Integration by Parts

If u and v are function of x, then

$$\int uv\,dx = u\int v dx - \int\left(\int v dx\right)\cdot u'\,dx$$

Question # 1

Evaluate the following integrals by parts add a word representing all the functions are defined.

(i) $\int x\sin x\,dx$ (ii) $\int \ln x\,dx$

(iii) $\int x\ln x\ dx$ (iv) $\int x^2\ln x\,dx$

(v) $\int x^3\ln x\,dx$ (vi) $\int x^4\ln x\,dx$

(vii) $\int \tan^{-1}x\ dx$ (viii) $\int x^2\sin x\ dx$

(ix) $\int x^2\tan^{-1}x\,dx$ (x) $\int x\tan^{-1}x\,dx$

(xi) $\int x^3\tan^{-1}x\ dx$ (xii) $\int x^3\cos x\,dx$

(xiii) $\int \sin^{-1}x\,dx$ (xiv) $\int x\ \sin^{-1}x\ dx$

(xv) $\int e^x\sin x\cos x\,dx$ (xvi) $\int x\sin x\cos x\,dx$

(xvii) $\int x\cos^2 x\,dx$ (xviii) $\int x\sin^2 x\ dx$

(xix) $\int (\ln x)^2\,dx$ (xx) $\int \ln(\tan x)\sec^2 x dx$

(xxi) $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution

(i) Let $I = \int x\sin x\,dx$

Integration by parts

$$I = x\cdot(-\cos x) - \int(-\cos x)\cdot(1)\,dx$$
$$= -x\cos x + \int\cos x\,dx$$
$$= -x\cos x + \sin x + c$$

(ii) Let $I = \int \ln x\,dx$
$= \int \ln x\cdot 1\ dx$

$u = \ln x$
$v = 1$

Integrating by parts

$$I = \ln x\cdot x - \int x\cdot\frac{1}{x}\ dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + c$$

(iii) Let $I = \int x\ln x\ dx$

$u = \ln x$
$v = x$

Integrating by parts

$$I = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\ dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + c$$
$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

(iv) *Do yourself*

(v) *Do yourself*

(vi) *Do yourself*

(vii) Let $I = \int\tan^{-1}x\ dx$
$= \int\tan^{-1}x\cdot 1\ dx$

$u = \tan^{-1}x$
$v = 1$

Integrating by parts

$$I = \tan^{-1}x\cdot x - \int x\cdot\frac{1}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\ln\left|1+x^2\right| + c$$

(viii) Let $I = \int x^2\sin x\ dx$

$u = x^2$
$v = \sin x$

Integrating by parts

$$I = x^2(-\cos x) - \int(-\cos x)\cdot 2x\ dx$$
$$= -x^2\cos x + 2\int x\ \cos x\,dx$$

$u = x$
$v = \cos x$

Again integrating by parts

$$I = -x^2\cos x + 2\left(x\ \sin x - \int\sin x\ (1)\ dx\right)$$
$$= -x^2\cos x + 2x\ \sin x - 2(-\cos x) + c$$
$$= -x^2\cos x + 2x\ \sin x + 2\cos x + c$$

(ix) Let $I = \int x^2\tan^{-1}x\,dx$

$u = \tan^{-1}x$
$v = x^2$

Integrating by parts

$$I = \tan^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int x\,dx+-\frac{1}{3}\int\frac{x}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\cdot\frac{x^2}{2}+-\frac{1}{3}\cdot\frac{1}{2}\int\frac{2x}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+-\frac{1}{6}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+-\frac{1}{6}\ln\left|1+x^2\right|+c$$

(x) Let $I=\int x\tan^{-1}x\,dx$

Integrating by parts

| $u=\tan^{-1}x$
| $v=x$

$$I=\frac{x^2}{2}\tan^{-1}x-\int\frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(\frac{1+x^2}{1+x^2}-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int dx+\frac{1}{2}\int\frac{1}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c \quad \textit{Ans.}$$

