19

The *F*-Distribution and Statistical Inference

19.1 INTRODUCTION

In the preceding chapter, we used the t-distribution to test the hypothesis about the difference between two means under the *assumption* that the two random samples are drawn independently from two normal populations that have *equal* variances. But in actual practice, the variances may or may not be equal. To check the assumption that the two normally distributed populations have equal variances, we use an important distribution, called the F-distribution, which is the sampling distribution of the ratio of two independent and unbiased estimates of the population variances. If the unbiased estimates denoted by s_1^2 and s_2^2, have been obtained from two random samples of sizes n_1 and n_2, drawn from normal populations having the same variances, then (assuming that s_1^2 is larger than s_2^2) the ratio is given by

$$F = \frac{s_1^2}{s_2^2}.$$

This ratio was named F by G.W. Snedecor (1882–1974) in honour of the great British statistician, Sir R.A. Fisher (1890–1962), who in 1924 developed its distribution as the Z-distribution which was later transformed into the F-distribution, using the relation $F = e^{2z}$.

Dividing the two estimates by the population variance σ^2, the ratio becomes

Similarly, the quantity $\frac{(n_2-1)s_2^2}{\sigma^2}$ is distributed as a χ^2-variable with $\upsilon_2 = n_2 - 1$ degrees of freedom. Hence, F is a ratio of two independent chi-square random variables, each divided by its respective degrees of freedom. The distribution of this statistic (ratio) is known as the Snedecor's F-distribution or sometimes, the variance ratio distribution. The F-distribution has $\upsilon_1 = n_1 - 1$ degrees of freedom for the numerator and $\upsilon_2 = n_2 - 1$ degrees of freedom for the denominator. It is interesting to note that the F-distribution has two parameters, namely υ_1 and υ_2, the degrees of freedom in that order. The F-distribution is extremely important as it has broad applications in modern statistical analysis.

19.2 THE F-DISTRIBUTION

Let s_1^2 and s_2^2 be the unbiased estimated variances of two random samples of sizes n_1 and n_2, drawn from normal populations with same variances. Then the ratio $F = \frac{s_1^2}{s_2^2}$ may be written as

$$F = \frac{s_1^2/\sigma^2}{s_2^2/\sigma^2} = \frac{U/\upsilon_1}{U/\upsilon_2},$$

where $U = \frac{(n_1-1)s_1^2}{\sigma^2}$ is a χ^2-variable with $\upsilon_1 = n_1 - 1$ $d.f.$ and

$V = \frac{(n_2-1)s_2^2}{\sigma^2}$ is a χ^2-variable with $\upsilon_2 = n_2 - 1$ $d.f.$

To find the distribution of F, we proceed as follows:

Since U and V are independent χ^2-variables with υ_1 and υ_2 degrees of freedom respectively, therefore their joint distribution is

$$f(u, v) = \frac{u^{(v_1/2)-1}\, e^{-u/2}}{2^{v_1/2}\,\Gamma(v_1/2)} \cdot \frac{v^{(v_2/2)-1}\, e^{-v/2}}{2^{v_2/2}\,\Gamma(v_2/2)}, \quad 0 < u,\ v < \infty$$

$$= \frac{u^{(v_1/2)-1}\, v^{(v_2/2)-1}}{2^{(v_1+v_2)/2}\,\Gamma(v_1/2)\,\Gamma(v_2/2)}\, e^{-(u+v)/2}$$

To obtain the distribution of F, we make the change of variables as $F = \frac{u/v_1}{v/v_2}$ and $v = v$ so that $u = \frac{v v_1}{v_2} F$, $v = v$ and the Jacobian of transformation is

$$J = \begin{vmatrix} \frac{\partial u}{\partial F} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial F} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v v_1}{v_2} & \frac{v_1 F}{v_2} \\ 0 & 1 \end{vmatrix} = \frac{v v_1}{v_2}.$$

Substituting these values, we get

$$f(F, v) = \frac{\left(\frac{v v_1}{v_2} F\right)^{(v_1/2)-1}\, v^{(v_2/2)-1}}{2^{(v_1+v_2)/2}\,\Gamma(v_1/2)\,\Gamma(v_2/2)}\, e^{-[(v v_1/v_2)F + v]/2}\, \frac{v v_1}{v_2}$$

