Example 4

Find the area bounded by the curve

$$f(x) = x^3 - 2x^2 + 1$$

and the x-axis in the first quadrant.

Solution

Put $f(x) = 0$

$\Rightarrow x^3 - 2x + 1 = 0$

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -2 & 0 & 1 \\ & \downarrow & 1 & -1 & -1 \\ \hline & 1 & -1 & -1 & \boxed{0} \end{array}$$

$\Rightarrow (x-1)(x^2 - x - 1) = 0$

$\Rightarrow x - 1 = 0$ or $x^2 - x - 1 = 0$

$\Rightarrow x = 1$ or $x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Thus the curve cuts the x-axis at $x = 1$, $\dfrac{1 \pm \sqrt{5}}{2}$

Since we are taking area in the first quad. only

$\therefore x = 1$, $\dfrac{1+\sqrt{5}}{2}$ ignoring $\dfrac{1-\sqrt{5}}{2}$ as it is $-$ive.

Intervals in 1st quad. are $[0,1]$ & $\left[1, \dfrac{1+\sqrt{5}}{2}\right]$

Since $f(x) \geq 0$ whenever $x \in [0,1]$

and $f(x) \leq 0$ whenever $x \in \left[1, \dfrac{1+\sqrt{5}}{2}\right]$

$\therefore$ Area in 1st quad. $= \int_0^1 \left(x^3 - 2x^2 + 1\right) dx$

$$= \left| \frac{x^4}{4} - 2\frac{x^3}{3} + x \right|_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3} + 1\right) - 0$$

$$= \frac{7}{12} \text{ sq. unit}$$

Question # 1

Find the area between the x-asis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$.

Solution

$y = x^2 + 1$; $x = 1$ to $x = 2$

$\therefore y \geq 0$ whenever $x \in [1,2]$

$\therefore$ Area $= \int_1^2 (x^2 + 1) dx$

$$= \int_1^2 x^2 dx + \int_1^2 dx$$

$$= \left| \frac{x^3}{3} \right|_1^2 + \left| x \right|_1^2$$

$$= \left(\frac{(2)^3}{3} - \frac{(1)^3}{3} \right) + (2 - 1)$$

$$= \left(\frac{8}{3} - \frac{1}{3} \right) + 1$$

$$= \frac{7}{3} + 1 = \frac{10}{3} \text{ sq. unit.}$$

Question # 2

Find the area above the x-asis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

Solution

$y = 5 - x^2$; $x = -1$ to $x = 2$

$\therefore y > 0$ whenever $x \in (-1, 2)$

$\therefore$ Area $= \int_{-1}^2 (5 - x^2) dx$

$$= \left| 5x - \frac{x^3}{3} \right|_{-1}^2$$

$$= \left(5(2) - \frac{(2)^3}{3} \right) - \left(5(-1) - \frac{(-1)^3}{3} \right)$$

$$= \left(10 - \frac{8}{3} \right) - \left(-5 + \frac{1}{3} \right)$$

$$= \frac{22}{3} - \left(-\frac{14}{3} \right) = \frac{22}{3} + \frac{14}{3}$$

$$= \frac{36}{3} = 12 \text{ sq. unit}$$

Question # 3

Find the area below the curve $y = 3\sqrt{x}$ and above the x-axis between $x = 1$ to $x = 4$.

Solution

$y = 3\sqrt{x}$; $x=1$ to $x=4$

Since $y \ge 0$ when $x \in [1,4]$

$\therefore$ Area $= \int_1^4 3\sqrt{x}\, dx$

$= \int_1^4 3x^{\frac{1}{2}}\, dx = 3\int_1^4 x^{\frac{1}{2}}\, dx$

$= 3\left|\dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^4 = 3\left|\dfrac{x^{\frac{3}{2}}}{3/2}\right|_1^4$

$= 3\times\dfrac{2}{3}\left|x^{\frac{3}{2}}\right|_1^4 = 2\left((4)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right)$

$= \dfrac{3}{4}\left((4)^{\frac{4}{3}}-(1)^{\frac{4}{3}}\right) = 2\left((2^2)^{\frac{3}{2}}-1\right)$

$= 2(8-1) = 14$ sq. unit

Question # 4

Find the area bounded by cos function from $x=-\dfrac{\pi}{2}$ to $x=\dfrac{\pi}{2}$

Solution

$y = \cos x$; $x=-\dfrac{\pi}{2}$ to $x=\dfrac{\pi}{2}$

$\because$ $y>0$ whenever $x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

