

Question # 1

Check each of the following equations written against the differential equation is its solution.

(i) $x \frac{dy}{dx} = 1 + y$, $y = cx - 1$

(ii) $x^2(2y+1) \frac{dy}{dx} - 1 = 0$, $y^2 + y = c - \frac{1}{x}$

(iii) $y \frac{dy}{dx} - e^{2x} = 1$, $y^2 = 2x + e^{2x} + c$

(iv) $\frac{1}{x} \frac{dy}{dx} - 2y = 0$, $y = ce^{x^2}$

(v) $\frac{dy}{dx} = \frac{y^2 + 1}{e^{-x}}$, $y = \text{Tan}(e^x + c)$

Solution

(i) $x \frac{dy}{dx} = 1 + y$

$$\Rightarrow x dy = (1 + y) dx \Rightarrow \frac{dy}{1 + y} = \frac{dx}{x}$$

Integrating both sides

$$\int \frac{dy}{1 + y} = \int \frac{dx}{x}$$

$$\Rightarrow \ln(1 + y) = \ln x + \ln c$$

$$= \ln cx$$

$$\Rightarrow 1 + y = cx$$

$$\Rightarrow y = cx - 1 \quad \text{Proved}$$

(ii) $x^2(2y+1) \frac{dy}{dx} - 1 = 0$

$$\Rightarrow x^2(2y+1) \frac{dy}{dx} = 1 \Rightarrow x^2(2y+1) dy = dx$$

$$\Rightarrow (2y+1) dy = \frac{1}{x^2} dx$$

On integrating

$$\int (2y+1) dy = \int \frac{1}{x^2} dx$$

$$\Rightarrow 2 \int y dy + \int dy = \int x^{-2} dx$$

$$\Rightarrow 2 \cdot \frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1} + c$$

$$\Rightarrow y^2 + y = \frac{x^{-1}}{-1} + c$$

$$\Rightarrow y^2 + y = c - \frac{1}{x} \quad \text{Proved}$$

(iii) $y \frac{dy}{dx} - e^{2x} = 1$

$$\Rightarrow y \frac{dy}{dx} = 1 + e^{2x} \Rightarrow y dy = (1 + e^{2x}) dx$$

On integrating

$$\int y dy = \int (1 + e^{2x}) dx$$

$$\Rightarrow \frac{y^2}{2} = x + \frac{e^{2x}}{2} + \frac{c}{2} \Rightarrow y^2 = 2x + e^{2x} + c$$

$$\Rightarrow y^2 = 2x + e^{2x} + c$$

(iv) $\frac{1}{x} \frac{dy}{dx} - 2y = 0$

$$\Rightarrow \frac{1}{x} \frac{dy}{dx} = 2y \Rightarrow \frac{dy}{dx} = 2xy$$

$$\Rightarrow \frac{dy}{y} = 2x dx$$

On integrating

$$\int \frac{dy}{y} = 2 \int x dx$$

$$\Rightarrow \ln y = 2 \cdot \frac{x^2}{2} + \ln c$$

$$= x^2 + \ln c$$

$$= x^2 \ln e + \ln c \quad \because \ln e = 1$$

$$= \ln e^{x^2} + \ln c$$

$$\Rightarrow \ln y = \ln ce^{x^2}$$

$$\Rightarrow y = ce^{x^2} \quad \text{Proved}$$

(v) $\frac{dy}{dx} = \frac{y^2 + 1}{e^{-x}} \Rightarrow \frac{dy}{y^2 + 1} = e^x dx$

Integrating both sides

$$\Rightarrow \int \frac{dy}{y^2 + 1} = \int e^x dx$$

$$\Rightarrow \text{Tan}^{-1} y = e^x + c$$

$$\Rightarrow y = \text{Tan}(e^x + c)$$

Solve the following differential equations:

Question # 2

$$\frac{dy}{dx} = -y$$

Solution

$$\frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} = -dx$$

On integrating

$$\int \frac{dy}{y} = - \int dx$$

$$\begin{aligned} \ln y &= -x + \ln c \\ &= -x \ln e + \ln c \quad \because \ln e = 1 \\ &= \ln e^{-x} + \ln c \\ \Rightarrow \ln y &= \ln ce^{-x} \quad \Rightarrow y = ce^{-x} \end{aligned}$$

Question # 3

$$ydx + xdy = 0$$

Solution

$$\begin{aligned} ydx + xdy &= 0 \quad \Rightarrow ydx = -xdy \\ \Rightarrow \frac{dx}{x} &= -\frac{dy}{y} \end{aligned}$$