(xi) Let $I=\int x^3\tan^{-1}x\,dx$

Integrating by parts

| $u=\tan^{-1}x$
| $v=x^3$

$$I=\tan^{-1}x\cdot\frac{x^4}{4}-\int\frac{x^4}{4}\cdot\frac{1}{1+x^2}dx$$

$$=\frac{x^4}{4}\tan^{-1}x--\frac{1}{4}\int\frac{x^4}{1+x^2}dx$$

$$=\frac{x^4}{4}\tan^{-1}x$$
$$-\frac{1}{4}\int\left(x^2-1+\frac{1}{1+x^2}\right)dx$$

$$\begin{array}{r|l} & x^2-1 \\ 1+x^2 & x^4 \\ & \underline{\mp x^4 \pm x^2} \\ & -x^2 \\ & \underline{\mp 1 \mp x^2} \\ & 1 \end{array}$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int x^2dx+\frac{1}{4}\int dx-\frac{1}{4}\int\frac{1}{1+x^2}dx$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\frac{x^3}{3}+\frac{1}{4}x-\frac{1}{4}\tan^{-1}x+c$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{x^3}{12}+\frac{1}{4}x-\frac{1}{4}\tan^{-1}x+c$$

(xii) *Do yourself as Question # 1(viii).*

(xiii) $I=\int\sin^{-1}x\,dx$

$=\int\sin^{-1}x\cdot 1\,dx$

| $u=\sin^{-1}x$
| $v=1$

Integrating by parts

$$I=\sin^{-1}x\cdot x-\int x\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$=x\sin^{-1}x-\int(1-x^2)^{-\frac{1}{2}}(x)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}\frac{d}{dx}(1-x^2)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{1/2}+c$$

$$=x\sin^{-1}x+\sqrt{1-x^2}+c$$

(xiv) Let $I=\int x\sin^{-1}x\,dx$

Integrating by parts

$$I=\sin^{-1}x\cdot\frac{x^2}{2}-\int\frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left(\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left(\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\sqrt{1-x^2}\,dx-\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx$$

$\Rightarrow\ I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}I_1-\frac{1}{2}\sin^{-1}x$ ….. (i)

Where $I_1=\int\sqrt{1-x^2}\,dx$

Put $x=\sin\theta \quad \Rightarrow \quad dx=\cos\theta\, d\theta$

$\Rightarrow\ I_1=\int\sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$

$=\int\sqrt{\cos^2\theta}\,\cos\theta\, d\theta$

$=\int\cos^2\theta\, d\theta \;=\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$=\frac{1}{2}\int(1+\cos 2\theta)\,d\theta$

$=\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+c$

$=\frac{1}{2}\left[\theta+\frac{2\sin\theta\cos\theta}{2}\right]+c$

$=\frac{1}{2}\left[\theta+\sin\theta\sqrt{1-\sin^2\theta}\right]+c$

$=\frac{1}{2}\left[\sin^{-1}x+x\sqrt{1-x^2}\right]+c$

Using value of I_1 in (i)

$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\frac{1}{2}\left(\sin^{-1}x+x\sqrt{1-x^2}+c\right)\right]-\frac{1}{2}\sin^{-1}x$

$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{2}c-\frac{1}{2}\sin^{-1}x$

$\Rightarrow\ I=\frac{x^2}{2}\sin^{-1}x-\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{2}c$

(xv) Let $I=\int e^x\sin x\cos x\,dx$

$=\frac{1}{2}\int e^x\cdot 2\sin x\cos x\,dx$

$=\frac{1}{2}\int e^x\sin 2x\,dx \quad \because\ \sin 2x=2\sin x\cos x$

Integrating by parts

$I=\frac{1}{2}\left[e^x\cdot\frac{-\cos 2x}{2}-\int\frac{-\cos 2x}{2}\cdot e^x dx\right]$

$=-\frac{1}{4}e^x\cos 2x+\frac{1}{4}\int e^x\cos 2x\,dx$

Again integrating by parts

$I=-\frac{1}{4}e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\,e^x\right)$

$=-\frac{1}{4}e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-\frac{1}{2}\int e^x\sin 2x\right)$