Now $f(F) = \int_0^\infty f(F, v)\, dv$

$$= \frac{(v_1/v_2)^{v_1/2}\, F^{(v_1/2)-1}}{2^{(v_1+v_2)/2}\,\Gamma(v_1/2)\,\Gamma(v_2/2)} \int_0^\infty v^{(v_1+v_2)/2-1}\, e^{-[(v_1/v_2)F+1]v/2}\, dv$$

Let $y = \frac{v}{2}\left(\frac{v_1}{v_2}F + 1\right)$, so that $v = 2y\left(\frac{v_1}{v_2}F + 1\right)^{-1}$ and

$$dv = 2\left(\frac{v_1}{v_2}F + 1\right)^{-1} dy.$$

Then after simplification, we get

$$f(F) = \frac{\Gamma[(v_1+v_2)/2]\,(v_1/v_2)^{v_1/2}\, F^{(v_1/2)-1}}{\Gamma(v_1/2)\,\Gamma(v_2/2)\,[1 + v_1 F/v_2]^{(v_1+v_2)/2}}, \quad 0 < F < \infty$$

is the required F-distribution with v_1 degrees of freedom in the numerator and v_2 degrees of freedom in the denominator. It is usually abbreviated …

Fisher's z-distribution can be obtained by writing $F = e^{2z}$ and $dF = 2F\,dz$ in the above distribution. Fisher's z-distribution should not be confused with Fisher's z-transformation of r, the correlation co-efficient. In practice, we generally use F-statistic as it is more easy to compute and more easy to apply.

19.2.1. Properties of the *F*-Distribution. The F-distribution has the following important properties:

(i) The F-distribution always ranges from zero to infinity.

(ii) The mean and the variance of the F-distribution with v_1 and v_2 degrees of freedom are

$$\mu = \frac{v_2}{v_2 - 2} \text{ for } v_2 > 2, \text{ and}$$

$$\sigma^2 = \frac{2v_2^2\,(v_1 + v_2 - 2)}{v_1(v_2 - 2)^2\,(v_2 - 4)} \text{ for } v_2 > 4.$$

Now the mean of the random variable F, defined as $F = \dfrac{U/v_1}{V/v_2}$, where U and V are independent chi-square variables with v_1 and v_2 degrees of freedom respectively, is given by

$$\mu = E(F) = E\left[\frac{U/v_1}{V/v_2}\right] = \frac{v_2}{v_1} E(U)\, E\left(\frac{1}{V}\right)$$

$$= v_2 \,.\, E\left(\frac{1}{V}\right) \qquad \text{as } E(U) = v_1$$

Now $\quad E\left(\dfrac{1}{V}\right) = \dfrac{1}{2^{v_2/2}\,\Gamma(v_2/2)} \displaystyle\int_0^{\infty} \frac{1}{v} \,.\, v^{(v_2/2)-1}\, e^{-v/2}\, dv$

Let $\quad y = \dfrac{v}{2}$, so that $v = 2y$ and $dv = 2dy$. Then

$$E\left(\frac{1}{V}\right) = \frac{2^{(v_2/2)-1}}{2^{v_2/2}\,\Gamma(v_2/2)} \int_0^{\infty} y^{(v_2-2)/2-1}\, e^{-y}\, dy$$

$$= \frac{2^{(v_2/2)-1}}{2^{v_2/2}\,\Gamma(v_2/2)}\, \Gamma\left(\frac{v_2-2}{2}\right)$$

$$= \frac{1}{v_2 - 2} \qquad \left[\because \Gamma\left(\frac{v_2}{2}\right) = \frac{v_2-2}{2}\,\Gamma\left(\frac{v_2-2}{2}\right)\right]$$

Thus $\mu = \upsilon_2 E\left(\frac{1}{V}\right) = \frac{\upsilon_2}{\upsilon_2 - 2}$

Hence, we see that there is no mean for $\upsilon_2 \leq 2$. Also the mean is independent of υ_1 and is always greater than 1. The result for variance is similarly established.

(iii) The F-distribution for $\upsilon_1 > 2, \upsilon_2 > 2$ is *unimodal* and the mode of the distribution with υ_1 (≥2) is at $F = \frac{\upsilon_2 (\upsilon_1 - 2)}{\upsilon_1 (\upsilon_2 + 2)}$, which is always less than 1.

(iv) The F-distribution is skewed to the right. But as the degrees of freedom υ_1 and υ_2 become large, the F-distribution approaches the normal distribution.

