

Definite Integral

$$\text{Let } \int f(x) dx = \varphi(x) + c$$

$$\begin{aligned} \text{Then } \int_a^b f(x) dx &= \left. \varphi(x) \right|_a^b \text{ or } \left. [\varphi(x)]_a^b \right. \\ &= \varphi(b) - \varphi(a) \end{aligned}$$

Also

- $\int_a^b f(x) dx = -\int_b^a f(x) dx$
- $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

where $a < c < b$

Evaluate the following definite integrals:

Question # 1

$$\int_1^2 (x^2 + 1) dx$$

Solution

$$\begin{aligned} &\int_1^2 (x^2 + 1) dx \\ &= \int_1^2 x^2 dx + \int_1^2 1 dx \\ &= \left. \left[\frac{x^3}{3} + x \right]_1^2 \right. = \left(\frac{2^3}{3} + 2 \right) - \left(\frac{1^3}{3} + 1 \right) \\ &= \frac{8}{3} + 2 - \frac{1}{3} - 1 = \frac{10}{3} \end{aligned}$$

Question # 2

$$\int_{-1}^1 \left(x^{\frac{1}{3}} + 1 \right) dx$$

Solution

$$\begin{aligned} &\int_{-1}^1 \left(x^{\frac{1}{3}} + 1 \right) dx \\ &= \int_{-1}^1 x^{\frac{1}{3}} dx + \int_{-1}^1 1 dx \\ &= \left. \left[\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + x \right]_{-1}^1 \right. \end{aligned}$$

$$\begin{aligned} &= \left. \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + (1 - (-1)) \right]_{-1}^1 \right. \\ &= \frac{3}{4} \left((1)^{\frac{4}{3}} - (-1)^{\frac{4}{3}} \right) + (1+1) \\ &= \frac{3}{4} (1-1) + 2 = 2 \end{aligned}$$

Question # 3

$$\int_{-2}^0 \frac{1}{(2x-1)^2} dx$$

Solution

$$\begin{aligned} &\int_{-2}^0 \frac{1}{(2x-1)^2} dx \\ &= \int_{-2}^0 (2x-1)^{-2} dx \\ &= \left. \frac{(2x-1)^{-2+1}}{(-2+1) \cdot 2} \right|_{-2}^0 \\ &= \left. \frac{(2x-1)^{-1}}{(-1) \cdot 2} \right|_{-2}^0 \\ &= \frac{(2(0)-1)^{-1}}{-2} - \frac{(2(-2)-1)^{-1}}{-2} \\ &= \frac{(0-1)^{-1}}{-2} - \frac{(-4-1)^{-1}}{-2} \\ &= \frac{(-1)^{-1}}{-2} - \frac{(-5)^{-1}}{-2} \\ &= \frac{1}{(-2)(-1)} - \frac{1}{(-2)(-5)} \\ &= \frac{1}{2} - \frac{1}{10} = \frac{2}{5} \end{aligned}$$

Question # 4

$$\int_{-6}^2 \sqrt{3-x} dx$$

Solution

$$\begin{aligned}
 & \int_{-6}^2 \sqrt{3-x} \, dx \\
 &= \int_{-6}^2 (3-x)^{\frac{1}{2}} \, dx \\
 &= \left[\frac{(3-x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(-1)} \right]_{-6}^2 = \left[\frac{(3-x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(-1)} \right]_{-6}^2 \\
 &= -\frac{2}{3} \left[(3-x)^{\frac{3}{2}} \right]_{-6}^2 \\
 &= -\frac{2}{3} \left((3-2)^{\frac{3}{2}} - (3+6)^{\frac{3}{2}} \right) \\
 &= -\frac{2}{3} \left((1)^{\frac{3}{2}} - (9)^{\frac{3}{2}} \right) = -\frac{2}{3} (1-27) = \frac{52}{3}.
 \end{aligned}$$

Question # 5

$$\int_1^{\sqrt{5}} \sqrt{(2t-1)^3} \, dt$$

Solution

$$\begin{aligned}
 & \int_1^{\sqrt{5}} \sqrt{(2t-1)^3} \, dt \\
 &= \int_1^{\sqrt{5}} (2t-1)^{\frac{3}{2}} \, dt \\
 &= \left[\frac{(2t-1)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right) \cdot 2} \right]_1^{\sqrt{5}} = \left[\frac{(2t-1)^{\frac{5}{2}}}{\left(\frac{5}{2}\right) \cdot 2} \right]_1^{\sqrt{5}} \\
 &= \left[\frac{(2t-1)^{\frac{5}{2}}}{5} \right]_1^{\sqrt{5}} = \frac{(2\sqrt{5}-1)^{\frac{5}{2}}}{5} - \frac{(2(1)-1)^{\frac{5}{2}}}{5} \\
 &= \frac{(2\sqrt{5}-1)^{\frac{5}{2}}}{5} - \frac{1}{5} \\
 &= \frac{\sqrt{(2\sqrt{5}-1)^5}}{5} - \frac{1}{5} \quad \text{Ans.}
 \end{aligned}$$

