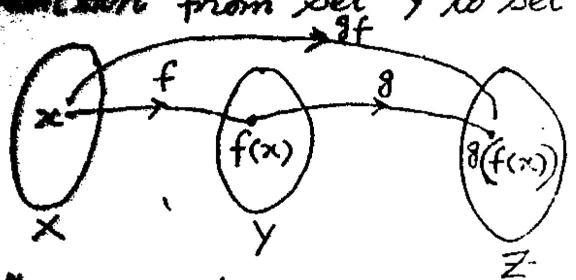
Composition of Functions:

Let f be a function from a set X to set Y and g be a function from set Y to set Z.



The composition of f and g is a function, denoted by $g \circ f$, from X to Z and is defined by

$$(g \circ f)(x) = g(f(x)) = gf(x) \quad \forall\, x \in X.$$

Example 1

Given that $f(x) = 2x+1$ and
$g(x) = x^2 - 1$

(i) $fg(x) = f(g(x)) = f(x^2-1)$
$= 2(x^2-1)+1 = 2x^2-2+1$
$= 2x^2-1$ Ans.

(ii) $gf(x) = g(f(x)) = g(2x+1)$
$= (2x+1)^2 - 1 = 4x^2 + 2(2x)(1) + 1 - 1$
$= 4x^2 + 4x$ Ans.

(iii) $f^2(x) = f(f(x)) = f(2x+1)$
$= 2(2x+1)+1 = 4x+2+1$
$= 4x+3$ Ans.

(iv) $g^2(x) = g(g(x)) = g(x^2-1)$
$= (x^2-1)^2 - 1$
$= x^4 - 2x^2 + 1 - 1 = x^4 - 2x^2$
Ans.

Note: 1. $gf(x) \neq fg(x)$ in general.
2. $ff = f^2$

EXERCISE 1.2

1. (i) Given that

$f(x) = 2x+1$ & $g(x) = \dfrac{3}{x-1}$, $x \neq 1$

(a) $f \circ g(x) = f(g(x)) = f\left(\dfrac{3}{x-1}\right)$
$= 2\left(\dfrac{3}{x-1}\right) + 1 = \dfrac{6}{x-1} + 1$
$= \dfrac{6+x-1}{x-1} = \dfrac{x+5}{x-1}$ Ans.

(b) $g \circ f(x) = g(f(x)) = g(2x+1) = \dfrac{3}{2x+1-1}$
$= \dfrac{3}{2x}$ Ans.

(c) $f \circ f(x) = f(f(x)) = f(2x+1)$
$= 2(2x+1)+1 = 4x+2+1$
$= 4x+3$ Ans.

(d) $g \circ g(x) = g(g(x)) = g\left(\dfrac{3}{x-1}\right) = \dfrac{3}{\frac{3}{x-1} - 1} = \dfrac{3}{\frac{3-x+1}{x-1}} = \dfrac{3(x-1)}{4-x}$

(ii) $f(x) = \sqrt{x+1}$ & $g(x) = \dfrac{1}{x^2}$, $x \neq 0$

(a) $f \circ g(x) = f(g(x)) = f\left(\dfrac{1}{x^2}\right)$
$= \sqrt{\dfrac{1}{x^2} + 1} = \sqrt{\dfrac{1+x^2}{x^2}} = \dfrac{\sqrt{1+x^2}}{x}$

(b) $g \circ f(x) = g(f(x)) = g(\sqrt{x+1})$
$= \dfrac{1}{(\sqrt{x+1})^2} = \dfrac{1}{x+1}$ Ans.

(c) $f \circ f(x) = f(f(x)) = f(\sqrt{x+1})$
$= \sqrt{\sqrt{x+1} + 1}$

(d) $g \circ g(x) = g(g(x)) = g\left(\dfrac{1}{x^2}\right)$
$= \dfrac{1}{\left(\frac{1}{x^2}\right)^2} = \dfrac{1}{\frac{1}{x^4}} = x^4$ Ans.

(iii) $f(x) = \dfrac{1}{\sqrt{x-1}}$, $x \neq 1$ and

$g(x) = (x^2+1)^2$

a) $fog(x) = f(g(x)) = f\left((x^2+1)^2\right)$

$= \dfrac{1}{\sqrt{(x^2+1)^2 - 1}} = \dfrac{1}{\sqrt{x^4+2x^2+1-1}}$

$= \dfrac{1}{\sqrt{x^4+2x^2}} = \dfrac{1}{\sqrt{x^2(x^2+2)}}$

$= \dfrac{1}{x\sqrt{x^2+2}}$ Ans.