(v) If F has an F-distribution with υ_1 and υ_2 degrees of freedom, then $\frac{1}{F}$ has an F distribution with υ_2 and υ_1 degrees of freedom. This implies that the critical value of F that cuts off a specified area of α in the lower tail of the distribution, turns out to be the reciprocal of the F value that cuts off the same area in the upper tail of the distribution with the degrees of freedom υ_1 and υ_2 interchanged, that is the lower and upper tail points are related by

$$F_{1-\alpha}(\upsilon_1, \upsilon_2) = \frac{1}{F_\alpha (\upsilon_2, \upsilon_1)}$$

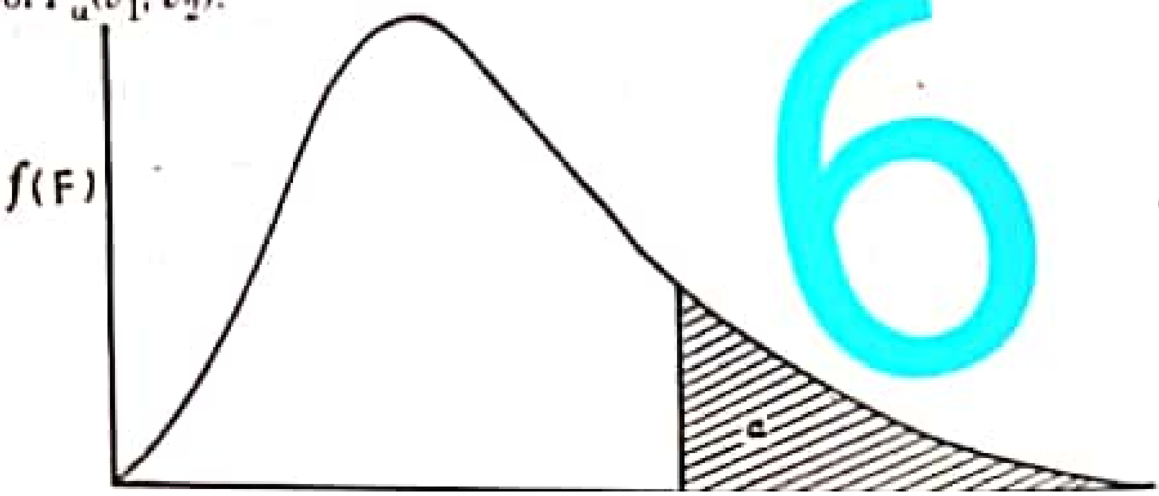
This property is useful for testing a value of $F<1$, where we take the reciprocal of F and interchange the degrees of freedom υ_1, υ_2. When $\upsilon_1 = \upsilon_2$, the distribution $\frac{1}{F}$ is the same as that of F. It is important to note that the degrees of freedom associated with the sample variance in the numerator is always stated first.

(vi) The F-distribution constitutes a wide class of distributions, which is an F-variable with 1 and n degrees of freedom. Hence the square of a t-variable with n degrees of freedom has an F-distribution with $\upsilon_1 = 1$ d.f in the numerator and $\upsilon_2 = n$ df in the denominator, *i.e.* $F(1,n) = t^2_{(n)}$. For this relation, the numerator degree of freedom must be 1. Thus $F(1, 6) = t^2_{(6)}$ or $F(1, 12) = t^2_{(12)}$, etc.

When υ_2 tends to infinity, the variance ratio reduces to $\frac{\chi^2}{\upsilon_1}$, so that $\upsilon_1 F$ is distributed as a χ^2-variable with υ_1 degrees of freedom.

Moreover, when $\upsilon_1 = 1$ and $\upsilon_2 = \infty$, the distribution of $\sqrt{F}$ is normal.

19.2.2. The F-Table of Areas. In view of its importance, the F-distribution has been tabulated. A value of F for a right tail area of α of the distribution with υ_1 and υ_2 degrees of freedom is denoted by the symbol $F_\alpha(\upsilon_1, \upsilon_2)$.



