

Suppose the position function of an object moving along a number line is given by $s = f(t) = 3t^2 + 5$, where t is in seconds and s is in meters.

a. Find the average velocity over the interval [10, 10.1].

b. Find the velocity when $t = 10$.

Solution:

a. Here $t = 10$ and $\Delta t = 10.1 - 10 = 0.1$. So we have

$$\begin{aligned} v_{\text{ave}} &= \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} \\ &= \frac{f(10+0.1) - f(10)}{0.1} \\ &= \frac{f(10.1) - f(10)}{0.1} \\ &= \frac{311.03 - 305}{0.1} = \frac{6.03}{0.1} = 60.3 \text{ m/s} \end{aligned}$$

b. The velocity at time t is given by

$$v = \frac{ds}{dt} = 6t$$

When $t = 10$, the velocity is

$$\left.\frac{ds}{dt}\right|_{t=10} = 6(10) = 60 \text{ m/s}$$

Notice that the average velocity over the interval [10, 10.1] is close to the velocity at $t = 10$. This is to be expected because the length of the interval is small.

Now Work Problem 1 ◁

Our discussion of the rate of change of s with respect to t applies equally well to *any* function $y = f(x)$. This means that we have the following:

If $y = f(x)$, then

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \begin{cases} \text{average rate of change} \\ \text{of } y \text{ with respect to } x \\ \text{over the interval from} \\ x \text{ to } x + \Delta x \end{cases}$$

and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \begin{cases} \text{instantaneous rate of change} \\ \text{of } y \text{ with respect to } x \end{cases} \qquad (2)$$

Because the instantaneous rate of change of $y = f(x)$ at a point is a derivative, it is also the *slope of the tangent line* to the graph of $y = f(x)$ at that point. For convenience, we usually refer to the instantaneous rate of change simply as the **rate of change.** The interpretation of a derivative as a rate of change is extremely important.

Let us now consider the significance of the rate of change of y with respect to x. From Equation (2), if Δx (a change in x) is close to 0, then $\Delta y/\Delta x$ is close to dy/dx. That is,

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

Therefore,

$$\Delta y \approx \frac{dy}{dx}\Delta x \tag{3}$$

That is, if x changes by Δx, then the change in y, Δy, is approximately dy/dx times the change in x. In particular,

if x changes by 1, an estimate of the change in y is $\dfrac{dy}{dx}$

APPLY IT ▷

3. Suppose that the profit P made by selling a certain product at a price of p per unit is given by $P = f(p)$ and the rate of change of that profit with respect to change in price is $\dfrac{dP}{dp} = 5$ at $p = 25$. Estimate the change in the profit P if the price changes from 25 to 25.5.

EXAMPLE 2 Estimating Δy by Using dy/dx

Suppose that $y = f(x)$ and $\dfrac{dy}{dx} = 8$ when $x = 3$. Estimate the change in y if x changes from 3 to 3.5.

Solution: We have $dy/dx = 8$ and $\Delta x = 3.5 - 3 = 0.5$. The change in y is given by Δy, and, from Equation (3),

$$\Delta y \approx \frac{dy}{dx}\Delta x = 8(0.5) = 4$$

We remark that, since $\Delta y = f(3.5) - f(3)$, we have $f(3.5) = f(3) + \Delta y$. For example, if $f(3) = 5$, then $f(3.5)$ can be estimated by $5 + 4 = 9$.

◁

APPLY IT ▷

4. The position of an object thrown upward at a speed of 16 feet/s from a height of 0 feet is given by $y(t) = 16t - 16t^2$. Find the rate of change of y with respect to t, and evaluate it when $t = 0.5$. Use your graphing calculator to graph $y(t)$. Use the graph to interpret the behavior of the object when $t = 0.5$.

EXAMPLE 3 Finding a Rate of Change

Find the rate of change of $y = x^4$ with respect to x, and evaluate it when $x = 2$ and when $x = -1$. Interpret your results.

Solution: The rate of change is

$$\frac{dy}{dx} = 4x^3$$

When $x = 2$, $dy/dx = 4(2)^3 = 32$. This means that if x increases, from 2, by a small amount, then y increases approximately 32 times as much. More simply, we say that, when $x = 2$, y is increasing 32 times as fast as x does. When $x = -1$, $dy/dx = 4(-1)^3 = -4$. The significance of the minus sign on -4 is that, when $x = -1$, y is *decreasing* 4 times as fast as x increases.

Now Work Problem 11 ◁

EXAMPLE 4 Rate of Change of Price with Respect to Quantity

Let $p = 100 - q^2$ be the demand function for a manufacturer's product. Find the rate of change of price p per unit with respect to quantity q. How fast is the price changing with respect to q when $q = 5$? Assume that p is in dollars.

Solution: The rate of change of p with respect to q is

$$\frac{dp}{dq} = \frac{d}{dq}(100 - q^2) = -2q$$

Thus,

$$\left.\frac{dp}{dq}\right|_{q=5} = -2(5) = -10$$

This means that when five units are demanded, an *increase* of one extra unit demanded corresponds to a decrease of approximately \$10 in the price per unit that consumers are willing to pay.

◁

EXAMPLE 5 **Rate of Change of Volume**

A spherical balloon is being filled with air. Find the rate of change of the volume of air in the balloon with respect to its radius. Evaluate this rate of change when the radius is 2 ft.

Solution: The formula for the volume V of a ball of radius r is $V = \frac{4}{3}\pi r^3$. The rate of change of V with respect to r is

$$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$$

When $r = 2$ ft, the rate of change is

$$\left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = 16\pi\frac{\text{ft}^3}{\text{ft}}$$

This means that when the radius is 2 ft, changing the radius by 1 ft will change the volume by approximately 16π ft^3.

◁

EXAMPLE 6 **Rate of Change of Enrollment**

A sociologist is studying various suggested programs that can aid in the education of preschool-age children in a certain city. The sociologist believes that x years after the beginning of a particular program, $f(x)$ thousand preschoolers will be enrolled, where

$$f(x) = \frac{10}{9}(12x - x^2) \quad 0 \le x \le 12$$

At what rate would enrollment change (a) after three years from the start of this program and (b) after nine years?

Solution: The rate of change of $f(x)$ is

$$f'(x) = \frac{10}{9}(12 - 2x)$$

a. After three years, the rate of change is

$$f'(3) = \frac{10}{9}(12 - 2(3)) = \frac{10}{9} \cdot 6 = \frac{20}{3} = 6\frac{2}{3}$$

Thus, enrollment would be increasing at the rate of $6\frac{2}{3}$ thousand preschoolers per year.

b. After nine years, the rate is

$$f'(9) = \frac{10}{9}(12 - 2(9)) = \frac{10}{9}(-6) = -\frac{20}{3} = -6\frac{2}{3}$$

Thus, enrollment would be *decreasing* at the rate of $6\frac{2}{3}$ thousand preschoolers per year.

Now Work Problem 9 ◁

Applications of Rate of Change to Economics

A manufacturer's **total-cost function,** $c = f(q)$, gives the total cost c of producing and marketing q units of a product. The rate of change of c with respect to q is called the **marginal cost.** Thus,

$$\text{marginal cost} = \frac{dc}{dq}$$

For example, suppose $c = f(q) = 0.1q^2 + 3$ is a cost function, where c is in dollars and q is in pounds. Then

$$\frac{dc}{dq} = 0.2q$$

The marginal cost when 4 lb are produced is dc/dq, evaluated when $q = 4$:

$$\left.\frac{dc}{dq}\right|_{q=4} = 0.2(4) = 0.80$$

This means that if production is increased by 1 lb, from 4 lb to 5 lb, then the change in cost is approximately \$0.80. That is, the additional pound costs about \$0.80. In general, *we interpret marginal cost as the approximate cost of one additional unit of output.* After all, the difference $f(q+1) - f(q)$ can be seen as a difference quotient

$$\frac{f(q+1) - f(q)}{1}$$

(the case where $h = 1$). Any difference quotient can be regarded as an approximation of the corresponding derivative and, conversely, any derivative can be regarded as an approximation of any of its corresponding difference quotients. Thus, for any function f of q we can always regard $f'(q)$ and $f(q+1) - f(q)$ as approximations of each other. In economics, the latter can usually be regarded as the exact value of the cost, or profit depending upon the function, of the $(q+1)$th item when q are produced. The derivative is often easier to compute than the exact value. [In the case at hand, the actual cost of producing one more pound beyond 4 lb is $f(5) - f(4) = 5.5 - 4.6 = \$0.90$.]

If c is the total cost of producing q units of a product, then the **average cost per unit,** $\bar{c}$, is

$$\bar{c} = \frac{c}{q} \qquad (4)$$

For example, if the total cost of 20 units is \$100, then the average cost per unit is $\bar{c} = 100/20 = \$5$. By multiplying both sides of Equation (4) by q, we have

$$c = q\bar{c}$$

That is, total cost is the product of the number of units produced and the average cost per unit.

EXAMPLE 7 Marginal Cost

If a manufacturer's average-cost equation is

$$\bar{c} = 0.0001q^2 - 0.02q + 5 + \frac{5000}{q}$$

find the marginal-cost function. What is the marginal cost when 50 units are produced?