(b) $gof(x) = g(f(x)) = g\left(\dfrac{1}{\sqrt{x-1}}\right)$

$= \left[\left(\dfrac{1}{\sqrt{x-1}}\right)^2 + 1\right]^2 = \left(\dfrac{1}{x-1} + 1\right)^2$

$= \left(\dfrac{1+x-1}{x-1}\right)^2 = \left(\dfrac{x}{x-1}\right)^2$ Ans.

(c) $fof(x) = f(f(x)) = f\left(\dfrac{1}{\sqrt{x-1}}\right)$

$= \dfrac{1}{\sqrt{\dfrac{1}{\sqrt{x-1}} - 1}} = \dfrac{1}{\sqrt{\dfrac{1-\sqrt{x-1}}{\sqrt{x-1}}}}$

$= \sqrt{\dfrac{\sqrt{x-1}}{1-\sqrt{x-1}}}$ Ans.

(d) $gog(x) = g(g(x)) = g\left((x^2+1)^2\right)$

$= \left[\{(x^2+1)^2\}^2 + 1\right]^2$

$= \left[\{x^4+2x^2+1\}^2 + 1\right]^2$

$= \left(x^8 + 4x^4 + 1 + 4x^6 + 4x^2 + 2x^4 + 1\right)^2$

$= \left(x^8 + 4x^6 + 6x^4 + 4x^2 + 2\right)^2$ Ans.

(iv) $f(x) = 3x^4 - 2x^2$, $g(x) = \dfrac{2}{\sqrt{x}}$, $x \neq 0$

(a) $fog(x) = f(g(x)) = f\left(\dfrac{2}{\sqrt{x}}\right)$

$= 3\left(\dfrac{2}{\sqrt{x}}\right)^4 - 2\left(\dfrac{2}{\sqrt{x}}\right)^2 = 3\left(\dfrac{16}{x^2}\right) - 2\left(\dfrac{4}{x}\right)$

$= \dfrac{48}{x^2} - \dfrac{8}{x} = \dfrac{48-8x}{x^2} = \dfrac{8(6-x)}{x^2}$ Ans.

(b) $gof(x) = g(f(x))$

$= g(3x^4 - 2x^2)$

$= \dfrac{2}{\sqrt{3x^4-2x^2}} = \dfrac{2}{\sqrt{x^2(3x^2-2)}}$

$= \dfrac{2}{x\sqrt{3x^2-2}}$ Ans.

(c) $fof(x) = f(f(x)) = f(3x^4 - 2x^2)$

$= 3(3x^4-2x^2)^4 - 2(3x^4-2x^2)^2$ Ans.

(d) $gog(x) = g(g(x)) = g\left(\dfrac{2}{\sqrt{x}}\right)$

$= \dfrac{2}{\sqrt{\dfrac{2}{\sqrt{x}}}} = 2 \times \sqrt{\dfrac{\sqrt{x}}{2}}$

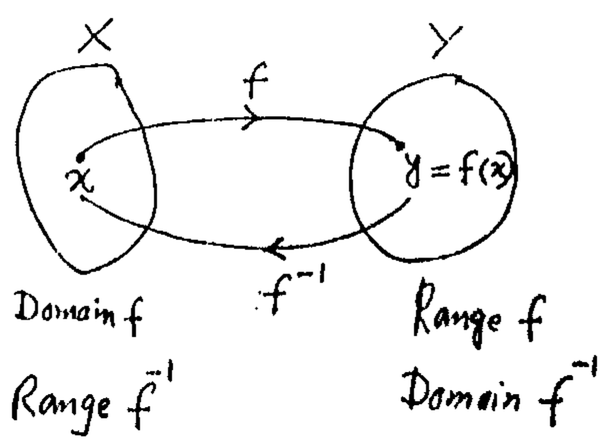
$= \dfrac{2\sqrt{\sqrt{x}}}{\sqrt{2}} = \dfrac{2\sqrt{\sqrt{x}}}{\sqrt{2}\times\sqrt{2}} \times \sqrt{2} = \dfrac{2\sqrt{2}\sqrt{\sqrt{x}}}{2}$

$= \sqrt{2}\sqrt{\sqrt{x}}$ Ans.

Inverse of a function

Let f be a one-one function from X onto Y. The inverse function of f, denoted by f^{-1}, is a function from Y onto X and is defined by:

$x = f^{-1}(y)\ \forall\ y \in Y$ if and only if $y = f(x)\ \forall\ x \in X$.