Question # 6

$$\int_2^{\sqrt{5}} x\sqrt{x^2-1} \, dx$$

Solution

$$\begin{aligned}
 & \int_2^{\sqrt{5}} x\sqrt{x^2-1} \, dx \\
 &= \int_2^{\sqrt{5}} (x^2-1)^{\frac{1}{2}} \cdot x \, dx \\
 &= \frac{1}{2} \int_2^{\sqrt{5}} (x^2-1)^{\frac{1}{2}} \cdot 2x \, dx \\
 &= \frac{1}{2} \int_2^{\sqrt{5}} (x^2-1)^{\frac{1}{2}} \cdot \frac{d}{dx}(x^2-1) \, dx \\
 &= \frac{1}{2} \left[\frac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_2^{\sqrt{5}} = \frac{1}{2} \left[\frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_2^{\sqrt{5}} \\
 &= \frac{1}{2} \cdot \frac{2}{3} \left[\left((\sqrt{5})^2 - 1 \right)^{\frac{3}{2}} - \left((2)^2 - 1 \right)^{\frac{3}{2}} \right] \\
 &= \frac{1}{3} \left[(5-1)^{\frac{3}{2}} - (4-1)^{\frac{3}{2}} \right] = \frac{1}{3} \left[(4)^{\frac{3}{2}} - (3)^{\frac{3}{2}} \right] \\
 &= \frac{1}{3} \left[(2^2)^{\frac{3}{2}} - (3)^{1+\frac{1}{2}} \right] = \frac{1}{3} \left[(2)^3 - 3(3)^{\frac{1}{2}} \right] \\
 &= \frac{1}{3} [8 - 3\sqrt{3}]
 \end{aligned}$$

Question # 7

$$\int_1^2 \frac{x}{x^2+2} \, dx$$

Solution

$$\begin{aligned}
 & \int_1^2 \frac{x}{x^2+2} \, dx \\
 &= \frac{1}{2} \int_1^2 \frac{2x}{x^2+2} \, dx \\
 &= \frac{1}{2} \int_1^2 \frac{\frac{d}{dx}(x^2+2)}{x^2+2} \, dx = \frac{1}{2} \left[\ln|x^2+2| \right]_1^2 \\
 &= \frac{1}{2} \left(\ln|2^2+2| - \ln|1^2+2| \right) \\
 &= \frac{1}{2} (\ln 6 - \ln 3) \\
 &= \frac{1}{2} \ln\left(\frac{6}{3}\right) = \frac{1}{2} \ln 2
 \end{aligned}$$

Question # 8

$$\int_2^3 \left(x - \frac{1}{x}\right)^2 dx$$

Solution

$$\begin{aligned} \int_2^3 \left(x - \frac{1}{x}\right)^2 dx &= \int_2^3 \left(x^2 + \frac{1}{x^2} - 2\right) dx \\ &= \int_2^3 x^2 dx + \int_2^3 x^{-2} dx - 2 \int_2^3 dx \end{aligned}$$

Now do yourself

Question # 9

$$\int_{-1}^1 \left(x + \frac{1}{2}\right) \sqrt{x^2 + x + 1} dx$$

Solution

$$\begin{aligned} &\int_{-1}^1 \left(x + \frac{1}{2}\right) \sqrt{x^2 + x + 1} dx \\ &= \int_{-1}^1 \left(\frac{2x+1}{2}\right) (x^2 + x + 1)^{\frac{1}{2}} dx \\ &= \frac{1}{2} \int_{-1}^1 (x^2 + x + 1)^{\frac{1}{2}} (2x+1) dx \\ &= \frac{1}{2} \int_{-1}^1 (x^2 + x + 1)^{\frac{1}{2}} \frac{d}{dx} (2x+1) dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left| \frac{(x^2 + x + 1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_{-1}^1 \\ &= \frac{1}{2} \left| \frac{(x^2 + x + 1)^{\frac{3}{2}}}{\frac{3}{2}} \right|_{-1}^1 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \left| (x^2 + x + 1)^{\frac{3}{2}} \right|_{-1}^1 \\ &= \frac{1}{3} \left[\left((1)^2 + (1) + 1\right)^{\frac{3}{2}} - \left((-1)^2 + (-1) + 1\right)^{\frac{3}{2}} \right] \\ &= \frac{1}{3} \left[(1+1+1)^{\frac{3}{2}} - (1-1+1)^{\frac{3}{2}} \right] \\ &= \frac{1}{3} \left[(3)^{\frac{3}{2}} - (1)^{\frac{3}{2}} \right] = \frac{1}{3} [3\sqrt{3} - 1] \end{aligned}$$