Solution:

Strategy The marginal-cost function is the derivative of the total-cost function c. Thus, we first find c by multiplying $\bar{c}$ by q. We have

$$\begin{aligned} c &= q\bar{c} \\ &= q\left(0.0001q^2 - 0.02q + 5 + \frac{5000}{q}\right) \\ c &= 0.0001q^3 - 0.02q^2 + 5q + 5000 \end{aligned}$$

Differentiating c, we have the marginal-cost function:

$$\frac{dc}{dq} = 0.0001(3q^2) - 0.02(2q) + 5(1) + 0$$
$$= 0.0003q^2 - 0.04q + 5$$

The marginal cost when 50 units are produced is

$$\left.\frac{dc}{dq}\right|_{q=50} = 0.0003(50)^2 - 0.04(50) + 5 = 3.75$$

If c is in dollars and production is increased by one unit, from $q = 50$ to $q = 51$, then the cost of the additional unit is approximately \$3.75. If production is increased by $\frac{1}{3}$ unit, from $q = 50$, then the cost of the additional output is approximately $\left(\frac{1}{3}\right)(3.75) = \$1.25$.

Now Work Problem 21 ◁

Suppose $r = f(q)$ is the **total-revenue function** for a manufacturer. The equation $r = f(q)$ states that the total dollar value received for selling q units of a product is r. The **marginal revenue** is defined as the rate of change of the total dollar value received with respect to the total number of units sold. Hence, marginal revenue is merely the derivative of r with respect to q:

$$\text{marginal revenue} = \frac{dr}{dq}$$

Marginal revenue indicates the rate at which revenue changes with respect to units sold. We interpret it as *the approximate revenue received from selling one additional unit of output.*

EXAMPLE 8 **Marginal Revenue**

Suppose a manufacturer sells a product at \$2 per unit. If q units are sold, the total revenue is given by

$$r = 2q$$

The marginal-revenue function is

$$\frac{dr}{dq} = \frac{d}{dq}(2q) = 2$$

which is a constant function. Thus, the marginal revenue is 2 regardless of the number of units sold. This is what we would expect, because the manufacturer receives \$2 for each unit sold.

Now Work Problem 23 ◁

Relative and Percentage Rates of Change

For the total-revenue function in Example 8, namely, $r = f(q) = 2q$, we have

$$\frac{dr}{dq} = 2$$

This means that revenue is changing at the rate of \$2 per unit, regardless of the number of units sold. Although this is valuable information, it may be more significant when compared to r itself. For example, if $q = 50$, then $r = 2(50) = 100$. Thus, the rate of change of revenue is $2/100 = 0.02$ *of* r. On the other hand, if $q = 5000$, then $r = 2(5000) = \$10{,}000$, so the rate of change of r is $2/10{,}000 = 0.0002$ *of* r.

Although r changes at the same rate at each level, compared to r itself, this rate is relatively smaller when $r = 10{,}000$ than when $r = 100$. By considering the ratio

$$\frac{dr/dq}{r}$$

we have a means of comparing the rate of change of r with r itself. This ratio is called the *relative rate of change* of r. We have shown that the relative rate of change when $q = 50$ is

$$\frac{dr/dq}{r} = \frac{2}{100} = 0.02$$

and when $q = 5000$, it is

$$\frac{dr/dq}{r} = \frac{2}{10{,}000} = 0.0002$$

CAUTION!

Percentages can be confusing! Remember that *percent* means "per hundred." Thus $100\% = \frac{100}{100} = 1$, $2\% = \frac{2}{100} = 0.02$, and so on.

By multiplying relative rates by 100%, we obtain the so-called *percentage rates of change*. The percentage rate of change when $q = 50$ is $(0.02)(100\%) = 2\%$; when $q = 5000$, it is $(0.0002)(100\%) = 0.02\%$. For example, if an additional unit beyond 50 is sold, then revenue increases by approximately 2%.

In general, for any function f, we have the following definition:

Definition

The ***relative rate of change*** of $f(x)$ is

$$\frac{f'(x)}{f(x)}$$

The ***percentage rate of change*** of $f(x)$ is

$$\frac{f'(x)}{f(x)} \cdot 100\%$$

APPLY IT ▷

5. The volume V enclosed by a capsule-shaped container with a cylindrical height of 4 feet and radius r is given by

$$V(r) = \frac{4}{3}\pi r^3 + 4\pi r^2$$

Determine the relative and percentage rates of change of volume with respect to the radius when the radius is 2 feet.

EXAMPLE 9 Relative and Percentage Rates of Change

Determine the relative and percentage rates of change of

$$y = f(x) = 3x^2 - 5x + 25$$

when $x = 5$.

Solution: Here

$$f'(x) = 6x - 5$$

Since $f'(5) = 6(5) - 5 = 25$ and $f(5) = 3(5)^2 - 5(5) + 25 = 75$, the relative rate of change of y when $x = 5$ is

$$\frac{f'(5)}{f(5)} = \frac{25}{75} \approx 0.333$$

Multiplying 0.333 by 100% gives the percentage rate of change: $(0.333)(100) = 33.3\%$.

Now Work Problem 35 ◁

PROBLEMS 11.3

1. Suppose that the position function of an object moving along a straight line is $s = f(t) = 2t^2 + 3t$, where t is in seconds and s is in meters. Find the average velocity $\Delta s/\Delta t$ over the interval $[1, 1 + \Delta t]$, where Δt is given in the following table:

Δt	1	0.5	0.2	0.1	0.01	0.001
$\Delta s/\Delta t$						

From your results, estimate the velocity when $t = 1$. Verify your estimate by using differentiation.

2. If $y = f(x) = \sqrt{2x+5}$, find the average rate of change of y with respect to x over the interval $[3, 3+\Delta x]$, where Δx is given in the following table:

Δx	1	0.5	0.2	0.1	0.01	0.001
$\Delta y/\Delta x$						

From your result, estimate the rate of change of y with respect to x when $x = 3$.

In each of Problems 3–8, a position function is given, where t is in seconds and s is in meters.
(a) *Find the position at the given t-value.*
(b) *Find the average velocity over the given interval.*
(c) *Find the velocity at the given t-value.*

3. $s = 2t^2 - 4t; [7, 7.5]; t = 7$

4. $s = \frac{1}{2}t + 1; [2, 2.1]; t = 2$

5. $s = 5t^3 + 3t + 24; [1, 1.01]; t = 1$

6. $s = -3t^2 + 2t + 1; [1, 1.25]; t = 1$

7. $s = t^4 - 2t^3 + t; [2, 2.1]; t = 2$

8. $s = 3t^4 - t^{7/2}; [0, \frac{1}{4}]; t = 0$

9. Income–Education Sociologists studied the relation between income and number of years of education for members of a particular urban group. They found that a person with x years of education before seeking regular employment can expect to receive an average yearly income of y dollars per year, where

$$y = 5x^{5/2} + 5900 \qquad 4 \le x \le 16$$

Find the rate of change of income with respect to number of years of education. Evaluation the expression when $x = 9$.

10. Find the rate of change of the volume V of a ball, with respect to its radius r, when $r = 1.5$ m. The volume V of a ball as a function of its radius r is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

11. Skin Temperature The approximate temperature T of the skin in terms of the temperature T_e of the environment is given by

$$T = 32.8 + 0.27(T_e - 20)$$

where T and T_e are in degrees Celsius.[3] Find the rate of change of T with respect to T_e.

12. Biology The volume V of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Find the rate of change of volume with respect to the radius when $r = 6.3 \times 10^{-4}$ cm.

In Problems 13–18, cost functions are given, where c is the cost of producing q units of a product. In each case, find the marginal-cost function. What is the marginal cost at the given value(s) of q?

13. $c = 500 + 10q; q = 100$

14. $c = 5000 + 6q; q = 36$

15. $c = 0.2q^2 + 4q + 50; q = 10$

16. $c = 0.1q^2 + 3q + 2; q = 3$

17. $c = q^2 + 50q + 1000; q = 15, q = 16, q = 17$

18. $c = 0.04q^3 - 0.5q^2 + 4.4q + 7500; q = 5, q = 25, q = 1000$

In Problems 19–22, $\bar{c}$ represents average cost per unit, which is a function of the number q of units produced. Find the marginal-cost function and the marginal cost for the indicated values of q.

19. $\bar{c} = 0.01q + 5 + \frac{500}{q}; q = 50, q = 100$

20. $\bar{c} = 5 + \frac{2000}{q}; q = 25, q = 250$

21. $\bar{c} = 0.00002q^2 - 0.01q + 6 + \frac{20{,}000}{q}; q = 100, q = 500$

22. $\bar{c} = 0.002q^2 - 0.5q + 60 + \frac{7000}{q}; q = 15, q = 25$

In Problems 23–26, r represents total revenue and is a function of the number q of units sold. Find the marginal-revenue function and the marginal revenue for the indicated values of q.

23. $r = 0.8q; q = 9, q = 300, q = 500$

24. $r = q\left(15 - \frac{1}{30}q\right); q = 5, q = 15, q = 150$

25. $r = 240q + 40q^2 - 2q^3; q = 10; q = 15; q = 20$

26. $r = 2q(30 - 0.1q); q = 10, q = 20$

27. Hosiery Mill The total-cost function for a hosiery mill is estimated by Dean[4] to be

$$c = -10{,}484.69 + 6.750q - 0.000328q^2$$

where q is output in dozens of pairs and c is total cost in dollars. Find the marginal-cost function and the average cost function and evaluate each when $q = 2000$.