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Thus we can say that Domain f^{-1} = Range f and Range f^{-1} = Domain f

$f^{-1}(y) = x$ when $f(x) = y$

and $f(x) = y$ when $f^{-1}(y) = x$

Also $(f^{-1}of)(x) = f^{-1}(f(x)) = f^{-1}(y)$
$= x$

and $(fof^{-1})(y) = f(f^{-1}(y)) = f(x) = y$

$\therefore f^{-1}of$ and fof^{-1} are identity mappings on the domain and range of f and f^{-1} respectively.

EXAMPLE 2.

Let $f: R \to R$ be the function defined by

$f(x) = 2x+1$. Find $f^{-1}(x)$

Sol. Let $y = f(x) = 2x+1$

Here y is the image of x under f.

$y = 2x+1$

$\Rightarrow 2x = y-1 \Rightarrow x = \frac{y-1}{2}$

$\Rightarrow f^{-1}(y) = \frac{y-1}{2}$

To find $f^{-1}(x)$, replacing y by x, we get

$f^{-1}(x) = \frac{x-1}{2}$

Verification

$f(f^{-1}(x)) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right)+1$
$= x-1+1 = x$

$f^{-1}(f(x)) = f^{-1}(2x+1) = \frac{2x+1-1}{2}$
$= x$

Hence $f^{-1}(x) = \frac{x-1}{2}$ is the required inverse of $f(x) = 2x+1$

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② i) $f(x) = -2x+8$

a) Let $y = f(x) = -2x+8$

So that y is image of x under f.

Now $y = -2x+8$

$\Rightarrow 2x = 8-y$

$\Rightarrow x = \frac{8-y}{2}$

$\Rightarrow f^{-1}(y) = \frac{8-y}{2}$

Amir Mahmood
Lecturer.
Govt. College Farooka (Sgd)

To find $f^{-1}(x)$, replacing y by x

$f^{-1}(x) = \frac{8-x}{2}$

b) $f^{-1}(-1) = \frac{8-(-1)}{2} = \frac{8+1}{2} = \frac{9}{2}$

To verify that

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = f\left(\frac{8-x}{2}\right) = -2\left(\frac{8-x}{2}\right)+8$
$= -8+x+8 = x$

$\Rightarrow f(f^{-1}(x)) = x \longrightarrow$ ①

Now $f^{-1}(f(x)) = f^{-1}(-2x+8)$
$= \frac{8-(-2x+8)}{2} = \frac{8+2x-8}{2} = \frac{2x}{2} = x$

$\Rightarrow f^{-1}(f(x)) = x \longrightarrow$ ②

$\therefore$ From ① and ②, we get

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$ (proved)

(ii) $f(x) = 3x^3+7$

a) Let $y = f(x) = 3x^3+7$

So that y is image of x under f.

Now $y = 3x^3+7$

$\Rightarrow 3x^3 = y-7 \Rightarrow x^3 = \frac{y-7}{3}$

$\Rightarrow x = \left(\frac{y-7}{3}\right)^{1/3}$

$\Rightarrow f^{-1}(y) = \left(\frac{y-7}{3}\right)^{1/3}$

To find $f^{-1}(x)$, replacing y by x

$f^{-1}(x) = \left(\frac{x-7}{3}\right)^{1/3}$

b) $f^{-1}(-1) = \left(\frac{-1-7}{3}\right)^{1/3} = \left(-\frac{8}{3}\right)^{1/3}$

To verify that

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = f\left(\left(\frac{x-7}{3}\right)^{1/3}\right)$

$= 3\left[\left(\frac{x-7}{3}\right)^{1/3}\right]^3 + 7$

$= 3\left(\frac{x-7}{3}\right) + 7 = x-7+7$

$= x$

$\Rightarrow f(f^{-1}(x)) = x \longrightarrow (1)$

Now $f^{-1}(f(x)) = f^{-1}(3x^3+7)$

$= \left(\frac{3x^3+7-7}{3}\right)^{1/3} = \left(\frac{3x^3}{3}\right)^{1/3}$

$= (x^3)^{1/3} = x$

$\Rightarrow f^{-1}(f(x)) = x \longrightarrow (2)$

$\therefore$ From (1) and (2), we get

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$ (proved)

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iii) $f(x) = (-x+9)^3$

a) Let $y = f(x) = (-x+9)^3$

So that y is the image of x under f.