<p>NOTE</p> $(3)^{\frac{3}{2}} = (3)^{1+\frac{1}{2}}$ $= 3^1 \cdot 3^{\frac{1}{2}} = 3\sqrt{3}$
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$$= \sqrt{3} - \frac{1}{3}$$

Question # 10

$$\int_0^3 \frac{dx}{x^2 + 9}$$

Solution

$$\begin{aligned} \int_0^3 \frac{dx}{x^2 + 9} &= \int_0^3 \frac{dx}{x^2 + 3^2} \\ &= \left| \frac{1}{3} \tan^{-1} \frac{x}{3} \right|_0^3 \\ &= \frac{1}{3} \tan^{-1} \left(\frac{3}{3}\right) - \frac{1}{3} \tan^{-1} \left(\frac{0}{3}\right) \\ &= \frac{1}{3} \tan^{-1}(1) - \frac{1}{3} \tan^{-1}(0) \\ &= \frac{1}{3} \left(\frac{\pi}{4}\right) - \frac{1}{3}(0) = \frac{\pi}{12} \end{aligned}$$

Question # 11

$$\int_{\pi/6}^{\pi/3} \cos t dt$$

Solution Do yourself**Question # 12**

$$\int_1^2 \left(x + \frac{1}{x}\right)^{\frac{1}{2}} \left(1 - \frac{1}{x^2}\right) dx$$

Solution Do yourself**Question # 13**

$$\int_1^2 \ln x dx$$

Solution

$$\text{Let } I = \int_1^2 \ln x dx = \int_1^2 \ln x \cdot 1 dx$$

Integrating by parts

$$\begin{aligned} I &= \left| \ln x \cdot x \right|_1^2 - \int_1^2 x \cdot \frac{1}{x} dx \\ &= \left| x \ln x \right|_1^2 - \int_1^2 dx \\ &= (2 \cdot \ln 2 - 1 \cdot \ln 1) - \left| x \right|_1^2 \\ &= (2 \cdot \ln 2 - 1 \cdot (0)) - (2 - 1) \end{aligned}$$

$$= (2 \cdot \ln 2 - 0) - 1 = 2 \ln 2 - 1$$

Question # 14

$$\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}} \right) dx$$

Solution

$$\begin{aligned} & \int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}} \right) dx \\ &= \int_0^2 e^{\frac{x}{2}} dx - \int_0^2 e^{-\frac{x}{2}} dx \\ &= \left[\frac{e^{\frac{x}{2}}}{\frac{1}{2}} \right]_0^2 - \left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}} \right]_0^2 = 2 \left[e^{\frac{x}{2}} \right]_0^2 + 2 \left[e^{-\frac{x}{2}} \right]_0^2 \\ &= 2 \left(e^{\frac{2}{2}} - e^{\frac{0}{2}} \right) + 2 \left(e^{-\frac{2}{2}} - e^{-\frac{0}{2}} \right) \\ &= 2(e^1 - e^0) + 2(e^{-1} - e^0) \\ &= 2 \left(e - 1 + \frac{1}{e} - 1 \right) = 2 \left(e + \frac{1}{e} - 2 \right) \\ &= 2 \left(\frac{e^2 + 1 - 2e}{e} \right) = 2 \frac{(e-1)^2}{e} \end{aligned}$$

Question # 15

$$\int_0^{\pi/4} \frac{\cos \theta + \sin \theta}{\cos 2\theta + 1} d\theta$$

Solution

$$\begin{aligned} \text{Let } I &= \int_0^{\pi/4} \frac{\cos \theta + \sin \theta}{\cos 2\theta + 1} d\theta \\ &= \int_0^{\pi/4} \frac{\cos \theta + \sin \theta}{2 \cos^2 \theta} d\theta \\ \because \cos^2 \theta &= \frac{1 + \cos 2\theta}{2} \\ &= \int_0^{\pi/4} \left(\frac{\cos \theta}{2 \cos^2 \theta} + \frac{\sin \theta}{2 \cos^2 \theta} \right) d\theta \\ &= \int_0^{\pi/4} \frac{1}{2 \cos \theta} d\theta + \int_0^{\pi/4} \frac{\sin \theta}{2 \cos \theta \cdot \cos \theta} d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} \sec \theta d\theta + \frac{1}{2} \int_0^{\pi/4} \sec \theta \tan \theta d\theta \end{aligned}$$