28. Light and Power Plant The total-cost function for an electric light and power plant is estimated by Nordin[5] to be

$$c = 32.07 - 0.79q + 0.02142q^2 - 0.0001q^3 \qquad 20 \le q \le 90$$

where q is the eight-hour total output (as a percentage of capacity) and c is the total fuel cost in dollars. Find the marginal-cost function and evaluate it when $q = 70$.

29. Urban Concentration Suppose the 100 largest cities in the United States in 1920 are ranked according to magnitude (areas of cities). From Lotka,[6] the following relation holds approximately:

$$PR^{0.93} = 5{,}000{,}000$$

Here, P is the population of the city having respective rank R. This relation is called the *law of urban concentration* for 1920. Solve for P in terms of R, and then find how fast the population is changing with respect to rank.

30. Depreciation Under the straight-line method of depreciation, the value v of a certain machine after t years have elapsed is given by

$$v = 120{,}000 - 15{,}500t$$

where $0 \le t \le 6$. How fast is v changing with respect to t when $t = 2$? $t = 4$? at any time?

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[3] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).

[4] J. Dean, "Statistical Cost Functions of a Hosiery Mill," *Studies in Business Administration*, XI, no. 4 (Chicago: University of Chicago Press, 1941).

[5] J. A. Nordin, "Note on a Light Plant's Cost Curves," *Econometrica*, 15 (1947), 231–35.

[6] A. J. Lotka, *Elements of Mathematical Biology* (New York: Dover Publications, Inc., 1956).

31. Winter Moth A study of the winter moth was made in Nova Scotia (adapted from Embree).[7] The prepupae of the moth fall onto the ground from host trees. At a distance of x ft from the base of a host tree, the prepupal density (number of prepupae per square foot of soil) was y, where

$$y = 59.3 - 1.5x - 0.5x^2 \quad 1 \leq x \leq 9$$

(a) At what rate is the prepupal density changing with respect to distance from the base of the tree when $x = 6$?
(b) For what value of x is the prepupal density decreasing at the rate of 6 prepupae per square foot per foot?

32. Cost Function For the cost function

$$c = 0.4q^2 + 4q + 5$$

find the rate of change of c with respect to q when $q = 2$. Also, what is $\Delta c/\Delta q$ over the interval [2, 3]?

In Problems 33–38, find **(a)** *the rate of change of y with respect to x and* **(b)** *the relative rate of change of y. At the given value of x, find* **(c)** *the rate of change of y,* **(d)** *the relative rate of change of y, and* **(e)** *the percentage rate of change of y.*

33. $y = f(x) = x + 4; x = 5$ **34.** $y = f(x) = 7 - 3x; x = 6$

35. $y = 2x^2 + 5; x = 10$ **36.** $y = 5 - 3x^3; x = 1$

37. $y = 8 - x^3; x = 1$ **38.** $y = x^2 + 3x - 4; x = -1$

39. Cost Function For the cost function

$$c = 0.3q^2 + 3.5q + 9$$

how fast does c change with respect to q when $q = 10$? Determine the percentage rate of change of c with respect to q when $q = 10$.

40. Organic Matters/Species Diversity In a discussion of contemporary waters of shallows seas, Odum[8] claims that in such waters the total organic matter y (in milligrams per liter) is a function of species diversity x (in number of species per thousand individuals). If $y = 100/x$, at what rate is the total organic matter changing with respect to species diversity when $x = 10$? What is the percentage rate of change when $x = 10$?

41. Revenue For a certain manufacturer, the revenue obtained from the sale of q units of a product is given by

$$r = 30q - 0.3q^2$$

(a) How fast does r change with respect to q? When $q = 10$, (b) find the relative rate of change of r, and (c) to the nearest percent, find the percentage rate of change of r.

42. Revenue Repeat Problem 43 for the revenue function given by $r = 10q - 0.2q^2$ and $q = 25$.

43. Weight of Limb The weight of a limb of a tree is given by $W = 2t^{0.432}$, where t is time. Find the relative rate of change of W with respect to t.

44. Response to Shock A psychological experiment[9] was conducted to analyze human responses to electrical shocks (stimuli). The subjects received shocks of various intensities. The response R to a shock of intensity I (in microamperes) was to be a number that indicated the perceived magnitude relative to that of a "standard" shock. The standard shock was assigned a magnitude of 10. Two groups of subjects were tested under slightly different conditions. The responses R_1 and R_2 of the first and second groups to a shock of intensity I were given by

$$R_1 = \frac{I^{1.3}}{1855.24} \quad 800 \leq I \leq 3500$$

and

$$R_2 = \frac{I^{1.3}}{1101.29} \quad 800 \leq I \leq 3500$$

(a) For each group, determine the relative rate of change of response with respect to intensity.
(b) How do these changes compare with each other?
(c) In general, if $f(x) = C_1x^n$ and $g(x) = C_2x^n$, where C_1 and C_2 are constants, how do the relative rates of change of f and g compare?

45. Cost A manufacturer of mountain bikes has found that when 20 bikes are produced per day, the average cost is \$200 and the marginal cost is \$150. Based on that information, approximate the total cost of producing 21 bikes per day.

46. Marginal and Average Costs Suppose that the cost function for a certain product is $c = f(q)$. If the relative rate of change of c (with respect to q) is $\frac{1}{q}$, prove that the marginal-cost function and the average-cost function are equal.

In Problems 47 and 48, use the numerical derivative feature of your graphing calculator.

47. If the total-cost function for a manufacturer is given by

$$c = \frac{5q^2}{\sqrt{q^2+3}} + 5000$$

where c is in dollars, find the marginal cost when 10 units are produced. Round your answer to the nearest cent.

48. The population of a city t years from now is given by

$$P = 250{,}000e^{0.04t}$$

Find the rate of change of population with respect to time t three years from now. Round your answer to the nearest integer.

Objective

To find derivatives by applying the product and quotient rules, and to develop the concepts of marginal propensity to consume and marginal propensity to save.

11.4 The Product Rule and the Quotient Rule

The equation $F(x) = (x^2 + 3x)(4x + 5)$ expresses $F(x)$ as a product of two functions: $x^2 + 3x$ and $4x + 5$. To find $F'(x)$ by using only our previous rules, we first multiply

[7] D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada*, no. 46 (1965).

[8] H. T. Odum, "Biological Circuits and the Marine Systems of Texas," in *Pollution and Marine Biology*, eds T. A. Olsen and F. J. Burgess (New York: Interscience Publishers, 1967).

[9] H. Babkoff, "Magnitude Estimation of Short Electrocutaneous Pulses," *Psychological Research*, 39, no. 1 (1976), 39–49.

the functions. Then we differentiate the result, term by term:

$$F(x) = (x^2 + 3x)(4x + 5) = 4x^3 + 17x^2 + 15x$$

$$F'(x) = 12x^2 + 34x + 15 \qquad (1)$$

However, in many problems that involve differentiating a product of functions, the multiplication is not as simple as it is here. At times, it is not even practical to attempt it. Fortunately, there is a rule for differentiating a product, and the rule avoids such multiplications. Since the derivative of a sum of functions is the sum of their derivatives, you might expect a similar rule for products. However, the situation is rather subtle.

COMBINING RULE 3 The Product Rule

If f and g are differentiable functions, then the product fg is differentiable, and

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

That is, the derivative of the product of two functions is the derivative of the first function times the second, plus the first function times the derivative of the second.

$$\frac{d}{dx}(\text{product}) = \begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix}(\text{second}) + (\text{first})\begin{pmatrix}\text{derivative}\\ \text{of second}\end{pmatrix}$$

Proof. If $F(x) = f(x)g(x)$, then, by the definition of the derivative of F,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Now we use a "trick." Adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x)g(x+h) - f(x)g(x+h)}{h}$$

Regrouping gives

$$F'(x) = \lim_{h\to 0} \frac{(f(x+h)g(x+h) - f(x)g(x+h)) + (f(x)g(x+h) - f(x)g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Since we assumed that f and g are differentiable,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

and

$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

The differentiability of g implies that g is continuous, so, from Section 10.3,

$$\lim_{h\to 0} g(x+h) = g(x)$$

Thus,

$$F'(x) = f'(x)g(x) + f(x)g'(x)$$

EXAMPLE 1 **Applying the Product Rule**

If $F(x) = (x^2 + 3x)(4x + 5)$, find $F'(x)$.

Solution: We will consider F as a product of two functions:

$$F(x) = \underbrace{(x^2 + 3x)}_{f(x)}\underbrace{(4x + 5)}_{g(x)}$$

Therefore, we can apply the product rule:

$$\begin{aligned} F'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= \underbrace{\frac{d}{dx}(x^2 + 3x)}_{\text{Derivative of first}}\underbrace{(4x + 5)}_{\text{Second}} + \underbrace{(x^2 + 3x)}_{\text{First}}\underbrace{\frac{d}{dx}(4x + 5)}_{\text{Derivative of second}} \\ &= (2x + 3)(4x + 5) + (x^2 + 3x)(4) \\ &= 12x^2 + 34x + 15 \qquad \text{simplifying} \end{aligned}$$

This agrees with our previous result. [See Equation (1).] Although there doesn't seem to be much advantage to using the product rule here, there are times when it is impractical to avoid it.

Now Work Problem 1 ◁

CAUTION!

It is worthwhile to repeat that the derivative of the product of two functions is somewhat subtle. Do not be tempted to make up a simpler rule.

EXAMPLE 2 **Applying the Product Rule**

If $y = (x^{2/3} + 3)(x^{-1/3} + 5x)$, find dy/dx.