$\therefore$ Area $= \int_{-\pi/2}^{\pi/2}\cos x\,dx$

$= \left|\sin x\right|_{-\pi/2}^{\pi/2}$

$= \sin\left(\dfrac{\pi}{2}\right)-\sin\left(-\dfrac{\pi}{2}\right)$

$= 1+1 = 2$ sq. unit

Question # 5

Find the area between the x-asis and the curve $y = 4x-x^2$

Solution

$y = 4x-x^2$

Putting $y=0$, we have

$4x-x^2 = 0$

$\Rightarrow x(4-x) = 0$

$\Rightarrow x = 0$ or $x = 4$

Now $y>0$ when $x\in(0,4)$

$\therefore$ Area $= \int_0^4\left(4x-x^2\right)dx$

$= \left|\dfrac{4x^2}{2}-\dfrac{x^3}{3}\right|_0^4 = \left|2x^2-\dfrac{x^3}{3}\right|_0^4$

$= \left(2(4)^2-\dfrac{(4)^3}{3}\right)-\left(2(0)^2-\dfrac{(0)^3}{3}\right)$

$= \left(32-\dfrac{64}{3}\right)-(0-0)$

$= \dfrac{32}{2}$ sq. unit.

Question # 6

Determine the area bounded by the parabola $y = x^2+2x-3$ and the x-axis.

Solution

$y = x^2+2x-3$

Putting $y = 0$, we have

$x^2+2x-3 = 0$

$\Rightarrow x^2+3x-x-2 = 0$

$\Rightarrow x(x+3)-1(x+3) = 0$

$\Rightarrow (x+3)(x-1) = 0$

$\Rightarrow x = -3$ or $x = 1$

Now $y\le 0$ whenever $x\in[-3,1]$

$\therefore$ Area $= -\int_{-3}^{1}\left(x^2+2x-3\right)dx$

$= -\left|\dfrac{x^3}{3}+\dfrac{2x^2}{2}-3x\right|_{-3}^{1}$

$= -\left|\dfrac{x^3}{3}+x^2-3x\right|_{-3}^{1}$

$= -\left(\dfrac{(1)^3}{3}+(1)^2-3(1)\right)$
$+\left(\dfrac{(-3)^3}{3}+(-3)^2-3(-3)\right)$

$= -\left(\dfrac{1}{3}+1-3\right)+\left(\dfrac{-27}{3}+9+9\right)$

$= -\left(-\dfrac{5}{3}\right)+(-9+18)$

$= \dfrac{5}{3}+9 = \dfrac{32}{3}$ sq. unit

Question # 7
Find the area bounded by the curve $y = x^3+1$, the x-axis and line $x=2$.
Solution

$y = x^3+1$
Putting $y=0$, we have
$x^3+1 = 0$
$\Rightarrow (x+1)(x^2-x+1) = 0$
$\Rightarrow x+1=0$ or $x^2-x+1=0$
$\Rightarrow x=-1$ or $x=\dfrac{1\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$
$=\dfrac{1\pm\sqrt{1-4}}{2}$
$\Rightarrow x=\dfrac{1\pm\sqrt{-3}}{2}$
Which is not possible.
Now $y\geq 0$ when $x\in[-1,2]$

$\therefore$ Area $= \int_{-1}^{2}(x^3+1)dx$

$= \left|\dfrac{x^4}{4}+x\right|_{-1}^{2}$

$= \left(\dfrac{(2)^4}{4}+2\right)-\left(\dfrac{(-1)^4}{4}-1\right)$

$= \left(\dfrac{16}{4}+2\right)-\left(\dfrac{1}{4}-1\right)$

$= 6-\dfrac{3}{4} = \dfrac{27}{4}$ sq. unit

Question # 8
Find the area bounded by the curve $y = x^3-2x+4$ and the x-axis.
Solution

$y = x^3-2x+4$; $x=1$
Putting $y=0$, we have
$x^3-2x+4 = 0$
By synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & \downarrow & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$\Rightarrow (x+2)(x^2-2x+2) = 0$
$\Rightarrow x+2=0$ or $x^2-2x+2 = 0$
$\Rightarrow x=-2$ or $x=\dfrac{2\pm\sqrt{(-2)^2-4(1)(2)}}{2}$
$=\dfrac{2\pm\sqrt{4-8}}{2}$
$=\dfrac{2\pm\sqrt{-4}}{2}$
This is imaginary.
Now $y\geq 0$ when $x\in[-2,1]$