$$= \frac{1}{2} \left[\ln \left| \sec \theta + \tan \theta \right| \right]_0^{\pi/4} + \frac{1}{2} \left[\sec \theta \right]_0^{\pi/4}$$

$$\begin{aligned} &= \frac{1}{2} \left(\ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln \left| \sec(0) + \tan(0) \right| \right) \\ &\quad + \frac{1}{2} \left(\sec \frac{\pi}{4} - \sec(0) \right) \\ &= \frac{1}{2} \left(\ln \left| \sqrt{2} + 1 \right| - \ln \left| 1 + 0 \right| \right) + \frac{1}{2} (\sqrt{2} - 1) \\ &= \frac{1}{2} \left(\ln \left| \sqrt{2} + 1 \right| - 0 \right) + \frac{1}{2} (\sqrt{2} - 1) \\ &= \frac{1}{2} \left(\ln \left| \sqrt{2} + 1 \right| + \sqrt{2} - 1 \right) \quad \text{Ans.} \end{aligned}$$

Question # 16

$$\int_0^{\pi/6} \cos^3 \theta d\theta$$

Solution

$$\begin{aligned} \int_0^{\pi/6} \cos^3 \theta d\theta &= \int_0^{\pi/6} \cos^2 \theta \cdot \cos \theta d\theta \\ &= \int_0^{\pi/6} (1 - \sin^2 \theta) \cos \theta d\theta \\ &= \int_0^{\pi/6} \cos \theta d\theta - \int_0^{\pi/6} \sin^2 \theta \cos \theta d\theta \\ &= \left[\sin \theta \right]_0^{\pi/6} - \int_0^{\pi/6} \sin^2 \theta \frac{d}{d\theta} \sin \theta d\theta \\ &= \left(\sin \frac{\pi}{6} - \sin(0) \right) - \left[\frac{\sin^3 \theta}{3} \right]_0^{\pi/6} \\ &= \left(\frac{1}{2} - 0 \right) - \frac{1}{3} \left(\sin^3 \frac{\pi}{6} - \sin^3(0) \right) \\ &= \frac{1}{2} - \frac{1}{3} \left(\left(\frac{1}{2} \right)^2 - (0)^2 \right) \\ &= \frac{1}{2} - \frac{1}{3} \left(\frac{1}{4} \right) = \frac{1}{2} - \frac{1}{12} = \frac{5}{12} \end{aligned}$$

Question # 17

$$\int_0^{\pi/4} \cos^2 \theta \cdot \cot^2 \theta d\theta$$

Solution

$$\begin{aligned}
& \int_{\pi/6}^{\pi/4} \cos^2 \theta \cdot \cot^2 \theta \, d\theta \\
&= \int_{\pi/6}^{\pi/4} \cos^2 \theta (\operatorname{cosec}^2 \theta - 1) \, d\theta \\
&= \int_{\pi/6}^{\pi/4} (\cos^2 \theta \operatorname{cosec}^2 \theta - \cos^2 \theta) \, d\theta \\
&= \int_{\pi/6}^{\pi/4} \left(\cos^2 \theta \frac{1}{\sin^2 \theta} - \cos^2 \theta \right) \, d\theta \\
&= \int_{\pi/6}^{\pi/4} \cot^2 \theta \, d\theta - \int_{\pi/6}^{\pi/4} \cos^2 \theta \, d\theta \\
&= \int_{\pi/6}^{\pi/4} (\operatorname{cosec}^2 \theta - 1) \, d\theta - \int_{\pi/6}^{\pi/4} \left(\frac{1 + \cos 2\theta}{2} \right) \, d\theta \\
&= \int_{\pi/6}^{\pi/4} \operatorname{csc}^2 \theta \, d\theta - \int_{\pi/6}^{\pi/4} d\theta - \frac{1}{2} \int_{\pi/6}^{\pi/4} d\theta - \frac{1}{2} \int_{\pi/6}^{\pi/4} \cos 2\theta \, d\theta \\
&= \left| -\cot \theta \right|_{\pi/6}^{\pi/4} - \frac{3}{2} \int_{\pi/6}^{\pi/4} d\theta - \frac{1}{2} \left| \frac{\sin 2\theta}{2} \right|_{\pi/6}^{\pi/4} \\
&= \left(-\cot \frac{\pi}{4} + \cot \frac{\pi}{6} \right) - \frac{3}{2} \left| \theta \right|_{\pi/6}^{\pi/4} - \frac{1}{2} \left(\frac{\sin 2(\pi/4)}{2} - \frac{\sin 2(\pi/6)}{2} \right) \\
&= (-1 + \sqrt{3}) - \frac{3}{2} \left(\frac{\pi}{4} - \frac{\pi}{6} \right) - \frac{1}{2} \left(\frac{1}{2} - \frac{\sqrt{3}}{2} \right) \\
&= (-1 + \sqrt{3}) - \frac{3}{2} \left(\frac{\pi}{12} \right) - \frac{1}{2} \left(\frac{1}{2} - \frac{\sqrt{3}}{4} \right) \\
&= -1 + \sqrt{3} - \frac{\pi}{8} - \frac{1}{4} + \frac{\sqrt{3}}{8} = -\frac{5}{4} + \frac{9}{8}\sqrt{3} - \frac{\pi}{8} \\
&= \frac{9\sqrt{3} - 10 - \pi}{8}
\end{aligned}$$