Solution: Applying the product rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^{2/3} + 3)(x^{-1/3} + 5x) + (x^{2/3} + 3)\frac{d}{dx}(x^{-1/3} + 5x) \\ &= \left(\frac{2}{3}x^{-1/3}\right)(x^{-1/3} + 5x) + (x^{2/3} + 3)\left(\frac{-1}{3}x^{-4/3} + 5\right) \\ &= \frac{25}{3}x^{2/3} + \frac{1}{3}x^{-2/3} - x^{-4/3} + 15 \end{aligned}$$

Alternatively, we could have found the derivative without the product rule by first finding the product $(x^{2/3} + 3)(x^{-1/3} + 5x)$ and then differentiating the result, term by term.

Now Work Problem 15 ◁

APPLY IT ▷

6. A taco stand usually sells 225 tacos per day at \$2 each. A business student's research tells him that for every \$0.15 decrease in the price, the stand will sell 20 more tacos per day. The revenue function for the taco stand is $R(x) = (2 - 0.15x)(225 + 20x)$, where x is the number of \$0.15 reductions in price. Find $\frac{dR}{dx}$.

EXAMPLE 3 **Differentiating a Product of Three Factors**

If $y = (x + 2)(x + 3)(x + 4)$, find y'.

Solution:

Strategy We would like to use the product rule, but as given it applies only to *two* factors. By treating the first two factors as a single factor, we can consider y to be a product of two functions:

$$y = [(x + 2)(x + 3)](x + 4)$$

The product rule gives

$$\begin{aligned} y' &= \frac{d}{dx}[(x + 2)(x + 3)](x + 4) + [(x + 2)(x + 3)]\frac{d}{dx}(x + 4) \\ &= \frac{d}{dx}[(x + 2)(x + 3)](x + 4) + [(x + 2)(x + 3)](1) \end{aligned}$$

Applying the product rule again, we have

$$\begin{aligned} y' &= \left(\frac{d}{dx}(x+2)(x+3) + (x+2)\frac{d}{dx}(x+3)\right)(x+4) + (x+2)(x+3) \\ &= [(1)(x+3) + (x+2)(1)](x+4) + (x+2)(x+3) \end{aligned}$$

After simplifying, we obtain

$$y' = 3x^2 + 18x + 26$$

Two other ways of finding the derivative are as follows:

1. Multiply the first two factors of y to obtain

$$y = (x^2 + 5x + 6)(x + 4)$$

and then apply the product rule.

2. Multiply all three factors to obtain

$$y = x^3 + 9x^2 + 26x + 24$$

and then differentiate term by term.

Now Work Problem 19 ◁

It is sometimes helpful to remember differentiation rules in more streamlined notation. For example,

$$(fg)' = f'g + fg'$$

is a correct equality of functions that expresses the product rule. We can then calculate

$$\begin{aligned} (fgh)' &= ((fg)h)' \\ &= (fg)'h + (fg)h' \\ &= (f'g + fg')h + (fg)h' \\ &= f'gh + fg'h + fgh' \end{aligned}$$

It is not suggested that you try to commit to memory derived rules like

$$(fgh)' = f'gh + fg'h + fgh'$$

Because $f'g + fg' = gf' + fg'$, using commutativity of the product of functions, we can express the product rule with the derivatives as second factors:

$$(fg)' = gf' + fg'$$

and using commutativity of addition

$$(fg)' = fg' + gf'$$

Some people prefer these forms.

APPLY IT ▷

7. One hour after x milligrams of a particular drug are given to a person, the change in body temperature $T(x)$, in degrees Fahrenheit, is given approximately by $T(x) = x^2\left(1 - \frac{x}{3}\right)$. The rate at which T changes with respect to the size of the dosage x, $T'(x)$, is called the *sensitivity* of the body to the dosage. Find the sensitivity when the dosage is 1 milligram. Do not use the product rule.

EXAMPLE 4 Using the Product Rule to Find Slope

Find the slope of the graph of $f(x) = (7x^3 - 5x + 2)(2x^4 + 7)$ when $x = 1$.

Solution:

Strategy We find the slope by evaluating the derivative when $x = 1$. Because f is a product of two functions, we can find the derivative by using the product rule.

We have

$$\begin{aligned} f'(x) &= (7x^3 - 5x + 2)\frac{d}{dx}(2x^4 + 7) + (2x^4 + 7)\frac{d}{dx}(7x^3 - 5x + 2) \\ &= (7x^3 - 5x + 2)(8x^3) + (2x^4 + 7)(21x^2 - 5) \end{aligned}$$

Since we must compute $f'(x)$ when $x = 1$, *there is no need to simplify $f'(x)$ before evaluating it*. Substituting into $f'(x)$, we obtain

$$f'(1) = 4(8) + 9(16) = 176$$

Now Work Problem 49 ◁

Usually, we do not use the product rule when simpler ways are obvious. For example, if $f(x) = 2x(x+3)$, then it is quicker to write $f(x) = 2x^2 + 6x$, from which $f'(x) = 4x + 6$. Similarly, we do not usually use the product rule to differentiate $y = 4(x^2 - 3)$. Since the 4 is a constant factor, by the constant-factor rule we have $y' = 4(2x) = 8x$.

The product rule (and quotient rule that follows) should not be applied when a more direct and efficient method is available.

The next rule is used for differentiating a *quotient* of two functions.

COMBINING RULE 4 The Quotient Rule

If f and g are differentiable functions and $g(x) \neq 0$, then the quotient f/g is also differentiable, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

With the understanding about the denominator not being zero, we can write

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

That is, the derivative of the quotient of two functions is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}(\text{quotient}) = \frac{(\text{denominator})\begin{pmatrix}\text{derivative}\\ \text{of numerator}\end{pmatrix} - (\text{numerator})\begin{pmatrix}\text{derivative}\\ \text{of denominator}\end{pmatrix}}{(\text{denominator})^2}$$

Proof. If $F(x) = \dfrac{f(x)}{g(x)}$, then

$$F(x)g(x) = f(x)$$

By the product rule,

$$F(x)g'(x) + g(x)F'(x) = f'(x)$$

Solving for $F'(x)$, we have

$$F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)}$$

But $F(x) = f(x)/g(x)$. Thus,

$$F'(x) = \frac{f'(x) - \dfrac{f(x)g'(x)}{g(x)}}{g(x)}$$

CAUTION!

The derivative of the quotient of two functions is trickier still than the product rule. We must remember where the minus sign goes!

Simplifying gives[10]

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

[10]The proof given assumes the existence of $F'(x)$. However, the rule can be proved without this assumption.

EXAMPLE 5 Applying the Quotient Rule

If $F(x) = \dfrac{4x^2+3}{2x-1}$, find $F'(x)$.

Solution:

Strategy We recognize F as a quotient, so we can apply the quotient rule.

Let $f(x) = 4x^2 + 3$ and $g(x) = 2x - 1$. Then

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{\overbrace{(2x-1)}^{\text{Denominator}} \overbrace{\frac{d}{dx}(4x^2+3)}^{\text{Derivative of numerator}} - \overbrace{(4x^2+3)}^{\text{Numerator}} \overbrace{\frac{d}{dx}(2x-1)}^{\text{Derivative of numerator}}}{\underbrace{(2x-1)^2}_{\text{Square of denominator}}}$$

$$= \frac{(2x-1)(8x) - (4x^2+3)(2)}{(2x-1)^2}$$

$$= \frac{8x^2 - 8x - 6}{(2x-1)^2} = \frac{2(2x+1)(2x-3)}{(2x-1)^2}$$

Now Work Problem 21 ◁

EXAMPLE 6 Rewriting before Differentiating

Differentiate $y = \dfrac{1}{x + \frac{1}{x+1}}$.

Solution:

Strategy To simplify the differentiation, we will rewrite the function so that no fraction appears in the denominator.

We have

$$y = \frac{1}{x + \frac{1}{x+1}} = \frac{1}{\frac{x(x+1)+1}{x+1}} = \frac{x+1}{x^2+x+1}$$

$$\frac{dy}{dx} = \frac{(x^2+x+1)(1) - (x+1)(2x+1)}{(x^2+x+1)^2} \qquad \text{quotient rule}$$

$$= \frac{(x^2+x+1) - (2x^2+3x+1)}{(x^2+x+1)^2}$$

$$= \frac{-x^2-2x}{(x^2+x+1)^2} = -\frac{x^2+2x}{(x^2+x+1)^2}$$

Now Work Problem 45 ◁

Although a function may have the form of a quotient, this does not necessarily mean that the quotient rule must be used to find the derivative. The next example illustrates some typical situations in which, although the quotient rule can be used, a simpler and more efficient method is available.

EXAMPLE 7 Differentiating Quotients without Using the Quotient Rule

Differentiate the following functions.

a. $f(x) = \dfrac{2x^3}{5}$

Solution: Rewriting, we have $f(x) = \frac{2}{5}x^3$. By the constant-factor rule,

$$f'(x) = \frac{2}{5}(3x^2) = \frac{6x^2}{5}$$

b. $f(x) = \dfrac{4}{7x^3}$

Solution: Rewriting, we have $f(x) = \frac{4}{7}(x^{-3})$. Thus,

$$f'(x) = \frac{4}{7}(-3x^{-4}) = -\frac{12}{7x^4}$$

c. $f(x) = \dfrac{5x^2 - 3x}{4x}$

Solution: Rewriting, we have $f(x) = \dfrac{1}{4}\left(\dfrac{5x^2 - 3x}{x}\right) = \dfrac{1}{4}(5x - 3)$ for $x \neq 0$. Thus,

$$f'(x) = \frac{1}{4}(5) = \frac{5}{4} \qquad \text{for } x \neq 0$$

Since the function f is not defined for $x = 0$, f' is not defined for $x = 0$ either.