$=-\frac{1}{4}e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-I\right)+c$

$=-\frac{1}{4}e^x\cos 2x+\frac{1}{8}e^x\sin 2x-\frac{1}{4}I+c$

$\Rightarrow\ I+\frac{1}{4}I=-\frac{1}{4}e^x\cos 2x+\frac{1}{8}e^x\sin 2x+c$

$\Rightarrow\ \frac{5}{4}I=-\frac{1}{4}e^x\cos 2x+\frac{1}{8}e^x\sin 2x+c$

$\Rightarrow\ I=-\frac{1}{5}e^x\cos 2x+\frac{1}{10}e^x\sin 2x+\frac{4}{5}c$

(xvi) Let $I=\int x\sin x\cos x\,dx$

$=\frac{1}{2}\int x\cdot 2\sin x\cos x\,dx$

$=\frac{1}{2}\int x\cdot\sin 2x\,dx$ | $u=x$, $v=\sin 2x$

Integrating by parts

$I=\frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right)-\int\left(\frac{-\cos 2x}{2}\right)(1)dx\right]$

(xvii)Let $I=\int x\cos^2 x\,dx$

$=\int x\left(\frac{1+\cos 2x}{2}\right)dx$

$=\frac{1}{2}\int x(1+\cos 2x)\,dx$

$=\frac{1}{2}\int x\,dx+\frac{1}{2}\int x\cos 2x dx$ | $u=x$, $v=\cos 2x$

$=\frac{1}{2}\cdot\frac{x^2}{2}+\frac{1}{2}\left[x\,\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\cdot(1)dx\right]$

$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x-\frac{1}{4}\int\sin 2x\,dx$

$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x-\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)+c$

$=\frac{x^2}{4}+\frac{1}{4}x\cdot\sin 2x+\frac{1}{8}\cos 2x+c$

(xviii) Let $I=\int x\sin^2 x\,dx$

$$=\int x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$=\frac{1}{2}\int x(1-\cos 2x)\,dx$$

$$=\frac{1}{2}\int x\,dx-\frac{1}{2}\int x\cos 2x\,dx \qquad \left|\begin{array}{l} u=x \\ v=\cos 2x\end{array}\right.$$

Integrating by parts

$$I=\frac{1}{2}\frac{x^2}{2}-\frac{1}{2}\left[x\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}.(1)\,dx\right]$$

$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{4}\int\sin 2x\,dx$$

$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)+c$$

$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+c$$

(xix) Let $I=\int(\ln x)^2dx$

$$=\int(\ln x)^2\cdot 1\,dx \qquad \left|\begin{array}{l} u=(\ln x)^2 \\ v=1\end{array}\right.$$

Integrating by parts

$$I=(\ln x)^2\cdot x-\int x\cdot 2(\ln x)\cdot\frac{1}{x}\,dx$$

$$=x(\ln x)^2-2\int(\ln x)\,dx$$

Again integrating by parts

$$I=x(\ln x)^2-2\left[\ln x\cdot x-\int x\cdot\frac{1}{x}\,dx\right]$$

$$=x(\ln x)^2-2x\ln x+2\int dx$$

$$=x(\ln x)^2-2x\ln x+2x+c$$

(xx) Let $I=\int\ln(\tan x)\sec^2 x\,dx$

Integrating by parts

$$I=\ln(\tan x)\cdot\tan x-\int\tan x\cdot\frac{1}{\tan x}\cdot\sec^2 x\,dx$$

$$=\tan x\ln(\tan x)-\int\sec^2 x\,dx$$

$$=\tan x\ln(\tan x)-\tan x+c$$

(xxi) Let $I=\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

$$=\int\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}(x)\,dx \qquad \left|\begin{array}{l} u=\sin^{-1}x \\ v=\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\end{array}\right.$$

$$=-\frac{1}{2}\int\sin^{-1}x\cdot\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\,dx$$

Integrating by parts

$$I=-\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-\int\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\cdot\frac{1}{\sqrt{1-x^2}}dx\right]$$

$$=-\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}-\int\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\cdot\frac{1}{\sqrt{1-x^2}}\,dx\right]$$

$$=-\frac{1}{2}\left[2(1-x^2)^{\frac{1}{2}}\sin^{-1}x-2\int dx\right]$$

$$=-\sqrt{1-x^2}\sin^{-1}x+\int dx$$

$$=-\sqrt{1-x^2}\sin^{-1}x+x+c$$

$$=x-\sqrt{1-x^2}\sin^{-1}x+c$$

Question # 2

Evaluate the following integrals.