1	161.4	[illegible]	[illegible]	[illegible]	[illegible]	234.0	238.9	243.9	249.0	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.37	19.41	19.45	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	2.95	2.79	2.61	2.40
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
13	4.67	3.80	3.41	3.18	3.03	2.92	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.40	2.23	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.38	2.20	2.00	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.36	2.18	1.98	1.73
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
26	4.22	3.37	2.98	2.74	2.59	2.47	2.32	2.15	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.30	2.13	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.44	2.29	2.12	1.91	1.65
29	4.18	3.33	2.93	2.70	2.54	2.43	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.02	1.83	1.61	1.25
∞	3.84	2.99	2.60	2.37	2.21	2.10	1.94	1.73	1.52	1.00

Lower

1	647.8	799.5	864.2	899.6	921.8	937.1	956.7	976.7	997.2	1018
2	38.51	39.00	39.17	39.25	39.30	39.33	39.37	39.41	39.46	39.50
3	17.44	16.04	15.44	15.10	14.88	14.73	14.54	14.34	14.12	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	8.98	8.75	8.51	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.76	6.52	6.28	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.60	5.37	5.12	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.90	4.67	4.42	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.43	4.20	3.95	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.10	3.87	3.61	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.85	3.62	3.37	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.66	3.43	3.17	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.51	3.28	3.02	2.72
13	6.41	4.97	4.35	4.00	3.77	3.60	3.39	3.15	2.89	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.29	3.05	2.79	2.49
15	6.20	4.77	4.15	3.80	3.58	3.41	3.20	2.96	2.70	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.12	2.89	2.63	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.06	2.82	2.56	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.01	2.77	2.50	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	2.96	2.72	2.45	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	2.91	2.68	2.41	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.87	2.64	2.37	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.84	2.60	2.33	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.81	2.57	2.30	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.78	2.54	2.27	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.75	2.51	2.24	1.91
26	5.66	4.27	3.67	3.33	3.10	2.94	2.73	2.49	2.22	1.88
27	5.63	4.24	3.65	3.31	3.08	2.92	2.71	2.47	2.19	1.85
28	5.61	4.22	3.63	3.29	3.06	2.90	2.69	2.45	2.17	1.83
29	5.59	4.20	3.61	3.27	3.04	2.88	2.67	2.43	2.15	1.81
30	5.57	4.18	3.59	3.25	3.03	2.87	2.65	2.41	2.14	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.53	2.29	2.01	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.41	2.17	1.88	1.48
120	5.15	3.80	3.23	2.89	2.67	2.52	2.30	2.05	1.76	1.31
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.19	1.94	1.64	1.00

Table 19.3 Percent Points of the F-Distribution

1 Per cent Points of F, i.e. $F_{0.01}(\nu_1, \nu_2)$

ν_2 \ ν_1	1	2	3	4	5	6	8	12	24	∞
1	4052	4999	5403	5625	5764	5859	5982	6106	6234	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.37	99.42	99.46	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.29	9.89	9.47	9.02
6	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
11	9.65	7.20	6.22	5.67	5.32	5.07	4.74	4.40	4.02	3.61
12	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.20	4.86	4.62	4.30	3.96	3.59	3.16
14	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.79	3.45	3.08	2.65
18	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.03	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.51	3.17	2.80	2.36
22	7.94	5.72	4.82	4.31	3.99	3.76	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.23	2.90	2.52	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.20	2.87	2.49	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.66	2.34	1.94	1.38
∞	6.64	4.60	3.78	3.32	3.02	2.80	2.51	2.18	1.79	1.00

19.3 CONFIDENCE INTERVAL FOR THE VARIANCE RATIO σ_1^2/σ_2^2

Let two independent random samples of size n_1 and n_2 be taken from two normal populations with variances σ_1^2 and σ_2^2, and let s_1^2 and s_2^2 be the unbiased sample estimates of σ_1^2 and σ_2^2. Then we know that

$$U = \frac{(n_1 - 1)\, s_1^2}{\sigma_1^2} \text{ is a } \chi^2\text{-variable with } \nu_1 = n_1 - 1 \; d.f. \text{ and}$$

$$V = \frac{(n_2 - 1)\, s_2^2}{\sigma_2^2} \text{ is a } \chi^2\text{-variable with } \nu_2 = n_2 - 1 \; d.f.$$