On integrating

$$\begin{aligned} \ln x &= -\ln y + \ln c \\ \Rightarrow \ln x &= \ln \frac{c}{y} \\ \Rightarrow x &= \frac{c}{y} \quad \Rightarrow xy = c \end{aligned}$$

Question # 4

$$\frac{dy}{dx} = \frac{1-x}{y}$$

Solution *Do yourself*

Question # 5

$$\frac{dy}{dx} = \frac{y}{x^2}, (y > 0)$$

Solution

$$\frac{dy}{dx} = \frac{y}{x^2} \quad \Rightarrow \frac{dy}{y} = x^{-2} dx$$

Integrating

$$\begin{aligned} \int \frac{dy}{y} &= \int x^{-2} dx \\ \Rightarrow \ln y &= \frac{x^{-2+1}}{-2+1} + \ln c \\ \Rightarrow \ln y &= \frac{x^{-1}}{-1} + \ln c \\ \Rightarrow \ln y &= -\frac{1}{x} + \ln c \\ \Rightarrow \ln y &= -\frac{1}{x} \ln e + \ln c \\ &= \ln e^{-\frac{1}{x}} + \ln c \\ \Rightarrow \ln y &= \ln ce^{-\frac{1}{x}} \quad \Rightarrow y = ce^{-\frac{1}{x}} \end{aligned}$$

Question # 6

$$\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$$

Solution

$$\begin{aligned} \sin y \operatorname{cosec} x \frac{dy}{dx} &= 1 \\ \Rightarrow \sin y dy &= \frac{dx}{\operatorname{cosec} x} \\ \Rightarrow \sin y dy &= \sin x dx \end{aligned}$$

Integrating

$$\begin{aligned} \int \sin y dy &= \int \sin x dx \\ \Rightarrow -\cos y &= -\cos x - c \\ \Rightarrow \cos y &= \cos x + c \end{aligned}$$

Question # 7

$$xdy + y(x-1)dx = 0$$

Solution

$$\begin{aligned} xdy + y(x-1)dx &= 0 \\ \Rightarrow xdy &= -y(x-1)dx \\ \Rightarrow \frac{dy}{y} &= -\frac{x-1}{x} dx \\ \Rightarrow \frac{dy}{y} &= -\left(\frac{x}{x} - \frac{1}{x}\right) dx \\ \Rightarrow \frac{dy}{y} &= -\left(1 - \frac{1}{x}\right) dx \end{aligned}$$

On integrating

$$\begin{aligned} \int \frac{dy}{y} &= -\int \left(1 - \frac{1}{x}\right) dx \\ \Rightarrow \ln y &= -x + \ln x + \ln c \\ &= -x \ln e + \ln x + \ln c \\ &= \ln e^{-x} + \ln x + \ln c \\ \Rightarrow \ln y &= \ln cxe^{-x} \quad \Rightarrow y = cxe^{-x} \end{aligned}$$

Question # 8

$$\frac{x^2+1}{y+1} = \frac{x}{y} \frac{dy}{dx}, (x, y > 0)$$

Solution

$$\begin{aligned} \frac{x^2+1}{y+1} &= \frac{x}{y} \frac{dy}{dx} \\ \Rightarrow \frac{x^2+1}{x} dx &= \frac{y+1}{y} dy \end{aligned}$$

On integrating

$$\begin{aligned} \int \frac{x^2+1}{x} dx &= \int \frac{y+1}{y} dy \\ \Rightarrow \int \left(\frac{x^2}{x} + \frac{1}{x}\right) dx &= \int \left(\frac{y}{y} + \frac{1}{y}\right) dy \end{aligned}$$

$$\Rightarrow \int \left(x + \frac{1}{x}\right) dx = \int \left(1 + \frac{1}{y}\right) dy$$

$$\Rightarrow \int x dx + \int \frac{1}{x} dx = \int dy + \int \frac{1}{y} dy$$

$$\Rightarrow \frac{x^2}{2} + \ln x = y + \ln y - \ln c$$

$$\Rightarrow \frac{x^2}{2} \ln e + \ln x + \ln c = y \ln e + \ln y$$

$$\Rightarrow \ln e^{\frac{x^2}{2}} + \ln x + \ln c = \ln e^y + \ln y$$

$$\Rightarrow \ln cxe^{\frac{x^2}{2}} = \ln ye^y$$

$$\Rightarrow cxe^{\frac{x^2}{2}} = ye^y \quad \text{i.e. } ye^y = cxe^{\frac{x^2}{2}}$$