Now Work Problem 17 ◁

CAUTION!

To differentiate $f(x) = \dfrac{1}{x^2 - 2}$, we might be tempted first to rewrite the quotient as $(x^2 - 2)^{-1}$. Currently it would be a mistake to do this because we do not yet have a rule for differentiating that form. In short, we have no choice now but to use the quotient rule. However, in the next section we will develop a rule that allows us to differentiate $(x^2 - 2)^{-1}$ in a direct and efficient way.

EXAMPLE 8 Marginal Revenue

If the demand equation for a manufacturer's product is

$$p = \frac{1000}{q + 5}$$

where p is in dollars, find the marginal-revenue function and evaluate it when $q = 45$.

Solution:

Strategy First we must find the revenue function. The revenue r received for selling q units when the price per unit is p is given by

$$\textbf{revenue} = \textbf{(price)(quantity)}; \quad \text{that is,} \quad r = pq$$

Using the demand equation, we will express r in terms of q only. Then we will differentiate to find the marginal-revenue function, dr/dq.

The revenue function is

$$r = \left(\frac{1000}{q + 5}\right)q = \frac{1000q}{q + 5}$$

Thus, the marginal-revenue function is given by

$$\frac{dr}{dq} = \frac{(q + 5)\frac{d}{dq}(1000q) - (1000q)\frac{d}{dq}(q + 5)}{(q + 5)^2}$$

$$= \frac{(q + 5)(1000) - (1000q)(1)}{(q + 5)^2} = \frac{5000}{(q + 5)^2}$$

and

$$\left.\frac{dr}{dq}\right|_{q=45} = \frac{5000}{(45+5)^2} = \frac{5000}{2500} = 2$$

This means that selling one additional unit beyond 45 results in approximately \$2 more in revenue.

Now Work Problem 59 ◁

Consumption Function

A function that plays an important role in economic analysis is the **consumption function.** The consumption function $C = f(I)$ expresses a relationship between the total national income I and the total national consumption C. Usually, both I and C are expressed in billions of dollars and I is restricted to some interval. The *marginal propensity to consume* is defined as the rate of change of consumption with respect to income. It is merely the derivative of C with respect to I:

$$\text{Marginal propensity to consume} = \frac{dC}{dI}$$

If we assume that the difference between income I and consumption C is savings S, then

$$S = I - C$$

Differentiating both sides with respect to I gives

$$\frac{dS}{dI} = \frac{d}{dI}(I) - \frac{d}{dI}(C) = 1 - \frac{dC}{dI}$$

We define dS/dI as the **marginal propensity to save.** Thus, the marginal propensity to save indicates how fast savings change with respect to income, and

$$\begin{array}{c}\text{Marginal propensity}\\ \text{to save}\end{array} = 1 - \begin{array}{c}\text{Marginal propensity}\\ \text{to consume}\end{array}$$

EXAMPLE 9 Finding Marginal Propensities to Consume and to Save

If the consumption function is given by

$$C = \frac{5(2\sqrt{I^3}+3)}{I+10}$$

determine the marginal propensity to consume and the marginal propensity to save when $I = 100$.

Solution:

$$\frac{dC}{dI} = 5\left(\frac{(I+10)\frac{d}{dI}(2I^{3/2}+3) - (2\sqrt{I^3}+3)\frac{d}{dI}(I+10)}{(I+10)^2}\right)$$

$$= 5\left(\frac{(I+10)(3I^{1/2}) - (2\sqrt{I^3}+3)(1)}{(I+10)^2}\right)$$

When $I = 100$, the marginal propensity to consume is

$$\left.\frac{dC}{dI}\right|_{I=100} = 5\left(\frac{1297}{12{,}100}\right) \approx 0.536$$

The marginal propensity to save when $I = 100$ is $1 - 0.536 = 0.464$. This means that if a current income of \$100 billion increases by \$1 billion, the nation consumes approximately 53.6% (536/1000) and saves 46.4% (464/1000) of that increase.

Now Work Problem 69 ◁

PROBLEMS 11.4

In Problems 1–48, differentiate the functions.

1. $f(x) = (4x+1)(6x+3)$
2. $f(x) = (3x-1)(7x+2)$
3. $s(t) = (5-3t)(t^3-2t^2)$
4. $Q(x) = (x^2+3x)(7x^2-5)$
5. $f(r) = (3r^2-4)(r^2-5r+1)$
6. $C(I) = (2I^2-3)(3I^2-4I+1)$
7. $f(x) = x^2(2x^2-5)$
8. $f(x) = 3x^3(x^2-2x+2)$
9. $y = (x^2+5x-7)(6x^2-5x+4)$
10. $\phi(x) = (3-5x+2x^2)(2+x-4x^2)$
11. $f(w) = (w^2+3w-7)(2w^3-4)$
12. $f(x) = (3x-x^2)(3-x-x^2)$
13. $y = (x^2-1)(3x^3-6x+5) - 4(4x^2+2x+1)$
14. $h(x) = 5(x^7+4) + 4(5x^3-2)(4x^2+7x)$
15. $F(p) = \frac{3}{2}(5\sqrt{p}-2)(3p-1)$
16. $g(x) = (\sqrt{x}+5x-2)(\sqrt[3]{x}-3\sqrt{x})$
17. $y = 7 \cdot \frac{2}{3}$
18. $y = (x-1)(x-2)(x-3)$
19. $y = (5x+3)(2x-5)(7x+9)$
20. $y = \dfrac{2x-3}{4x+1}$
21. $f(x) = \dfrac{5x}{x-1}$
22. $H(x) = \dfrac{-5x}{5-x}$
23. $f(x) = \dfrac{-13}{3x^5}$
24. $f(x) = \dfrac{3(5x^2-7)}{4}$
25. $y = \dfrac{x+2}{x-1}$
26. $h(w) = \dfrac{3w^2+5w-1}{w-3}$
27. $h(z) = \dfrac{6-2z}{z^2-4}$
28. $z = \dfrac{2x^2+5x-2}{3x^2+5x+3}$
29. $y = \dfrac{4x^2+3x+2}{3x^2-2x+1}$
30. $f(x) = \dfrac{x^3-x^2+1}{x^2+1}$
31. $y = \dfrac{x^2-4x+3}{2x^2-3x+2}$
32. $F(z) = \dfrac{z^4+4}{3z}$
33. $g(x) = \dfrac{1}{x^{100}+7}$
34. $y = \dfrac{-8}{7x^6}$
35. $u(v) = \dfrac{v^3-8}{v}$
36. $y = \dfrac{x-5}{8\sqrt{x}}$
37. $y = \dfrac{3x^2-x-1}{\sqrt[3]{x}}$
38. $y = \dfrac{x^{0.3}-2}{2x^{2.1}+1}$
39. $y = 1 - \dfrac{5}{2x+5} + \dfrac{2x}{3x+1}$
40. $q(x) = 2x^3 + \dfrac{5x+1}{3x-5} - \dfrac{2}{x^3}$
41. $y = \dfrac{x-5}{(x+2)(x-4)}$
42. $y = \dfrac{(9x-1)(3x+2)}{4-5x}$
43. $s(t) = \dfrac{t^2+3t}{(t^2-1)(t^3+7)}$
44. $f(s) = \dfrac{17}{s(4s^3+5s-23)}$
45. $y = 3x - \dfrac{\frac{2}{x}-\frac{3}{x-1}}{x-2}$
46. $y = 3 - 12x^3 + \dfrac{1-\frac{5}{x^2+2}}{x^2+5}$
47. $f(x) = \dfrac{a+x}{a-x}$, where a is a constant
48. $f(x) = \dfrac{x^{-1}+a^{-1}}{x^{-1}-a^{-1}}$, where a is a constant
49. Find the slope of the curve $y = (2x^2-x+3)(x^3+x+1)$ at $(1, 12)$.
50. Find the slope of the curve $y = \dfrac{x^3}{x^4+1}$ at $(-1, -\frac{1}{2})$.

In Problems 51–54, find an equation of the tangent line to the curve at the given point.

51. $y = \dfrac{6}{x-1}$; $(3, 3)$
52. $y = \dfrac{x+5}{x^2}$; $(1, 6)$
53. $y = (2x+3)[2(x^4-5x^2+4)]$; $(0, 24)$
54. $y = \dfrac{x-1}{x(x^2+1)}$; $(2, \frac{1}{10})$

In Problems 55 and 56, determine the relative rate of change of y with respect to x for the given value of x.

55. $y = \dfrac{x}{2x-6}$; $x = 1$
56. $y = \dfrac{1-x}{1+x}$; $x = 5$

57. **Motion** The position function for an object moving in a straight line is

$$s = \frac{2}{t^3+1}$$

where t is in seconds and s is in meters. Find the position and velocity of the object at $t = 1$.

58. **Motion** The position function for an object moving in a straight-line path is

$$s = \frac{t+3}{t^2+7}$$

where t is in seconds and s is in meters. Find the positive value(s) of t for which the velocity of the object is 0.

In Problems 59–62, each equation represents a demand function for a certain product, where p denotes the price per unit for q units. Find the marginal-revenue function in each case. Recall that revenue = pq.