$\therefore$ Area $= \int_{-2}^{1}(x^3-2x+4)dx$

$= \int_{-2}^{1}x^3dx-2\int_{-2}^{1}x\,dx+4\int_{-2}^{1}dx$

$= \left|\dfrac{x^4}{4}\right|_{-2}^{1}-2\left|\dfrac{x^2}{2}\right|_{-2}^{1}+4|x|_{-2}^{1}$

$= \left(\dfrac{(1)^4}{4}-\dfrac{(-2)^4}{4}\right)-2\left(\dfrac{(1)^2}{2}-\dfrac{(-2)^2}{2}\right)+4(1-(-2))$

$= \left(\dfrac{1}{4}-\dfrac{16}{4}\right)-2\left(\dfrac{1}{2}-\dfrac{4}{2}\right)+4(1+2)$

$= \left(\dfrac{1}{4}-4\right)-2\left(\dfrac{1}{2}-2\right)+4(3)$

$= \left(-\dfrac{15}{4}\right)-2\left(-\dfrac{3}{2}\right)+12$

$= -\dfrac{15}{4}+3+12 = \dfrac{45}{4}$ sq. unit

Question # 9
Find the area between the curve
Solution

$y = x^3-4x$
Putting $y=0$, we have
$x^3-4x = 0$
$\Rightarrow x(x^2-4) = 0$
$\Rightarrow x(x+2)(x-2) = 0$
$\Rightarrow x = 0$ or $x=-2$ or $x=2$
Now $y\geq 0$ whenever $x\in[-2,0]$
And $y\leq 0$ whenever $x\in[0,2]$

$\therefore$ Area $= \int_{-2}^{0}y\,dx-\int_{0}^{2}y\,dx$

$$= \int_{-2}^{0}\left(x^3-4x\right)dx-\int_{0}^{2}\left(x^3-4x\right)dx$$

$$= \left|\frac{x^4}{4}-4\frac{x^2}{2}\right|_{-2}^{0}-\left|\frac{x^4}{4}-4\frac{x^2}{2}\right|_{0}^{2}$$

$$= \left|\frac{x^4}{4}-2x^2\right|_{-2}^{0}-\left|\frac{x^4}{4}-2x^2\right|_{0}^{2}$$

$$= \left(\frac{(0)^4}{4}-2(0)^2\right)-\left(\frac{(-2)^4}{4}-2(-2)^2\right)-\left(\frac{(2)^4}{4}-2(2)^2\right)+\left(\frac{(0)^4}{4}-2(0)^2\right)$$

$$= (0-0)-\left(\frac{16}{4}-8\right)-\left(\frac{16}{4}-8\right)+(0-0)$$

$$= -(4-8)-(4-8) \quad = -(-4)-(-4)$$

$$= 4+4 \quad = 8 \text{ sq. unit.}$$

Question # 9

Find the area between the curve
$y = x(x-1)(x+1)$ and the x-axis.

Solution

$$y = x(x-1)(x+1)$$

Putting $y = 0$, we have

$$x(x-1)(x+1) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -1$$

Now $y \geq 0$ whenever $x \in [-1,0]$

And $y \leq 0$ whenever $x \in [0,1]$

$$\therefore \text{ Area } = \int_{-1}^{0} y\,dx-\int_{0}^{1} y\,dx$$

$$= \int_{-1}^{0} x(x-1)(x+1)\,dx-\int_{0}^{1} x(x-1)(x+1)\,dx$$

$$= \int_{-1}^{0}\left(x^3-x\right)dx-\int_{0}^{1}\left(x^3-x\right)dx$$

$$= \left|\frac{x^4}{4}-\frac{x^2}{2}\right|_{-1}^{0}-\left|\frac{x^4}{4}-\frac{x^2}{2}\right|_{0}^{1}$$

$$= \left(\frac{(0)^4}{4}-\frac{(0)^2}{2}\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^2}{2}\right)-\left(\frac{(1)^4}{4}-\frac{(1)^2}{2}\right)+\left(\frac{(0)^4}{4}-\frac{(0)^2}{2}\right)$$

$$= (0-0)-\left(\frac{1}{4}-\frac{1}{2}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)+(0-0)$$

$$= 0-\left(-\frac{1}{4}\right)-\left(-\frac{1}{4}\right)+0$$

$$= \frac{1}{4}+\frac{1}{4} \quad = \frac{1}{2} \text{ sq. unit}$$

Question # 11

Find the area between the x-asis and the curve
$y = \cos\frac{1}{2}x$ from $x=-\pi$ to $x=\pi$