Now $y = (-x+9)^3$

$\Rightarrow y^{1/3} = -x+9$

$\Rightarrow x = 9 - y^{1/3}$

$\Rightarrow f^{-1}(y) = 9 - y^{1/3}$

To find $f^{-1}(x)$, replacing y by x

$\therefore f^{-1}(x) = 9 - x^{1/3}$

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b) $f^{-1}(-1) = 9 - (-1)^{1/3}$

To verify that

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = f(9 - x^{1/3}) = \left[-(9-x^{1/3})+9\right]^3$

$= (-9+x^{1/3}+9)^3 = (x^{1/3})^3 = x$

$\Rightarrow f(f^{-1}(x)) = x \longrightarrow (1)$

Now

$f^{-1}(f(x)) = f^{-1}((-x+9)^3)$

$= 9 - \left((-x+9)^3\right)^{1/3}$

$= 9-(-x+9) = 9+x-9 = x$

$\Rightarrow f^{-1}(f(x)) = x \longrightarrow (2)$

$\therefore$ From (1) and (2), we get

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$ (proved)

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iv) $f(x) = \frac{2x+1}{x-1}$, $x > 1$

Let $y = f(x) = \frac{2x+1}{x-1}$, so that

y is the image of x under f.

Now $y = \frac{2x+1}{x-1} \Rightarrow y(x-1) = 2x+1$

$\Rightarrow xy - y = 2x+1$

$\Rightarrow xy - 2x = 1+y$

$\Rightarrow x(y-2) = y+1$

$\Rightarrow x = \frac{y+1}{y-2}$

$\Rightarrow f^{-1}(y) = \frac{y+1}{y-2}$

To find $f^{-1}(x)$, relace y by x

$\therefore f^{-1}(x) = \frac{x+1}{x-2}$

b) $f^{-1}(-1) = \frac{-1+1}{-1-2} = \frac{0}{-3} = 0$

To verify that

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right)$

$= \dfrac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1} = \dfrac{\frac{2(x+1)+1(x-2)}{x-2}}{\frac{x+1-1(x-2)}{x-2}}$

$= \dfrac{2x+2+x-2}{x+1-x+2} = \dfrac{3x}{3} = x$

$\Rightarrow f(f^{-1}(x)) = x \longrightarrow (1)$

Now $f^{-1}(f(x)) = f^{-1}\left(\frac{2x+1}{x-1}\right)$

$= \dfrac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2} = \dfrac{\frac{2x+1+x-1}{x-1}}{\frac{2x+1-2x+2}{x-1}}$

$= \dfrac{3x}{3} = x$

$\Rightarrow f^{-1}(f(x)) = x \longrightarrow (2)$

$\therefore$ from (1) & (2), we get

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$

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(3)(i) $f(x) = \sqrt{x+2}$

Let $y = f(x) = \sqrt{x+2}$

y will be real if $x+2 \geq 0$

$\Rightarrow x \geq -2$

$\therefore$ Dom $f = [-2, +\infty)$

Range $f = [0, +\infty)$

By definition of inverse function f^{-1} we have

Dom f^{-1} = Range $f = [0, +\infty)$

Range f^{-1} = Dom $f = [-2, +\infty)$ Ans.

(ii) $f(x) = \dfrac{1}{x+3}$, $x \neq -3$

$f(x) = \dfrac{1}{x+3}$, $x \neq -3$

The function is not defined at $x = -3$

$\therefore$ Dom $f = \mathbb{R} - \{-3\}$

Range $f = \mathbb{R} - \{0\}$

By definition of Inverse function f^{-1}, we have

Dom f^{-1} = Range $f = \mathbb{R} - \{0\}$

Range f^{-1} = Dom $f = \mathbb{R} - \{-3\}$ Ans.

(iii) $f(x) = \dfrac{x-1}{x-4}$, $x \neq 4$

The function f is not defined at $x = 4$

$\therefore$ Dom $f = \mathbb{R} - \{4\}$

Range $f = \mathbb{R} - \{1\}$

By definition of inverse function f^{-1} we have

Dom f^{-1} = Range $f = \mathbb{R} - \{1\}$

Range f^{-1} = Dom $f = \mathbb{R} - \{4\}$ Ans.

(iv) $f(x) = (x-5)^2$, $x \geq 5$

Dom $f = [5, +\infty)$

Range $f = [0, +\infty)$

By definition of inverse function f^{-1}, we have

Dom f^{-1} = Range $f = [0, +\infty)$

Range f^{-1} = Dom $f = [5, +\infty)$

Ans.

Amir Mahmood
Lecturer