59. $p = 80 - 0.02q$
60. $p = 500/q$
61. $p = \dfrac{108}{q+2} - 3$
62. $p = \dfrac{q+750}{q+50}$

63. **Consumption Function** For the United States (1922–1942), the consumption function is estimated by[11]

$$C = 0.672I + 113.1$$

Find the marginal propensity to consume.

64. **Consumption Function** Repeat Problem 63 for $C = 0.836I + 127.2$.

In Problems 65–68, each equation represents a consumption function. Find the marginal propensity to consume and the marginal propensity to save for the given value of I.

65. $C = 3 + \sqrt{I} + 2\sqrt[3]{I}$; $I = 1$
66. $C = 6 + \dfrac{3I}{4} - \dfrac{\sqrt{I}}{3}$; $I = 25$

[11] T. Haavelmo, "Methods of Measuring the Marginal Propensity to Consume," *Journal of the American Statistical Association*, XLII (1947), 105–22.

67. $C = \dfrac{16\sqrt{I} + 0.8\sqrt{I^3} - 0.2I}{\sqrt{I} + 4}; I = 36$

68. $C = \dfrac{20\sqrt{I} + 0.5\sqrt{I^3} - 0.4I}{\sqrt{I} + 5}; I = 100$

69. Consumption Function Suppose that a country's consumption function is given by

$$C = \frac{9\sqrt{I} + 0.8\sqrt{I^3} - 0.3I}{\sqrt{I}}$$

where C and I are expressed in billions of dollars.

(a) Find the marginal propensity to save when income is \$25 billion.

(b) Determine the relative rate of change of C with respect to I when income is \$25 billion.

70. Marginal Propensities to Consume and to Save Suppose that the savings function of a country is

$$S = \frac{I - 2\sqrt{I} - 8}{\sqrt{I} + 2}$$

where the national income (I) and the national savings (S) are measured in billions of dollars. Find the country's marginal propensity to consume and its marginal propensity to save when the national income is \$150 billion. (*Hint:* It may be helpful to first factor the numerator.)

71. Marginal Cost If the total-cost function for a manufacturer is given by

$$c = \frac{6q^2}{q + 2} + 6000$$

find the marginal-cost function.

72. Marginal and Average Costs Given the cost function $c = f(q)$, show that if $\dfrac{d}{dq}(\bar{c}) = 0$, then the marginal-cost function and average-cost function are equal.

73. Host–Parasite Relation For a particular host–parasite relationship, it is determined that when the host density (number of hosts per unit of area) is x, the number of hosts that are parasitized is y, where

$$y = \frac{900x}{10 + 45x}$$

At what rate is the number of hosts parasitized changing with respect to host density when $x = 2$?

74. Acoustics The persistence of sound in a room after the source of the sound is turned off is called *reverberation*. The *reverberation time* RT of the room is the time it takes for the intensity level of the sound to fall 60 decibels. In the acoustical design of an auditorium, the following formula may be used to compute the RT of the room:[12]

$$\text{RT} = \frac{0.05V}{A + xV}$$

Here V is the room volume, A is the total room absorption, and x is the air absorption coefficient. Assuming that A and x are positive constants, show that the rate of change of RT with respect to V is always positive. If the total room volume increases by one unit, does the reverberation time increase or decrease?

75. Predator–Prey In a predator–prey experiment,[13] it was statistically determined that the number of prey consumed, y, by an individual predator was a function of the prey density x (the number of prey per unit of area), where

$$y = \frac{0.7355x}{1 + 0.02744x}$$

Determine the rate of change of prey consumed with respect to prey density.

76. Social Security Benefits In a discussion of social security benefits, Feldstein[14] differentiates a function of the form

$$f(x) = \frac{a(1 + x) - b(2 + n)x}{a(2 + n)(1 + x) - b(2 + n)x}$$

where a, b, and n are constants. He determines that

$$f'(x) = \frac{-1(1 + n)ab}{(a(1 + x) - bx)^2(2 + n)}$$

Verify this. (*Hint:* For convenience, let $2 + n = c$.) Next observe that Feldstein's function f is of the form

$$g(x) = \frac{A + Bx}{C + Dx}, \quad \text{where } A, B, C, \text{ and } D \text{ are constants}$$

Show that $g'(x)$ is a constant divided by a nonnegative function of x. What does this mean?

77. Business The manufacturer of a product has found that when 20 units are produced per day, the average cost is \$150 and the marginal cost is \$125. What is the relative rate of change of average cost with respect to quantity when $q = 20$?

78. Use the result $(fgh)' = f'gh + fg'h + fgh'$ to find dy/dx if

$$y = (3x + 1)(2x - 1)(x - 4)$$

Objective

To introduce and apply the chain rule, to derive a special case of the chain rule, and to develop the concept of the marginal-revenue product as an application of the chain rule.

11.5 The Chain Rule

Our next rule, the *chain rule,* is ultimately the most important rule for finding derivatives. It involves a situation in which y is a function of the variable u, but u is a function of x,

[12] L. L. Doelle, *Environmental Acoustics* (New York: McGraw-Hill Book Company, 1972).

[13] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist,* XCI, no. 7 (1959), 385–98.

[14] M. Feldstein, "The Optimal Level of Social Security Benefits," *The Quarterly Journal of Economics*, C, no. 2 (1985), 303–20.

and we want to find the derivative of y with respect to x. For example, the equations

$$y = u^2 \quad \text{and} \quad u = 2x + 1$$

define y as a function of u and u as a function of x. If we substitute $2x + 1$ for u in the first equation, we can consider y to be a function of x:

$$y = (2x + 1)^2$$

To find dy/dx, we first expand $(2x + 1)^2$:

$$y = 4x^2 + 4x + 1$$

Then

$$\frac{dy}{dx} = 8x + 4$$

From this example, you can see that finding dy/dx by first performing a substitution *could* be quite involved. For instance, if originally we had been given $y = u^{100}$ instead of $y = u^2$, we wouldn't even want to try substituting. Fortunately, the chain rule will allow us to handle such situations with ease.

COMBINING RULE 5 The Chain Rule

If y is a differentiable function of u and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We can show you why the chain rule is reasonable by considering rates of change. Suppose

$$y = 8u + 5 \quad \text{and} \quad u = 2x - 3$$

Let x change by one unit. How does u change? To answer this question, we differentiate and find $du/dx = 2$. But for *each* one-unit change in u, there is a change in y of $dy/du = 8$. Therefore, what is the change in y if x changes by one unit; that is, what is dy/dx? The answer is $8 \cdot 2$, which is $\frac{dy}{du} \cdot \frac{du}{dx}$. Thus, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

We will now use the chain rule to redo the problem at the beginning of this section. If

$$y = u^2 \quad \text{and} \quad u = 2x + 1$$

then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}(u^2) \cdot \frac{d}{dx}(2x + 1)$$
$$= (2u)2 = 4u$$

Replacing u by $2x + 1$ gives

$$\frac{dy}{dx} = 4(2x + 1) = 8x + 4$$

which agrees with our previous result.

APPLY IT ▷

8. If an object moves horizontally according to $x = 6t$, where t is in seconds, and vertically according to $y = 4x^2$, find its vertical velocity $\frac{dy}{dt}$.

EXAMPLE 1 Using the Chain Rule

a. If $y = 2u^2 - 3u - 2$ and $u = x^2 + 4$, find dy/dx.

Solution: By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}(2u^2 - 3u - 2) \cdot \frac{d}{dx}(x^2 + 4)$$
$$= (4u - 3)(2x)$$

We can write our answer in terms of x alone by replacing u by $x^2 + 4$.

$$\frac{dy}{dx} = [4(x^2 + 4) - 3](2x) = [4x^2 + 13](2x) = 8x^3 + 26x$$

b. If $y = \sqrt{w}$ and $w = 7 - t^3$, find dy/dt.

Solution: Here, y is a function of w and w is a function of t, so we can view y as a function of t. By the chain rule,

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dw}\cdot\frac{dw}{dt} = \frac{d}{dw}(\sqrt{w})\cdot\frac{d}{dt}(7-t^3)\\ &= \left(\frac{1}{2}w^{-1/2}\right)(-3t^2) = \frac{1}{2\sqrt{w}}(-3t^2)\\ &= -\frac{3t^2}{2\sqrt{w}} = -\frac{3t^2}{2\sqrt{7-t^3}}\end{aligned}$$

Now Work Problem 1 ◁

EXAMPLE 2 Using the Chain Rule

If $y = 4u^3 + 10u^2 - 3u - 7$ and $u = 4/(3x-5)$, find dy/dx when $x = 1$.

Solution: By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}(4u^3 + 10u^2 - 3u - 7)\cdot\frac{d}{dx}\left(\frac{4}{3x-5}\right)\\ &= (12u^2 + 20u - 3)\cdot\frac{(3x-5)\frac{d}{dx}(4) - 4\frac{d}{dx}(3x-5)}{(3x-5)^2}\\ &= (12u^2 + 20u - 3)\cdot\frac{-12}{(3x-5)^2}\end{aligned}$$

When x is replaced by a, $u = u(x)$ must be replaced by $u(a)$.