Solution

$$g(x) = \cos\frac{1}{2}x \quad ; \quad x=-\pi \text{ to } x=\pi$$

$$\because \quad g(x) \geq 0 \quad \text{when} \quad x\in[-\pi,\pi]$$

$$\therefore \text{ Area } = \int_{-\pi}^{\pi}\cos\frac{1}{2}x\,dx$$

$$= \left|\frac{\sin\frac{x}{2}}{1/2}\right|_{-\pi}^{\pi} \quad = 2\left|\sin\frac{x}{2}\right|_{-\pi}^{\pi}$$

$$= 2\left(\sin\left(\frac{\pi}{2}\right)-\sin\left(\frac{-\pi}{2}\right)\right)$$

$$= 2\left(1-(-1)\right) \quad = 2(1+1)$$

$$= 2(2) \quad = 4 \text{ sq. unit.}$$

Question # 12

Find the area between the x-asis and the curve
$y = \sin 2x$ from $x=0$ to $x=\frac{\pi}{3}$

Solution

$$y = \sin 2x \quad ; \quad x=0 \text{ to } x=\frac{\pi}{3}$$

$$\because \quad y\geq 0 \quad \text{when} \quad x\in\left[0,\frac{\pi}{3}\right]$$

$$\therefore \text{ Area } = \int_{0}^{\pi/3}\sin 2x\,dx$$

$$= \left|-\frac{\cos 2x}{2}\right|_{0}^{\pi/3} \quad = -\frac{1}{2}\left(\cos\frac{2\pi}{3}-\cos(0)\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{2}-1\right) = -\frac{1}{2}\left(-\frac{3}{2}\right) = \frac{3}{4} \text{ sq. unit.}$$

Question # 13

Find the area between the x-asis and the curve

$y = \sqrt{2ax - x^2}$ when $a > 0$

Solution

$$y = \sqrt{2ax - x^2}$$

Putting $y = 0$, we have

$$\sqrt{2ax - x^2} = 0$$

On squaring

$$2ax - x^2 = 0$$

$$\Rightarrow x(2a - x) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad 2a - x = 0 \Rightarrow x = 2a$$

$$\because y \geq 0 \text{ when } x \in [0, 2a]$$

$$\therefore \text{Area} = \int_0^{2a} \sqrt{2ax - x^2}\, dx$$

$$= \int_0^{2a} \sqrt{a^2 - a^2 + 2ax - x^2}\, dx$$

$$= \int_0^{2a} \sqrt{a^2 - \left(a^2 - 2ax + x^2\right)}\, dx$$

$$= \int_0^{2a} \sqrt{a^2 - (a - x)^2}\, dx$$

Put $a - x = a\sin\theta$

$\Rightarrow -dx = a\cos\theta\, d\theta$

$\Rightarrow dx = -a\cos\theta\, d\theta$

When $x = 0$

$a - 0 = a\sin\theta \Rightarrow a\sin\theta = a$

$\Rightarrow \sin\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2}$

When $x = 2a$

$a - 2a = a\sin\theta \Rightarrow -a = a\sin\theta$

$\Rightarrow -1 = \sin\theta \Rightarrow \theta = -\dfrac{\pi}{2}$

$$\text{So area} = \int_{\pi/2}^{-\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\,(-a\cos\theta d\theta)$$

$$= -a\int_{\pi/2}^{-\pi/2} \sqrt{a^2\left(1 - \sin^2\theta\right)}\cos\theta\, d\theta$$

$$= -a\int_{\pi/2}^{-\pi/2} \sqrt{a^2\cos^2\theta}\cos\theta\, d\theta$$

$$= -a\int_{\pi/2}^{-\pi/2} a\cos\theta \cdot \cos\theta\, d\theta$$

$$= -a^2\int_{\pi/2}^{-\pi/2} \cos^2\theta\, d\theta$$

$$= -a^2\int_{\pi/2}^{-\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= -\frac{a^2}{2}\int_{\pi/2}^{-\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= -\frac{a^2}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_{\pi/2}^{-\pi/2}$$

$$= -\frac{a^2}{2}\left(-\frac{\pi}{2} + \sin(-\pi) - \frac{\pi}{2} - \sin\pi\right)$$

$$= -\frac{a^2}{2}(-\pi - 0 - 0)$$

$$= -\frac{a^2}{2}(-\pi) = \frac{a^2\pi}{2} \text{ sq. unit}$$

Error Analyst: Adnan Moeen (2018)

Book: Exercise 3.7 (Page 167)
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