Question # 18

$$\int_0^{\pi/4} \cos^4 t \, dt$$

Solution

$$\begin{aligned}
\int_0^{\pi/4} \cos^4 t \, dt &= \int_0^{\pi/4} (\cos^2 t)^2 \, dt \\
&= \int_0^{\pi/4} \left(\frac{1 + \cos 2t}{2} \right)^2 \, dt \\
&= \int_0^{\pi/4} \left(\frac{1 + 2\cos 2t + \cos^2 2t}{4} \right) \, dt \\
&= \frac{1}{4} \int_0^{\pi/4} (1 + 2\cos 2t + \cos^2 2t) \, dt \\
&= \frac{1}{4} \int_0^{\pi/4} \left(1 + 2\cos 2t + \frac{1 + \cos 4t}{2} \right) \, dt \\
&= \frac{1}{4} \int_0^{\pi/4} \left(\frac{2 + 4\cos 2t + 1 + \cos 4t}{2} \right) \, dt \\
&= \frac{1}{8} \int_0^{\pi/4} (3 + 4\cos 2t + \cos 4t) \, dt \\
&= \frac{1}{8} \left| 3t + 4 \frac{\sin 2t}{2} + \frac{\sin 4t}{4} \right|_0^{\pi/4} \\
&= \frac{1}{8} \left(3 \left(\frac{\pi}{4} \right) + 2 \sin 2 \left(\frac{\pi}{4} \right) + \frac{\sin 4 \left(\frac{\pi}{4} \right)}{4} \right. \\
&\quad \left. - 3(0) - 2 \sin 2(0) - \frac{\sin 4(0)}{4} \right) \\
&= \frac{1}{8} \left(\frac{3\pi}{4} + 2 + \frac{0}{4} - 0 - 0 - \frac{0}{4} \right) = \frac{1}{8} \left(\frac{3\pi}{4} + 2 \right) \\
&= \frac{1}{8} \left(\frac{3\pi + 8}{4} \right) = \frac{3\pi + 8}{32}
\end{aligned}$$

Question # 19

$$\int_0^{\pi/3} \cos^2 \theta \sin \theta \, d\theta$$

Solution

$$\text{Let } I = \int_0^{\pi/3} \cos^2 \theta \sin \theta \, d\theta$$

$$\text{Put } t = \cos \theta \Rightarrow dt = -\sin \theta \, d\theta$$

$$\Rightarrow -dt = \sin \theta \, d\theta$$

$$\text{When } \theta = 0 \text{ then } t = 1$$

And when $\theta = \frac{\pi}{3}$ then $t = \frac{1}{2}$

$$\begin{aligned} \text{So } I &= \int_1^{\frac{1}{2}} t^2 (-dt) \\ &= -\int_1^{\frac{1}{2}} t^2 dt = -\left| \frac{t^3}{3} \right|_1^{\frac{1}{2}} \\ &= -\left(\frac{\left(\frac{1}{2}\right)^3}{3} - \frac{(1)^3}{3} \right) = -\left(\frac{\frac{1}{8}}{3} - \frac{1}{3} \right) \\ &= -\left(\frac{1}{24} - \frac{1}{3} \right) = -\left(-\frac{7}{24} \right) = \frac{7}{24} \end{aligned}$$