Even though dy/dx is in terms of x's and u's, we can evaluate it when $x = 1$ if we determine the corresponding value of u. When $x = 1$,

$$u = \frac{4}{3(1)-5} = -2$$

Thus,

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=1} &= [12(-2)^2 + 20(-2) - 3]\cdot\frac{-12}{[3(1)-5]^2}\\ &= 5\cdot(-3) = -15\end{aligned}$$

Now Work Problem 5 ◁

The chain rules states that if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Actually, the chain rule applies to a composite function, because

$$y = f(u) = f(g(x)) = (f \circ g)(x)$$

Thus y, as a function of x, is $f \circ g$. This means that we can use the chain rule to differentiate a function when we recognize the function as a composition. However, we must first break down the function into composite parts.

For example, to differentiate

$$y = (x^3 - x^2 + 6)^{100}$$

we think of the function as a composition. Let

$$y = f(u) = u^{100} \quad \text{and} \quad u = g(x) = x^3 - x^2 + 6$$

Then $y = (x^3 - x^2 + 6)^{100} = (g(x))^{100} = f(g(x))$. Now that we have a composite, we differentiate. Since $y = u^{100}$ and $u = x^3 - x^2 + 6$, by the chain rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (100u^{99})(3x^2 - 2x) \\ &= 100(x^3 - x^2 + 6)^{99}(3x^2 - 2x)\end{aligned}$$

We have just used the chain rule to differentiate $y = (x^3 - x^2 + 6)^{100}$, which is a power of a *function* of x, not simply a power of x. The following rule, called the **power rule,** generalizes our result and is a special case of the chain rule:

The Power Rule $$\frac{d}{dx}(u^a) = au^{a-1}\frac{du}{dx}$$

where it is understood that u is a differentiable function of x and a is a real number.
Proof. Let $y = u^a$. Since y is a differentiable function of u and u is a differentiable function of x, the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

But $dy/du = au^{a-1}$. Thus,

$$\frac{dy}{dx} = au^{a-1}\frac{du}{dx}$$

which is the power rule.

EXAMPLE 3 Using the Power Rule

If $y = (x^3 - 1)^7$, find y'.

Solution: Since y is a power of a *function* of x, the power rule applies. Letting $u(x) = x^3 - 1$ and $a = 7$, we have

$$\begin{aligned}y' &= a[u(x)]^{a-1}u'(x) \\ &= 7(x^3 - 1)^{7-1}\frac{d}{dx}(x^3 - 1) \\ &= 7(x^3 - 1)^6(3x^2) = 21x^2(x^3 - 1)^6\end{aligned}$$

Now Work Problem 9 ◁

EXAMPLE 4 Using the Power Rule

If $y = \sqrt[3]{(4x^2 + 3x - 2)^2}$, find dy/dx when $x = -2$.

Solution: Since $y = (4x^2 + 3x - 2)^{2/3}$, we use the power rule with

$$u = 4x^2 + 3x - 2$$

and $a = \frac{2}{3}$. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{3}(4x^2 + 3x - 2)^{(2/3)-1}\frac{d}{dx}(4x^2 + 3x - 2) \\ &= \frac{2}{3}(4x^2 + 3x - 2)^{-1/3}(8x + 3) \\ &= \frac{2(8x + 3)}{3\sqrt[3]{4x^2 + 3x - 2}}\end{aligned}$$

Thus,

$$\left.\frac{dy}{dx}\right|_{x=-2} = \frac{2(-13)}{3\sqrt[3]{8}} = -\frac{13}{3}$$

Now Work Problem 19 ◁

EXAMPLE 5 Using the Power Rule

The technique used in Example 5 is frequently used when the numerator of a quotient is a constant and the denominator is not.

If $y = \dfrac{1}{x^2 - 2}$, find $\dfrac{dy}{dx}$.

Solution: Although the quotient rule can be used here, a more efficient approach is to treat the right side as the power $(x^2 - 2)^{-1}$ and use the power rule. Let $u = x^2 - 2$. Then $y = u^{-1}$, and

$$\begin{aligned}\frac{dy}{dx} &= (-1)(x^2-2)^{-1-1}\frac{d}{dx}(x^2-2)\\ &= (-1)(x^2-2)^{-2}(2x)\\ &= -\frac{2x}{(x^2-2)^2}\end{aligned}$$

Now Work Problem 27 ◁

EXAMPLE 6 Differentiating a Power of a Quotient

If $z = \left(\dfrac{2s+5}{s^2+1}\right)^4$, find $\dfrac{dz}{ds}$.

The problem here is to recognize the form of the function to be differentiated. In this case it is a power, not a quotient.

Solution: Since z is a power of a function, we first use the power rule:

$$\frac{dz}{ds} = 4\left(\frac{2s+5}{s^2+1}\right)^{4-1}\frac{d}{ds}\left(\frac{2s+5}{s^2+1}\right)$$

Now we use the quotient rule:

$$\frac{dz}{ds} = 4\left(\frac{2s+5}{s^2+1}\right)^3\left(\frac{(s^2+1)(2)-(2s+5)(2s)}{(s^2+1)^2}\right)$$

Simplifying, we have

$$\begin{aligned}\frac{dz}{ds} &= 4\cdot\frac{(2s+5)^3}{(s^2+1)^3}\left(\frac{-2s^2-10s+2}{(s^2+1)^2}\right)\\ &= -\frac{8(s^2+5s-1)(2s+5)^3}{(s^2+1)^5}\end{aligned}$$

Now Work Problem 41 ◁

EXAMPLE 7 Differentiating a Product of Powers

If $y = (x^2-4)^5(3x+5)^4$, find y'.

Solution: Since y is a product, we first apply the product rule:

$$y' = (x^2-4)^5\frac{d}{dx}((3x+5)^4) + (3x+5)^4\frac{d}{dx}((x^2-4)^5)$$

Now we use the power rule:

$$\begin{aligned}y' &= (x^2-4)^5(4(3x+5)^3(3)) + (3x+5)^4(5(x^2-4)^4(2x))\\ &= 12(x^2-4)^5(3x+5)^3 + 10x(3x+5)^4(x^2-4)^4\end{aligned}$$

In differentiating a product in which at least one factor is a power, simplifying the derivative usually involves factoring.

To simplify, we first remove common factors:

$$\begin{aligned} y' &= 2(x^2-4)^4(3x+5)^3[6(x^2-4)+5x(3x+5)] \\ &= 2(x^2-4)^4(3x+5)^3(21x^2+25x-24) \end{aligned}$$

Now Work Problem 39 ◁

Usually, the power rule should be used to differentiate $y = [u(x)]^n$. Although a function such as $y = (x^2+2)^2$ can be written $y = x^4+4x^2+4$ and differentiated easily, this method is impractical for a function such as $y = (x^2+2)^{1000}$. Since $y = (x^2+2)^{1000}$ is of the form $y = [u(x)]^n$, we have

$$y' = 1000(x^2+2)^{999}(2x)$$

Marginal-Revenue Product

Let us now use our knowledge of calculus to develop a concept relevant to economic studies. Suppose a manufacturer hires m employees who produce a total of q units of a product per day. We can think of q as a function of m. If r is the total revenue the manufacturer receives for selling these units, then r can also be considered a function of m. Thus, we can look at dr/dm, the rate of change of revenue with respect to the number of employees. The derivative dr/dm is called the **marginal-revenue product.** It approximates the change in revenue that results when a manufacturer hires an extra employee.

EXAMPLE 8 Marginal-Revenue Product

A manufacturer determines that m employees will produce a total of q units of a product per day, where

$$q = \frac{10m^2}{\sqrt{m^2+19}} \tag{1}$$

If the demand equation for the product is $p = 900/(q+9)$, determine the marginal-revenue product when $m = 9$.

Solution: We must find dr/dm, where r is revenue. Note that, by the chain rule,

$$\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$$

Thus, we must find both dr/dq and dq/dm when $m = 9$. We begin with dr/dq. The revenue function is given by

$$r = pq = \left(\frac{900}{q+9}\right)q = \frac{900q}{q+9} \tag{2}$$

so, by the quotient rule,

$$\frac{dr}{dq} = \frac{(q+9)(900) - 900q(1)}{(q+9)^2} = \frac{8100}{(q+9)^2}$$

In order to evaluate this expression when $m = 9$, we first use the given equation $q = 10m^2/\sqrt{m^2+19}$ to find the corresponding value of q:

$$q = \frac{10(9)^2}{\sqrt{9^2+19}} = 81$$

Hence,

$$\left.\frac{dr}{dq}\right|_{m=9} = \left.\frac{dr}{dq}\right|_{q=81} = \frac{8100}{(81+9)^2} = 1$$

Now we turn to dq/dm. From the quotient and power rules, we have

$$\frac{dq}{dm} = \frac{d}{dm}\left(\frac{10m^2}{\sqrt{m^2+19}}\right)$$

$$= \frac{(m^2+19)^{1/2}\frac{d}{dm}(10m^2) - (10m^2)\frac{d}{dm}[(m^2+19)^{1/2}]}{[(m^2+19)^{1/2}]^2}$$

$$= \frac{(m^2+19)^{1/2}(20m) - (10m^2)[\frac{1}{2}(m^2+19)^{-1/2}(2m)]}{m^2+19}$$

so

$$\left.\frac{dq}{dm}\right|_{m=9} = \frac{(81+19)^{1/2}(20\cdot 9) - (10\cdot 81)[\frac{1}{2}(81+19)^{-1/2}(2\cdot 9)]}{81+19}$$

$$= 10.71$$

A direct formula for the marginal-revenue product is

$$\frac{dr}{dm} = \frac{dq}{dm}\left(p + q\frac{dp}{dq}\right)$$

Therefore, from the chain rule,

$$\left.\frac{dr}{dm}\right|_{m=9} = (1)(10.71) = 10.71$$

This means that if a tenth employee is hired, revenue will increase by approximately \$10.71 per day.