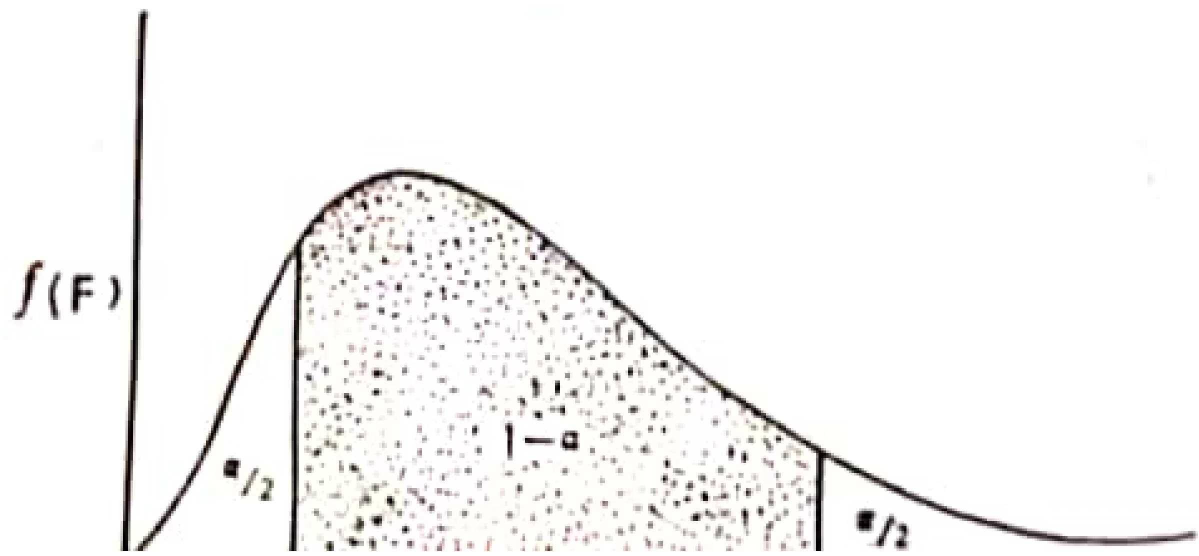
Thus, the ratio

$$F = \frac{U/\nu_1}{V/\nu_2} = \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} = \frac{\sigma_2^2\, s_1^2}{\sigma_1^2\, s_2^2}$$

has an F-distribution with ν_1 and ν_2 degrees of freedom.

To construct a $(1 - \alpha)$ 100 per cent confidence interval for the variance ratio σ_1^2/σ_2^2, we need two critical values that cut off an area of $\alpha/2$ in the lower tail and in the upper tail respectively of the F-distribution with ν_1 and ν_2 degrees of freedom. If these two values turn out to be $F_{1-\alpha/2}(\nu_1, \nu_2)$ and $F_{\alpha/2}(\nu_1, \nu_2)$, then we can make the following probability statement (see figure given below)

$$P\,[F_{1-\alpha/2}(\nu_1, \nu_2) < F < F_{\alpha/2}(\nu_1, \nu_2)] = 1-\alpha$$



$$P\left[F_{1-\alpha/2}(\upsilon_1, \upsilon_2) < \frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2} < F_{\alpha/2}(\upsilon_1, \upsilon_2)\right] = 1 - \alpha$$

Multiplying each term inside the bracket by s_2^2/s_1^2, and then inverting each term (we inverse the direction of inequality signs when terms are inverted), we obtain

$$P\left[\frac{s_1^2}{s_2^2}\frac{1}{F_{\alpha/2}(\upsilon_1, \upsilon_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2}\frac{1}{F_{1-\alpha/2}(\upsilon_1, \upsilon_2)}\right] = 1 - \alpha$$

But $\dfrac{1}{F_{1-\alpha/2}(\upsilon_1, \upsilon_2)} = F_{\alpha/2}(\upsilon_2, \upsilon_1)$ [property (v)]

$$\therefore \quad P\left[\frac{s_1^2}{s_2^2}\cdot\frac{1}{F_{\alpha/2}(\upsilon_1, \upsilon_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2}\cdot F_{\alpha/2}(\upsilon_2, \upsilon_1)\right] = 1 - \alpha$$

Hence a $(1-\alpha)$ 100 percent confidence interval for σ_1^2/σ_2^2 is given by

$$\left[\frac{s_1^2}{s_2^2}\cdot\frac{1}{F_{\alpha/2}(\upsilon_1, \upsilon_2)}, \frac{s_1^2}{s_2^2}\cdot F_{\alpha/2}(\upsilon_2, \upsilon_1)\right]$$

We can also find a confidence interval for σ_1/σ_2 by taking the square root of the endpoints of this interval.

Example 19.1. Given two random samples of size $n_1 = 12$ and $n_2 = 10$ from two independent normal populations, with $s_1 = 2.3$ and $s_2 = 1.5$, find a 90% confidence interval for σ_1^2/σ_2^2 and σ_1/σ_2.