Question # 20

$$\int_0^{\pi/4} (1 + \cos^2 \theta) \tan^2 \theta d\theta$$

Solution

$$\begin{aligned} &\int_0^{\pi/4} (1 + \cos^2 \theta) \tan^2 \theta d\theta \\ &= \int_0^{\pi/4} (1 + \cos^2 \theta) \frac{\sin^2 \theta}{\cos^2 \theta} d\theta \\ &= \int_0^{\pi/4} \left(\frac{\sin^2 \theta}{\cos^2 \theta} + \sin^2 \theta \right) d\theta \\ &= \int_0^{\pi/4} (\tan^2 \theta + \sin^2 \theta) d\theta \\ &= \int_0^{\pi/4} \left(\sec^2 \theta - 1 + \frac{1 - \cos 2\theta}{2} \right) d\theta \\ &= \int_0^{\pi/4} \left(\frac{2\sec^2 \theta - 2 + 1 - \cos 2\theta}{2} \right) d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} (2\sec^2 \theta - 1 - \cos 2\theta) d\theta \\ &= \frac{1}{2} \left| 2 \tan \theta - \theta - \frac{\sin 2\theta}{2} \right|_0^{\pi/4} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left(2 \tan \frac{\pi}{4} - \frac{\pi}{4} - \frac{\sin 2\left(\frac{\pi}{4}\right)}{2} \right. \\ &\quad \left. - 2 \tan(0) + 0 + \frac{\sin 2(0)}{2} \right) \\ &= \frac{1}{2} \left(2(1) - \frac{\pi}{4} - \frac{1}{2} - 2(0) + 0 + 0 \right) \\ &= \frac{1}{2} \left(\frac{3}{2} - \frac{\pi}{4} \right) = \frac{1}{2} \left(\frac{6 - \pi}{4} \right) = \frac{6 - \pi}{8} \end{aligned}$$

Question # 21

$$\int_0^{\pi/4} \frac{\sec \theta}{\sin \theta + \cos \theta} d\theta$$

Solution

$$\begin{aligned} \text{Let } I &= \int_0^{\pi/4} \frac{\sec \theta}{\sin \theta + \cos \theta} d\theta \\ &= \int_0^{\pi/4} \frac{\sec \theta}{\cos \theta \left(\frac{\sin \theta}{\cos \theta} + 1 \right)} d\theta \\ &= \int_0^{\pi/4} \frac{\sec^2 \theta}{(\tan \theta + 1)} d\theta \end{aligned}$$

Put $t = \tan \theta + 1 \Rightarrow dt = \sec^2 \theta d\theta$

When $x = 0$ then $t = 1$

Also when $x = \frac{\pi}{4}$ then $t = 2$

$$\begin{aligned} \text{So } I &= \int_1^2 \frac{dt}{t} \\ &= \left| \ln t \right|_1^2 \\ &= \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2 \end{aligned}$$

Review

$$\text{If } f(x) = \begin{cases} g(x) & : a \leq x \leq b \\ h(x) & : b \leq x \leq c \end{cases}$$

Then

$$\int_a^c f(x) dx = \int_a^b g(x) dx + \int_b^c h(x) dx$$

Question # 22

$$\int_{-1}^5 |x - 3| dx$$

Solution

$$\text{Let } I = \int_{-1}^5 |x-3| dx$$

Since

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \Rightarrow x \geq 3 \\ -(x-3) & \text{if } x-3 < 0 \Rightarrow x < 3 \end{cases}$$

$$\begin{aligned} \text{So } \int_{-1}^5 |x-3| dx &= \int_{-1}^3 [-(x-3)] dx + \int_3^5 (x-3) dx \\ &= -\int_{-1}^3 (x-3) dx + \int_3^5 (x-3) dx \\ &= -\left[\frac{(x-3)^2}{2} \right]_{-1}^3 + \left[\frac{(x-3)^2}{2} \right]_3^5 \\ &= -\left(\frac{(3-3)^2}{2} - \frac{(-1-3)^2}{2} \right) + \left(\frac{(5-3)^2}{2} - \frac{(3-3)^2}{2} \right) \\ &= -\left(\frac{0}{2} - \frac{16}{2} \right) + \left(\frac{4}{2} - \frac{0}{2} \right) = 8 + 2 = 10 \end{aligned}$$

Question # 23

$$\int_{\frac{1}{8}}^1 \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{2}{3}}} dx$$

Solution

$$\begin{aligned} \text{Let } I &= \int_{\frac{1}{8}}^1 \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{2}{3}}} dx \\ &= \int_{\frac{1}{8}}^1 (x^{\frac{1}{3}} + 2)^2 x^{-\frac{2}{3}} dx \end{aligned}$$

$$\text{Put } t = x^{\frac{1}{3}} + 2$$

$$\Rightarrow dt = \frac{1}{3} x^{-\frac{2}{3}} dx \Rightarrow 3 dt = x^{-\frac{2}{3}} dx$$

$$\text{When } x = \frac{1}{8} \text{ then } t = \frac{5}{2}$$

$$\text{And when } x = 1 \text{ then } t = 3$$

$$\begin{aligned} \text{So } I &= \int_{\frac{5}{2}}^3 (t)^2 3 dt = 3 \left[\frac{t^3}{3} \right]_{\frac{5}{2}}^3 \\ &= 3 \left(\frac{3^3}{3} - \frac{\left(\frac{5}{2}\right)^3}{3} \right) = 3 \left(\frac{27}{3} - \frac{125}{8} \right) \end{aligned}$$

$$= 3 \left(\frac{27}{3} - \frac{125}{24} \right) = 3 \left(\frac{91}{24} \right) = \frac{91}{8}$$