Question # 9

$$\frac{1}{x} \frac{dy}{dx} = \frac{1}{2} (1 + y^2)$$

Solution *Do yourself*

Question # 10

$$2x^2 y \frac{dy}{dx} = x^2 - 1$$

Solution *Do yourself*

Question # 11

$$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$$

Solution

$$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$$

$$\Rightarrow \frac{dy}{dx} = x - \frac{2xy}{2y+1}$$

$$= x \left(1 - \frac{2y}{2y+1}\right)$$

$$= x \left(\frac{2y+1-2y}{2y+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = x \left(\frac{1}{2y+1}\right) \Rightarrow (2y+1)dy = x dx$$

Now do yourself

Question # 12

$$(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0$$

Solution

$$(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0$$

$$\Rightarrow (x^2 - yx^2) \frac{dy}{dx} = -y^2 - xy^2$$

$$\Rightarrow x^2(1-y) \frac{dy}{dx} = -y^2(1+x)$$

$$\Rightarrow \frac{1-y}{y^2} dy = -\frac{1+x}{x^2} dx$$

Now do yourself

Question # 13

$$\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$$

Solution

$$\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$$

$$\Rightarrow \sec^2 x \tan y dx = -\sec^2 y \tan x dy$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

On integrating

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy$$

$$\Rightarrow \int \frac{d}{dx}(\tan x) dx = -\int \frac{d}{dy}(\tan y) dy$$

$$\Rightarrow \ln \tan x = -\ln \tan y + \ln c$$

$$\Rightarrow \ln \tan x + \ln \tan y = \ln c$$

$$\Rightarrow \ln(\tan x \tan y) = \ln c$$

$$\Rightarrow \tan x \tan y = c$$

Question # 14

$$\left(y - x \frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$$

Solution

$$\left(y - x \frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$$

$$\Rightarrow y - x \frac{dy}{dx} = 2y^2 + 2 \frac{dy}{dx}$$

$$\Rightarrow y - 2y^2 = 2 \frac{dy}{dx} + x \frac{dy}{dx}$$

$$\Rightarrow y(1-2y) = (2+x) \frac{dy}{dx}$$

$$\Rightarrow \frac{dx}{2+x} = \frac{dy}{y(1-2y)}$$

On integrating

$$\int \frac{dx}{2+x} = \int \frac{dy}{y(1-2y)} \dots\dots\dots (i)$$

Now consider

$$\frac{1}{y(1-2y)} = \frac{A}{y} + \frac{B}{1-2y}$$

$$\Rightarrow 1 = A(1-2y) + By \dots\dots\dots (ii)$$

Put $y=0$ in (ii)

$$1 = A(1-2(0)) + 0 \Rightarrow A = 1$$

Put $1-2y=0 \Rightarrow 2y=1 \Rightarrow y = \frac{1}{2}$ in (ii)

$$1 = 0 + B\left(\frac{1}{2}\right) \Rightarrow B = 2$$

So $\frac{1}{y(1-2y)} = \frac{1}{y} + \frac{2}{1-2y}$

Using in (i)

← *

$$\begin{aligned} \int \frac{dx}{2+x} &= \int \left(\frac{1}{y} + \frac{2}{1-2y} \right) dy \\ &= \int \frac{1}{y} dy + \int \frac{2}{1-2y} dy \\ &= \int \frac{1}{y} dy - \int \frac{-2}{1-2y} dy \end{aligned}$$

$$\Rightarrow \int \frac{dx}{2+x} = \int \frac{1}{y} dy - \int \frac{d}{dx}(1-2y) \frac{dy}{1-2y}$$

$$\Rightarrow \ln(2+x) = \ln y - \ln(1-2y) - \ln c$$

$$\Rightarrow \ln(2+x) + \ln c = \ln y - \ln(1-2y)$$

$$\Rightarrow \ln c(2+x) = \ln \frac{y}{(1-2y)}$$

$$\Rightarrow c(2+x) = \frac{y}{(1-2y)}$$

$$\Rightarrow y = c(2+x)(1-2y)$$

Alternative (← *)

$$\begin{aligned} \int \frac{dx}{2+x} &= \int \left(\frac{1}{y} + \frac{2}{1-2y} \right) dx \\ &= \int \frac{1}{y} dy + \int \frac{2}{1-2y} dy \\ &= \int \frac{1}{y} dy - \int \frac{2}{2y-1} dy \end{aligned}$$