The 90% confidence interval for the ratio σ_1^2/σ_2^2 is

$$\left[\frac{s_1^2}{s_2^2}\cdot\frac{1}{F_{\alpha/2}(\upsilon_1, \upsilon_2)}, \frac{s_1^2}{s_2^2}\cdot F_{\alpha/2}(\upsilon_2, \upsilon_1)\right]$$

Here $s_1^2 = (2.3)^2 = 5.29$, $s_2^2 = (1.5)^2 = 2.25$, $\alpha = 0.10$, $\upsilon_1 = 12-1 = 11$ and $\upsilon_2 = 10-1 = 9$.

Consulting the F-table, we find that $F_{0.05}(11, 9) = 3.10$ and $F_{0.05}(9, 11) = 2.90$. Substituting these values, we get

$$\left[\frac{5.29}{2.25}\left(\frac{1}{3.10}\right), \frac{5.29}{2.25}(2.90)\right] \text{ or } (0.76, 6.81)$$

Hence the 90% confidence interval for σ_1^2/σ_2^2 is (0.76, 6.81).

Taking the square root of the end points (0.76, 6.81), we get the 90% confidence interval for σ_1/σ_2 as (0.87, 2.61).

19.4 TESTS BASED ON *F*-DISTRIBUTION

The following tests of hypotheses are based on the *F*-distribution:

(i) Testing a hypothesis about the equality of two variances.

(ii) Testing a hypothesis about the equality of $k (k > 2)$ population means.

(iii) Testing a hypothesis about linearity of regression.

(iv) Testing hypotheses about various correlation co-efficients.

The discussion of the hypotheses (a) stated at (ii), *i.e.* the hypotheses about the equality of three or more population means, is postponed until the next chapter, where we discuss one of the most important techniques in statistical analysis, known as *analysis of variance*, and (b) stated at (iii) and (iv), will be considered in Chapter 21.

19.4.1. Testing Hypothesis about the Equality of Two Variances. Suppose that we have two independent random samples of size n_1 and n_2 from two normal populations with variances σ_1^2 and σ_2^2, and we wish to test the hypothesis that the two variances are equal (that is, $H_0 : \sigma_1^2/\sigma_2^2 = 1$ or equivalently $H_0 : \sigma_1^2 = \sigma_2^2$). Let s_1^2 and s_2^2 denote the unbiased estimates, based on $\upsilon_1 = n_1 - 1$ and $\upsilon_2 = n_2 - 1$ degrees of freedom. Then

$$U = \frac{(n_1 - 1)\, s_1^2}{\sigma_1^2}$$ is distributed as a χ^2-variable with υ_1 *d.f.*, and

$$V = \frac{(n_2 - 1)\, s_2^2}{\sigma_2^2}$$ is distributed as a χ^2-variable with υ_2 *d.f.*

By definition $$F = \frac{U/\upsilon_1}{V/\upsilon_2} = \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}, \qquad (s_1^2 > s_2^2).$$

Assuming that our null hypothesis $H_0 : \sigma_1^2/\sigma_2^2 = 1$ (that is, $H_0 : \sigma_1^2 = \sigma_2^2$) is true, the test-statistic becomes

$$F = \frac{s_1^2}{s_2^2}$$

which has an F-distribution with v_1 and v_2 degrees of freedom. The computed value of F, if H_0 is true, will be relatively close to 1. If it turns out to be considerably larger than 1 (or considerably smaller than 1 if larger s^2 is not placed in the numerator), it will suggest that $\sigma_1^2/\sigma_2^2 \neq 1$ that is, $\sigma_1^2 \neq \sigma_2^2$.

The *procedure* for testing a hypothesis that the population variances σ_1^2 and σ_2^2 are equal, consists of the following steps:

(i) Formulate the null hypothesis as $H_0 : \sigma_1^2/\sigma_2^2=1$ (that is H_0 : $\sigma_1^2=\sigma_2^2$). The alternative hypothesis may be

(a) $H_1 : \sigma_1^2/\sigma_2^2>1$, or (b) $H_1 : \sigma_1^2/\sigma_2^2<1$ or (c) $H_1 : \sigma_1^2/\sigma_2^2\neq1$.

(ii) Decide on the significance level α.

(iii) The test-statistic to use is

$$F = \frac{s_1^2}{s_2^2}, \text{ where } s_1^2 \text{ is larger than } s_2^2.$$

which, if H_0 is true, has an F-distribution with v_1 and v_2 degrees of freedom.