Question # 24

$$\int_1^3 \frac{x^2 - 2}{x+1} dx$$

Solution

$$\int_1^3 \frac{x^2 - 2}{x+1} dx$$

$$= \int_1^3 \left(x - 1 - \frac{1}{x+1} \right) dx$$

$$= \int_1^3 x dx - \int_1^3 dx - \int_1^3 \frac{dx}{x+1}$$

$$= \left[\frac{x^2}{2} \right]_1^3 - [x]_1^3 - [\ln|x+1|]_1^3$$

$$= \left(\frac{3^2}{2} - \frac{1^2}{2} \right) - (3-1) - (\ln|3+1| - \ln|1+1|)$$

$$= \left(\frac{9}{2} - \frac{1}{2} \right) - (2) - (\ln 4 - \ln 2)$$

$$= 4 - 2 - \ln \frac{4}{2} = 2 - \ln 2$$

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2 - 2} \\ \underline{x^2 + x} \\ -x-2 \\ \underline{-x-1} \\ -1 \end{array}$$

Question # 25

$$\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2 + 1)} dx$$

Solution

$$\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2 + 1)} dx$$

$$= \int_2^3 \frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1} dx$$

$$= \int_2^3 \frac{d}{dx} (x^3 - x^2 + x - 1) dx$$

$$= \left[\ln |x^3 - x^2 + x - 1| \right]_2^3$$

$$= \ln |3^3 - 3^2 + 3 - 1| - \ln |2^3 - 2^2 + 2 - 1|$$

$$= \ln |27 - 9 + 3 - 1| - \ln |8 - 4 + 2 - 1|$$

$$= \ln 20 - \ln 5 = \ln \frac{20}{5} = \ln 4$$

Question # 26

$$\int_0^{\pi/4} \frac{\sin x - 1}{\cos^2 x} dx$$

Solution

$$\begin{aligned} \int_0^{\pi/4} \frac{\sin x - 1}{\cos^2 x} dx &= \int_0^{\pi/4} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) dx \\ &= \int_0^{\pi/4} \left(\frac{\sin x}{\cos x \cdot \cos x} - \frac{1}{\cos^2 x} \right) dx \\ &= \int_0^{\pi/4} (\sec x \tan x - \sec^2 x) dx \\ &= |\sec x - \tan x|_0^{\pi/4} \\ &= \left(\sec \frac{\pi}{4} - \tan \frac{\pi}{4} \right) - (\sec(0) - \tan(0)) \\ &= \sqrt{2} - 1 - 1 + 0 = \sqrt{2} - 2 \end{aligned}$$

Question # 27

$$\int_0^{\pi/4} \frac{1}{1 + \sin x} dx$$

Solution

$$\begin{aligned} \text{Let } I &= \int_0^{\pi/4} \frac{1}{1 + \sin x} dx \\ &= \int_0^{\pi/4} \frac{1}{1 + \sin x} \cdot \frac{1 - \sin x}{1 - \sin x} dx \\ &= \int_0^{\pi/4} \frac{1 - \sin x}{1 - \sin^2 x} dx = \int_0^{\pi/4} \frac{1 - \sin x}{\cos^2 x} dx \\ &\text{Now same as Question # 24} \end{aligned}$$

Question # 28

$$\int_0^1 \frac{3x}{\sqrt{4-3x}} dx$$

Solution

$$\begin{aligned} \text{Let } I &= \int_0^1 \frac{3x}{\sqrt{4-3x}} dx \\ \text{Put } t &= 4 - 3x \Rightarrow 3x = 4 - t \\ \text{Also } dt &= -3dx \Rightarrow -\frac{1}{3} dt = dx \\ \text{When } x &= 0 \text{ then } t = 4 \\ \text{And when } x &= 1 \text{ then } t = 1 \end{aligned}$$

$$\begin{aligned} \text{So } I &= \int_4^1 \frac{4-t}{\sqrt{t}} \left(-\frac{1}{3} dt \right) \\ &= -\frac{1}{3} \int_4^1 \left(\frac{4}{t^{1/2}} - \frac{t}{t^{1/2}} \right) dt \\ &= +\frac{1}{3} \int_1^4 \left(4t^{-1/2} - t^{1/2} \right) dt \end{aligned}$$