Now Work Problem 80 ◁

PROBLEMS 11.5

In Problems 1–8, use the chain rule.

1. If $y = u^2 - 2u$ and $u = x^2 - x$, find dy/dx.
2. If $y = 2u^3 - 8u$ and $u = 7x - x^3$, find dy/dx.
3. If $y = \frac{1}{w}$ and $w = 3x - 5$, find dy/dx.
4. If $y = \sqrt[4]{z}$ and $z = x^5 - x^4 + 3$, find dy/dx.
5. If $w = u^3$ and $u = \frac{t-1}{t+1}$, find dw/dt when $t = 1$.
6. If $z = u^2 + \sqrt{u} + 9$ and $u = 2s^2 - 1$, find dz/ds when $s = -1$.
7. If $y = 3w^2 - 8w + 4$ and $w = 2x^2 + 1$, find dy/dx when $x = 0$.
8. If $y = 2u^3 + 3u^2 + 5u - 1$ and $u = 3x + 1$, find dy/dx when $x = 1$.

In Problems 9–52, find y'.

9. $y = (3x+2)^6$
10. $y = (x^2-4)^4$
11. $y = (3+2x^3)^5$
12. $y = (x^2+x)^4$
13. $y = 5(x^3-3x^2+2x)^{100}$
14. $y = \frac{(2x^2+1)^4}{2}$
15. $y = (x^2-2)^{-3}$
16. $y = (2x^3-8x)^{-12}$
17. $y = 2(x^2+5x-2)^{-5/7}$
18. $y = 3(5x-2x^3)^{-5/3}$
19. $y = \sqrt{5x^2-x}$
20. $y = \sqrt{3x^2-7}$
21. $y = \sqrt[4]{2x-1}$
22. $y = \sqrt[3]{8x^2-1}$
23. $y = 4\sqrt[7]{(x^2+1)^3}$
24. $y = 7\sqrt[3]{(x^5-3)^5}$
25. $y = \frac{6}{2x^2-x+1}$
26. $y = \frac{3}{x^4+2}$
27. $y = \frac{1}{(x^2-3x)^2}$
28. $y = \frac{1}{(3+5x)^3}$
29. $y = \frac{4}{\sqrt{9x^2+1}}$
30. $y = \frac{3}{(3x^2-x)^{2/3}}$
31. $y = \sqrt[3]{7x} + \sqrt[3]{7}x$
32. $y = \sqrt{2x} + \frac{1}{\sqrt{2x}}$
33. $y = x^3(2x+3)^7$
34. $y = x(x+4)^4$
35. $y = 4x^2\sqrt{5x+1}$
36. $y = 4x^3\sqrt{1-x^2}$
37. $y = (x^2+2x-1)^3(5x)$
38. $y = x^4(x^4-1)^5$
39. $y = (8x-1)^3(2x+1)^4$
40. $y = (3x+2)^5(4x-5)^2$
41. $y = \left(\frac{x-3}{x+2}\right)^{12}$
42. $y = \left(\frac{2x}{x+2}\right)^4$
43. $y = \sqrt{\frac{x+1}{x-5}}$
44. $y = \sqrt[3]{\frac{8x^2-3}{x^2+2}}$
45. $y = \frac{2x-5}{(x^2+4)^3}$
46. $y = \frac{(4x-2)^4}{3x^2+7}$
47. $y = \frac{(8x-1)^5}{(3x-1)^3}$
48. $y = \sqrt[3]{(x-3)^3(x+5)}$
49. $y = 6(5x^2+2)\sqrt{x^4+5}$
50. $y = 6+3x-4x(7x+1)^2$
51. $y = 8t + \frac{t-1}{t+4} - \left(\frac{8t-7}{4}\right)^2$
52. $y = \frac{(2x^3+6)(7x-5)}{(2x+4)^2}$

In Problems 53 and 54, use the quotient rule and power rule to find y'. Do not simplify your answer.

53. $y = \frac{(3x+2)^3(x+1)^4}{(x^2-7)^3}$
54. $y = \frac{\sqrt{x+2}(4x^2-1)^2}{9x-3}$
55. If $y = (5u+6)^3$ and $u = (x^2+1)^4$, find dy/dx when $x = 0$.
56. If $z = 2y^2 - 4y + 5$, $y = 6x - 5$, and $x = 2t$, find dz/dt when $t = 1$.
57. Find the slope of the curve $y = (x^2-7x-8)^3$ at the point $(8, 0)$.

58. Find the slope of the curve $y = \sqrt{x+2}$ at the point $(7, 3)$.

In Problems 59–62, find an equation of the tangent line to the curve at the given point.

59. $y = \sqrt[3]{(x^2-8)^2}$; $(3, 1)$

60. $y = (x+3)^3$; $(-1, 8)$

61. $y = \dfrac{\sqrt{7x+2}}{x+1}$; $\left(1, \dfrac{3}{2}\right)$

62. $y = \dfrac{-3}{(3x^2+1)^3}$; $(0, -3)$

In Problems 63 and 64, determine the percentage rate of change of y with respect to x for the given value of x.

63. $y = (x^2+1)^4$; $x = 1$

64. $y = \dfrac{1}{(x^2-1)^3}$; $x = 2$

In Problems 65–68, q is the total number of units produced per day by m employees of a manufacturer, and p is the price per unit at which the q units are sold. In each case, find the marginal-revenue product for the given value of m.

65. $q = 5m, p = -0.4q + 50$; $m = 6$

66. $q = (200m - m^2)/20, p = -0.1q + 70$; $m = 40$

67. $q = 10m^2/\sqrt{m^2+9}, p = 525/(q+3)$; $m = 4$

68. $q = 50m/\sqrt{m^2+11}, p = 100/(q+10)$; $m = 5$

69. Demand Equation Suppose $p = 100 - \sqrt{q^2+20}$ is a demand equation for a manufacturer's product.
(a) Find the rate of change of p with respect to q.
(b) Find the relative rate of change of p with respect to q.
(c) Find the marginal-revenue function.

70. Marginal-Revenue Product If $p = k/q$, where k is a constant, is the demand equation for a manufacturer's product and $q = f(m)$ defines a function that gives the total number of units produced per day by m employees, show that the marginal-revenue product is always zero.

71. Cost Function The cost c of producing q units of a product is given by

$$c = 5500 + 12q + 0.2q^2$$

If the price per unit p is given by the equation

$$q = 900 - 1.5p$$

use the chain rule to find the rate of change of cost with respect to price per unit when $p = 85$.

72. Hospital Discharges A governmental health agency examined the records of a group of individuals who were hospitalized with a particular illness. It was found that the total proportion that had been discharged at the end of t days of hospitalization was given by

$$f(t) = 1 - \left(\frac{250}{250+t}\right)^3$$

Find $f'(100)$ and interpret your answer.

73. Marginal Cost If the total-cost function for a manufacturer is given by

$$c = \frac{4q^2}{\sqrt{q^2+2}} + 6000$$

find the marginal-cost function.

74. Salary/Education For a certain population, if E is the number of years of a person's education and S represents average annual salary in dollars, then for $E \geq 7$,

$$S = 340E^2 - 4360E + 42{,}800$$

(a) How fast is salary changing with respect to education when $E = 16$?
(b) At what level of education does the rate of change of salary equal \$5000 per year of education?

75. Biology The volume of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. At time t seconds, the radius (in centimeters) is given by

$$r = 10^{-8}t^2 + 10^{-7}t$$

Use the chain rule to find dV/dt when $t = 10$.

76. Pressure in Body Tissue Under certain conditions, the pressure p developed in body tissue by ultrasonic beams is given as a function of the beam's intensity via the equation[15]

$$p = (2\rho VI)^{1/2}$$

where ρ (a Greek letter read "rho") is density of the affected tissue and V is the velocity of propagation of the beam. Here ρ and V are constants. (a) Find the rate of change of p with respect to I. (b) Find the relative rate of change of p with respect to I.

77. Demography Suppose that, for a certain group of 20,000 births, the number of people surviving to age x years is

$$l_x = -0.000354x^4 + 0.00452x^3 + 0.848x^2 - 34.9x + 20{,}000$$
$$0 \leq x \leq 95.2$$

(a) Find the rate of change of l_x with respect to x, and evaluate your answer for $x = 65$.
(b) Find the relative rate of change and the percentage rate of change of l_x when $x = 65$. Round your answers to three decimal places.

78. Muscle Contraction A muscle has the ability to shorten when a load, such as a weight, is imposed on it. The equation

$$(P + a)(v + b) = k$$

is called the "fundamental equation of muscle contraction."[16] Here P is the load imposed on the muscle, v is the velocity of the shortening of the muscle fibers, and a, b, and k are positive constants. Express v as a function of P. Use your result to find dv/dP.

79. Economics Suppose $pq = 100$ is the demand equation for a manufacturer's product. Let c be the total cost, and assume that the marginal cost is 0.01 when $q = 200$. Use the chain rule to find dc/dp when $q = 200$.

80. Marginal-Revenue Product A monopolist who employs m workers finds that they produce

$$q = 2m(2m+1)^{3/2}$$

units of product per day. The total revenue r (in dollars) is given by

$$r = \frac{50q}{\sqrt{1000+3q}}$$

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[15] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).
[16] Ibid.