(iv) Calculate the value of F from the sample data.

(v) Determine the critical region of size α from the right tail of F-distribution with v_1 and v_2 degrees of freedom.

(a) When H_1 is $\sigma_1^2/\sigma_2^2>1$ (*i.e.* $H_1 : \sigma_1^2>\sigma_2^2$), the critical region will be $F \geq F_\alpha(v_1, v_2)$.

(b) When H_1 is $\sigma_1^2/\sigma_2^2<1$ (*i.e.* $H_1 : \sigma_1^2<\sigma_2^2$), we interchange the role of two samples and use $F = s_2^2/s_1^2$, then the critical region will be $F \geq F_\alpha(v_2, v_1)$.

(c) When H_1 is $\sigma_1^2/\sigma_2^2\neq1$ (*i.e.* $H_1 : \sigma_1^2\neq\sigma_2^2$), the critical region will be $F \geq F_{\alpha/2}(v_1, v_2)$, when $s_1^2 > s_2^2$, or

$F \geq F_{\alpha/2}(v_2, v_1)$, when $s_2^2 > s_1^2$.

This procedure avoids the use of the left-hand tail. However, if one wishes to use the left hand tail (two-sided test) also, then the critical region will be $F \geq F_{\alpha/2}(\upsilon_1, \upsilon_2)$, and $F \leq 1/F_{\alpha/2}(\upsilon_2, \upsilon_1)$.

(vi) Decide as below:

Reject H_0 if the computed value of F falls in the critical region, accept H_0 otherwise.

Example 19.2. Given two random samples of size $n_1 = 12$ and $n_2 = 10$ from two independent normal populations, with $s_1 = 2.3$ and $s_2 = 1.5$, test at 0.05 level of significance, the hypothesis $H_0 : \sigma_1^2/\sigma_2^2 = 1$ against the alternative $H_1 : \sigma_1^2/\sigma_2^2 > 1$.

(i) We state our null and alternative hypotheses as

$H_0 : \sigma_1^2 / \sigma_2^2 = 1$ (that is, $H_0 : \sigma_1^2 = \sigma_2^2$), and

$H_1 : \sigma_1^2 / \sigma_2^2 > 1$ (that is, $H_1 : \sigma_1^2 > \sigma_2^2$).

(ii) The level of significance is set at $\alpha = 0.05$.

(iii) The test-statistic to use is

$$F = \frac{s_1^2}{s_2^2}, \text{ where } s_1^2 \text{ is larger than } s_2^2,$$

which, if H_0 is true, has an F-distribution with $\upsilon_1 = 11$ and $\upsilon_2 = 9$ degrees of freedom.

(iv) Computations. Substituting the values, we get

$$F = \frac{(2.3)^2}{(1.5)^2} = \frac{5.29}{2.25} = 2.35.$$

(v) The critical region is $F > F_{0.05}(11, 9) = 3.10$.

(vi) **Conclusion.** Since the computed value of F does not fall in the critical region, we therefore do not reject H_0 at $\alpha = 0.05$ and may conclude that there is sufficient evidence to indicate that the two variances are equal.

Example 19.3. Two random samples drawn from two normal populations are:

Sample *I*: 20, 16, 26, 27, 23, 22, 18, 24, 25 and 19.

Sample *II*: 27, 33, 42, 35, 32, 34, 38, 28, 41, 43, 30, and 37.

Obtain the estimates of variances of the populations and test whether the two populations have the same variance.

(P.U., B.A./B.Sc. 1974)

(i) We state our null and alternative hypotheses as

$H_0 : \sigma_1^2 / \sigma_2^2 = 1$ (that is, $H_0 : \sigma_1^2 = \sigma_2^2$), and

$H_1 : \sigma_1^2 / \sigma_2^2 \neq 1$ (that is, $H_1 : \sigma_1^2 \neq \sigma_2^2$).

(ii) We choose the level of significance at $\alpha = 0.05$.