Now do yourself

Question # 29

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(2 + \sin x)} dx$$

Solution

$$\text{Let } I = \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(2 + \sin x)} dx$$

$$\text{Put } t = \sin x \Rightarrow dt = \cos x dx$$

$$\text{When } x = \frac{\pi}{6} \text{ then } t = \frac{1}{2}$$

$$\text{When } x = \frac{\pi}{2} \text{ then } t = 1$$

$$\text{So } I = \int_{1/2}^1 \frac{dt}{t(2+t)}$$

Now consider

$$\frac{1}{t(2+t)} = \frac{A}{t} + \frac{B}{2+t}$$

$$\Rightarrow 1 = A(2+t) + Bt \dots\dots (i)$$

Put $t = 0$ in (i)

$$1 = A(2+0) + B(0) \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

Put $2+t = 0 \Rightarrow t = -2$ in eq. (i)

$$1 = 0 + B(-2) \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$$

$$\text{So } \frac{1}{t(2+t)} = \frac{1/2}{t} + \frac{-1/2}{2+t}$$

$$\begin{aligned} \Rightarrow \int_{1/2}^1 \frac{1}{t(2+t)} dt &= \int_{1/2}^1 \frac{1/2}{t} dt + \int_{1/2}^1 \frac{-1/2}{2+t} dt \\ &= \frac{1}{2} \int_{1/2}^1 \frac{1}{t} dt - \frac{1}{2} \int_{1/2}^1 \frac{1}{2+t} dt \\ &= \frac{1}{2} \left| \ln|t| \right|_{1/2}^1 - \frac{1}{2} \left| \ln|2+t| \right|_{1/2}^1 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[\ln|1| - \ln\left|\frac{1}{2}\right| \right] \\
&\quad - \frac{1}{2} \left[\ln|2+1| - \ln\left|2+\frac{1}{2}\right| \right] \\
&= \frac{1}{2} \left[0 - \ln\frac{1}{2} \right] - \frac{1}{2} \left[\ln 3 - \ln\frac{5}{2} \right] \\
&= \frac{1}{2} \left[-\ln\frac{1}{2} - \ln 3 + \ln\frac{5}{2} \right] \\
&= \frac{1}{2} \ln \left(\frac{5/2}{1/2 \times 3} \right) = \frac{1}{2} \ln \left(\frac{5}{3} \right)
\end{aligned}$$

Question # 30

$$I = \int_0^{\pi/2} \frac{\sin x dx}{(1 + \cos x)(2 + \cos x)}$$

Solution

$$\text{Let } I = \int_0^{\pi/2} \frac{\sin x dx}{(1 + \cos x)(2 + \cos x)}$$

$$\text{Put } t = \cos x \Rightarrow dt = -\sin x dx \\ \Rightarrow -dt = \sin x dx$$

$$\text{When } x=0 \text{ then } t=1$$

$$\text{And when } x = \frac{\pi}{2} \text{ then } t=0$$

$$\begin{aligned}
\text{So } I &= \int_1^0 \frac{-dt}{(1+t)(2+t)} \\
&= -\int_1^0 \frac{dt}{(1+t)(2+t)} = \int_0^1 \frac{dt}{(1+t)(2+t)}
\end{aligned}$$

Now consider

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$$

$$\Rightarrow 1 = A(2+t) + B(1+t) \dots (i)$$

$$\text{Put } 1+t=0 \Rightarrow t=-1 \text{ in (i)}$$

$$1 = A(2-1) + 0 \Rightarrow A=1$$

$$\text{Put } 2+t=0 \Rightarrow t=-2 \text{ in (i)}$$

$$1 = 0 + B(1-2) \Rightarrow 1 = -B \quad \text{i.e. } B = -1$$

So

$$\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} + \frac{-1}{2+t}$$

$$\int_0^1 \frac{1}{(1+t)(2+t)} dt = \int_0^1 \frac{1}{1+t} dt - \int_0^1 \frac{1}{2+t} dt$$

$$= \left| \ln|1+t| \right|_0^1 - \left| \ln|2+t| \right|_0^1$$

$$= (\ln|1+1| - \ln|1+0|)$$

$$- (\ln|2+1| - \ln|2+0|)$$

$$= \ln 2 - 0 - \ln 3 + \ln 2$$

$$= \ln \left(\frac{2 \times 2}{3} \right) = \ln \left(\frac{4}{3} \right)$$

Error Analyst

Usman Khilji - FAZMIC Sargodha 2004-06
Muzammil Ahsan - 2011
Sarah Sarwar - 2015

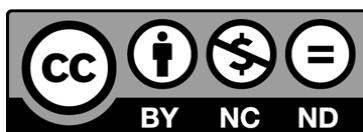
Book: Exercise 3.6 page163

Calculus and Analytic Geometry Mathematic 12
Punjab Textbook Board, Lahore.

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