(i) $\int\tan^4 x\,dx$ (ii) $\int\sec^4 x\,dx$

(iii) $\int e^x\sin 2x\cos x\,dx$ (iv) $\int\tan^3 x\cdot\sec x\,dx$

(iv) $\int\tan^3 x\cdot\sec x\,dx$ (v) $\int x^3e^{5x}dx$

(vi) $\int e^{-x}\sin 2x\,dx$ (vii) $\int e^{2x}\cdot\cos 3x\,dx$

(viii) $\int\text{cosec}^3 x\,dx$

Solution

(i) Let $I=\int\tan^4 x\,dx$

$$=\int\tan^2 x\cdot\tan^2 x\,dx$$

$$=\int\tan^2 x\left(\sec^2 x-1\right)dx$$

$$=\int\left(\tan^2 x\sec^2 x-\tan^2 x\right)dx$$

$$=\int\tan^2 x\sec^2 x\,dx-\int\tan^2 x\,dx$$

$$=\int\tan^2 x\frac{d}{dx}(\tan x)\,dx-\int(\sec^2 x-1)\,dx$$

$$=\frac{\tan^{2+1}x}{2+1}-\int\sec^2 x\,dx+\int dx$$

$$=\frac{1}{3}\tan^3 x-\tan x+x+c$$

(ii) Let $I=\int \sec^4 x\,dx$

$=\int (\sec^2 x)\cdot(\sec^2 x)\,dx$

$=\int (1+\tan^2 x)\cdot(\sec^2 x)\,dx$

$=\int \sec^2 x\,dx+\int \tan^2 x\sec^2 x\,dx$

$=\tan x+\int (\tan x)^2\frac{d}{dx}(\tan x)\,dx$

$=\tan x+\frac{\tan^3 x}{3}+c$

(iii) Let $I=\int e^x\sin 2x\cos x\,dx$

$=\frac{1}{2}\int e^x\left(2\sin 2x\cos x\right)dx$

$=\frac{1}{2}\int e^x\left(\sin(2x+x)+\sin(2x-x)\right)dx$

$=\frac{1}{2}\int e^x\left(\sin 3x+\sin x\right)dx$

$=\frac{1}{2}\int e^x\sin 3x\,dx+\frac{1}{2}\int e^x\sin x\,dx$

$=\frac{1}{2}I_1+\frac{1}{2}I_2$ ………. (i)

Where $I_1=\int e^x\sin 3x\,dx$ and $I_2=\int e^x\sin x\,dx$

Solve I_1 and I_2 as in Q # 1(xv) and put value of I_1 and I_2 in (i).

(iv) $I=\int \tan^3 x\cdot\sec x\,dx$

$=\int \tan^2 x\cdot\tan x\cdot\sec x\,dx$

$=\int \left(\sec^2 x-1\right)\cdot\sec x\tan x\,dx$

Put $t=\sec x \quad\Rightarrow\quad dt=\sec x\tan x\,dx$

So $I=\int\left(t^2-1\right)dt$

$=\frac{t^3}{3}-t+c$

$=\frac{\sec^3 x}{3}-\sec+c$

(v) Let $I=\int x^3e^{5x}dx$ | $u=x^3$, $v=e^x$

Integrating by parts

$I=x^3\cdot\frac{e^{5x}}{5}-\int\frac{e^{5x}}{5}\cdot 3x^2dx$

$=\frac{1}{5}x^3e^{5x}-\frac{3}{5}\int x^2e^{5x}dx$ | $u=x^2$, $v=e^x$

Again integrating by parts

$I=\frac{1}{5}x^3e^{5x}-\frac{3}{5}\left[x^2\cdot\frac{e^{5x}}{5}-\int\frac{e^{5x}}{5}\cdot 2x\,dx\right]$