$$\Rightarrow \int \frac{dx}{2+x} = \int \frac{1}{y} dy - \int \frac{d}{dx}(2y-1) \frac{dy}{2y-1}$$

$$\Rightarrow \ln(2+x) = \ln y - \ln(2y-1) - \ln c$$

$$\Rightarrow \ln(2+x) + \ln c = \ln y - \ln(2y-1)$$

$$\Rightarrow \ln c(2+x) = \ln \frac{y}{(2y-1)}$$

$$\Rightarrow c(2+x) = \frac{y}{(2y-1)}$$

i.e. $\frac{y}{(2y-1)} = c(2+x)$

Review

- $\int \tan x dx = \ln|\sec x| = -\ln|\cos x|$
- $\int \cot x dx = \ln|\sin x| = -\ln|\csc x|$
- $\int \sec x dx = \ln|\sec x + \tan x|$
- $\int \csc x dx = \ln|\csc x - \cot x|$

Question # 15

$$1 + \cos x \tan y \frac{dy}{dx} = 0$$

Solution

$$1 + \cos x \tan y \frac{dy}{dx} = 0$$

$$\Rightarrow \cos x \tan y \frac{dy}{dx} = -1$$

$$\Rightarrow \tan y dy = -\frac{1}{\cos x} dx$$

$$\Rightarrow \tan y dy = -\sec x dx$$

On integrating

$$\int \tan y dy = -\int \sec x dx$$

$$\Rightarrow -\ln|\cos y| = -\ln|\sec x + \tan x| - \ln c$$

$$\Rightarrow \ln|\cos y| = +\ln|\sec x + \tan x| + \ln c$$

$$\Rightarrow \ln|\cos y| = \ln|c(\sec x + \tan x)|$$

$$\Rightarrow \cos y = c(\sec x + \tan x)$$

Question # 16

$$y - x \frac{dy}{dx} = 3 \left(1 + x \frac{dy}{dx} \right)$$

Solution

$$y - x \frac{dy}{dx} = 3 \left(1 + x \frac{dy}{dx} \right)$$

$$\Rightarrow y - x \frac{dy}{dx} = 3 + 3x \frac{dy}{dx}$$

$$\Rightarrow y - 3 = 3x \frac{dy}{dx} + x \frac{dy}{dx}$$

$$= (3x + x) \frac{dy}{dx}$$

$$\Rightarrow y - 3 = 4x \frac{dy}{dx} \Rightarrow \frac{dx}{x} = 4 \frac{dy}{y-3}$$

Now do yourself

Question # 17

$$\sec x + \tan y \frac{dy}{dx} = 0$$

Solution

$$\sec x + \tan y \frac{dy}{dx} = 0$$

$$\Rightarrow \tan y \frac{dy}{dx} = -\sec x$$

$$\Rightarrow \tan y dy = -\sec x dx$$

Now do yourself as Question # 15

Question # 18

$$(e^x + e^{-x}) \frac{dy}{dx} = e^x - e^{-x}$$

Solution

$$(e^x + e^{-x}) \frac{dy}{dx} = e^x - e^{-x}$$

$$\Rightarrow dy = \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

On integrating

$$\int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$\Rightarrow y = \int \frac{d(e^x + e^{-x})}{e^x + e^{-x}} dx$$

$$\Rightarrow y = \ln(e^x + e^{-x}) + c$$

Question # 19

Find the general solution of the equation

$$\frac{dy}{dx} - x = xy^2. \text{ Also find the perpendicular}$$

solution if $y = 1$ when $x = 0$.

Solution

$$\frac{dy}{dx} - x = xy^2 \Rightarrow \frac{dy}{dx} = x + xy^2$$

$$\Rightarrow \frac{dy}{dx} = x(1 + y^2) \Rightarrow \frac{dy}{1 + y^2} = x dx$$

$$\Rightarrow \int \frac{dy}{1 + y^2} = \int x dx$$

$$\Rightarrow \tan^{-1} y = \frac{x^2}{2} + c$$

$$\Rightarrow y = \tan\left(\frac{x^2}{2} + c\right)$$

Question # 20

Solve the differential equation $\frac{dx}{dt} = 2x$ given

that $x = 4$ when $t = 0$

Solution

$$\frac{dx}{dt} = 2x \Rightarrow \frac{dx}{x} = 2dt$$

$$\Rightarrow \int \frac{dx}{x} = 2 \int dt$$

$$\Rightarrow \ln x = 2t + \ln c$$

$$= \ln e^{2t} + \ln c \quad \because \ln e^x = x$$

$$\Rightarrow \ln x = \ln ce^{2t}$$

$$\Rightarrow x = ce^{2t} \dots\dots (i)$$

When $t = 0$ then $x = 4$, putting in (i)

$$4 = ce^{2(0)} \Rightarrow 4 = ce^0$$

$$\Rightarrow 4 = c(1) \Rightarrow c = 4$$

Putting in (i)

$$\Rightarrow x = 4e^{2t}$$

Question # 21

Solve the differential equation $\frac{ds}{dt} + 2st = 0$.