(iii) The test-statistic to use is

$$F = \frac{s_1^2}{s_2^2} \quad (s_1^2 > s_2^2)$$

which, if H_0 is true, has an F-distribution with υ_1 and υ_2 *d.f.*

(iv) Computations. The two sums of squares are

$$\Sigma(X_{1i} - \bar{X}_1)^2 = \Sigma X_{1i}^2 - \frac{(\Sigma X_{1i})^2}{n_1}$$

$$= 4960 - \frac{(220)^2}{10} = 4960 - 4840 = 120, \text{ and}$$

$$\Sigma(X_{2j} - \bar{X}_2)^2 = \Sigma X_{2j}^2 - \frac{(\Sigma X_{2j})^2}{n_2}$$

$$= 15014 - \frac{(420)^2}{12} = 15014 - 14700 = 314.$$

Now we find the two estimates as

$$s_1^2 = \frac{\Sigma(X_{1i} - \bar{X}_1)^2}{n_1 - 1} = \frac{120}{9} = 13.33, \text{ and}$$

$$s_2^2 = \frac{\Sigma(X_{2j} - \bar{X}_2)^2}{n_2 - 1} = \frac{314}{11} = 28.55.$$

Since s_2^2 is larger than s_1^2, we therefore interchange the roles of the two samples and use the test statistic $F = \frac{s_2^2}{s_1^2}$. Substituting the values, we get $F = \frac{28.55}{13.33} = 2.14$.

(v) The critical region is $F \geq F_{0.025}(11, 9) = 3.92$ $\quad (\because v_2 = 11, v_1 = 9)$

(vi) **Conclusion.** Since the computed value of F does not fall in critical region, so we do not reject H_0 and may conclude that the two populations have the same variance.

Alternatively. If we wish to use a two sided test, then the critical region will be

$$F \geq F_{0.025}(9, 11) = 3.59, \text{ and}$$

$$F \leq \frac{1}{F_{0.025}(11, 9)} = \frac{1}{3.92} = 0.26.$$

Now $\quad F = \dfrac{s_1^2}{s_2^2} = \dfrac{13.33}{28.55} = 0.47$

Decision. The computed value $F = 0.47$ falls in the acceptance region so we do not reject H_0 and conclude that the two populations have equal variances.

Example 19.4. In an experiment on reaction times in seconds of two individuals A and B, measured under identical conditions, the following results were obtained:

A	A^2	B	B^2
0.41	0.1681	0.32	0.1024
0.38	0.1444	0.36	0.1296
0.37	0.1369	0.38	0.1444
0.42	0.1764	0.33	0.1089
0.35	0.1225	0.38	0.1444
0.38	0.1444		
2.31	0.8927	1.77	0.6297

$$\bar{A} = \frac{\sum A}{n_1} = \frac{2.31}{6} = 0.385, \quad \bar{B} = \frac{\sum B}{n_2} = \frac{1.77}{5} = 0.354.$$

$$s_A^2 = \frac{1}{n_1-1}\left[\sum A^2 - \frac{(\sum A)^2}{n_1}\right] = \frac{1}{5}\left[0.8927 - \frac{(2.31)^2}{6}\right] = 0$$

$$s_B^2 = \frac{1}{n_2-1}\left[\sum B^2 - \frac{(\sum B)^2}{n_2}\right] = \frac{1}{4}\left[0.6297 - \frac{(1.77)^2}{5}\right] = 0$$

$$F = \frac{s_A^2}{s_B^2} = \frac{0.00067}{0.00078} = 0.86$$

(v) The critical region is $F \geq F_{0.025}(5, 4) = 9.36$, and

$$\frac{1}{\quad} = 0.14$$

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$$H_0 : \mu_A = \mu_B \text{ against } H_1 : \mu_A \neq \mu_B$$

(ii) The level of significance is set at $\alpha = 0.05$.

(iii) The test-statistic to use is

$$t = \frac{\bar{A} - \bar{B}}{s_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}},$$

which, if H_0 is true, follows a t-distribution with $v = 9$ *d.f.*

(iv) **Computations:** To calculate t, we need to compute s_p^2, which i

$$s_p = \sqrt{\frac{(n_1 - 1)\, s_A^2 + (n_2 - 1)\, s_B^2}{n_1 + n_2 - 2}} = \sqrt{\frac{0.00335 + 0.0}{6 + 5 - 2}}$$

$$= 0.0268$$

$$\therefore \quad t = \frac{0.385 - 0.354}{0.0268\sqrt{1/6 + 1/5}} = \frac{0.031}{0.016} = 1.94$$

(v) The critical region is $|t| \geq t_{0.025,(9)} = 2.26$

(vi) **Conclusion:** Since the calculated value $t = 1.94$ does not fall i the critical region, so we accept H_0 and conclude that the tw means are equal.