Also find the perpendicular solution if $s = 4e$, when $t = 0$

Solution

$$\frac{ds}{dt} + 2st = 0$$

$$\Rightarrow \frac{ds}{dt} = -2st \Rightarrow \frac{ds}{s} = -2t dt$$

On integrating

$$\int \frac{ds}{s} = -2 \int t dt$$

$$\Rightarrow \ln s = -2 \frac{t^2}{2} + \ln c$$

$$= -t^2 + \ln c$$

$$= \ln e^{-t^2} + \ln c \quad \because \ln e^x = x$$

$$\Rightarrow \ln s = \ln ce^{-t^2}$$

$$\Rightarrow s = ce^{-t^2} \dots\dots (i)$$

When $t = 0$ then $s = 4e$, using in (i)

$$4e = ce^{-(0)^2} \Rightarrow 4e = c(1)$$

$$\Rightarrow c = 4e$$

Putting in (i)

$$s = 4e \cdot e^{-t^2}$$

$$\Rightarrow s = 4e^{1-t^2}$$

Question # 22

In a culture, bacteria increases at the rate proportional to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution

- Number of bacteria initially = 200
- No. of bacteria after two hours = 2(200) = 400
- No. of bacteria after four hours = 2(400) = 800 *Ans.*

Question # 23

A ball is thrown vertically upward with a velocity of 2450cm / sec. Neglecting air resistance, find

- (i) velocity of ball at any time t
- (ii) distance travelled in any time t
- (iii) maximum height attained by the ball.

Solution

i) When a body is projected upward its acceleration is $-g$. (where $g = 980 \text{ cm/sec}^2$)

i.e. acceleration = $\frac{dv}{dt} = -g$,
 where v is velocity of ball.

$\Rightarrow \frac{dv}{dt} = -980$
 $\Rightarrow dv = -980 dt$

On integrating

$\int dv = -980 \int dt$

$\Rightarrow v = -980t + c_1 \dots\dots\dots (i)$

Initially, when $t = 0$ then $v = 2450 \text{ cm/sec}$

$2450 = -980(0) + c_1$

$\Rightarrow c_1 = 2450$

Putting in (i)

$v = -980t + 2450$

ii) Since velocity = $v = \frac{dx}{dt}$
 where x is height of ball.

$\Rightarrow \frac{dx}{dt} = -980t + 2450$

$\Rightarrow dx = (-980t + 2450) dt$

Integrating

$\int dx = \int (-980t + 2450) dt$

$\Rightarrow x = -980 \frac{t^2}{2} + 2450t + c_2$

$\Rightarrow x = -490t^2 + 2450t + c_2 \dots\dots\dots (ii)$

Initially, when $t = 0$ then $x = 0$

$0 = -490(0) + 2450(0) + c_2$

$\Rightarrow c_2 = 0$

Putting value of c_2 in (ii)

$\Rightarrow x = -490t^2 + 2450t + 0$

$\Rightarrow x = 2450t - 490t^2$

iii)

$\therefore v = -980t + 2450$

When body is at max. height then $v = 0$

$\Rightarrow -980t + 2450 = 0$

$\Rightarrow 980t = 2450 \Rightarrow t = \frac{2450}{980}$

$\Rightarrow t = 2.5 \text{ sec}$

Since $x = 2450t - 490t^2$

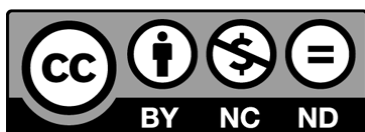
When $t = 2.5 \text{ sec}$

$x = 2450(2.5) - 490(2.5)^2$
 $= 6125 - 3062.5$
 $= 3062.5$

Hence ball attains max. height of 3062.5 cm.

Book: Exercise 3.8, page 177.
Calculus and Analytic Geometry Mathematic 12
Punjab Textbook Board, Lahore.

Available online at <http://www.MathCity.org> in PDF Format
 (Picture format to view online).
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