$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{25}\int x\,e^{5x}\,dx$

Again integrating by parts

$I=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}$
$+\frac{6}{25}\left[x\cdot\frac{e^{5x}}{5}-\int\frac{e^{5x}}{5}\cdot(1)dx\right]$

$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{125}xe^{5x}-\frac{6}{125}\int e^{5x}\,dx$

$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{125}xe^{5x}-\frac{6}{125}\cdot\frac{e^{5x}}{5}+c$

$=\frac{e^{5x}}{5}\left(x^3-\frac{3}{5}x^2+\frac{6}{25}x-\frac{6}{125}\right)+c$

(vi) Let $I=\int e^{-x}\sin 2x\,dx$ | $u=e^{-x}$, $v=\sin 2x$

Integrating by parts

$I=e^{-x}\cdot\frac{-\cos 2x}{2}-\int\frac{-\cos 2x}{2}\cdot e^{-x}(-1)\,dx$

$=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{2}\int e^{-x}\cos 2x\,dx$

Again integrating by parts

$I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{2}\left[e^{-x}\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\cdot e^{-x}(-1)\,dx\right]$

$=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x-\frac{1}{4}\int e^{-x}\sin 2x\,dx$

$\Rightarrow\ I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x-\frac{1}{4}I+c$

$\Rightarrow\ I+\frac{1}{4}I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x+c$

$\Rightarrow\ \frac{5}{4}I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x+c$

$\Rightarrow\ I=-\frac{2}{5}e^{-x}\cos 2x-\frac{1}{5}e^{-x}\sin 2x+\frac{4}{5}c$

$=-\frac{1}{5}e^{-x}\left(2\cos 2x+\sin 2x\right)+\frac{4}{5}c$

(vii) *Do yourself as above*

(viii) $I=\int \operatorname{cosec}^3 x\,dx$ | $u=\operatorname{cosec} x$, $v=\operatorname{cosec}^2 x$

$$= \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x \, dx$$

Integrating by parts

$$I = \csc x(-\cot x) i \int (-\cot x)(-\csc x \cot x) \, dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x \, dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \left(\operatorname{cosec}^2 x - 1\right) dx$$

$$= -\operatorname{cosec} x \cot x - \int \left(\operatorname{cosec}^3 x - \operatorname{cosec} x\right) dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x \, dx + \int \operatorname{cosec} x \, dx$$

$$= -\operatorname{cosec} x \cot x - I + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad I + I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad 2I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad I = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln\left|\csc x - \cot x\right| + \frac{1}{2}c$$

Question # 3

Show that

$$\int e^{ax} \sin bx \, dx = \frac{1}{\sqrt{a^2 + b^2}} \, e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Solution

Let $I = \int e^{ax} \sin bx \, dx$ $u = e^{ax}$, $v = \sin bx$

Integrating by parts

$$I = e^{ax}\left(-\frac{\cos bx}{b}\right) - \int \left(-\frac{\cos bx}{b}\right) \cdot e^{ax}(a) \, dx$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax} \cos bx \, dx$$

Again integrating by parts

$$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \int \frac{\sin bx}{b} \cdot e^{ax} a \, dx\right]$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2}\int e^{ax} \sin bx \, dx$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} I + c_1$$

$$\Rightarrow \quad I + \frac{a^2}{b^2} I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx + c_1$$

$$\Rightarrow \quad \left(\frac{b^2 + a^2}{b^2}\right) I = \frac{e^{ax}}{b^2}\left(-b\cos bx + a \sin bx\right) + c_1$$

$$\Rightarrow \quad \left(b^2 + a^2\right) I = e^{ax}\left(a \sin bx - b \cos bx\right) + b^2 c_1$$

Put $a = r\cos\theta$ & $b = r\sin\theta$

Squaring and adding

$$a^2 + b^2 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

$$\Rightarrow \quad a^2 + b^2 = r^2 (1) \quad \Rightarrow \quad r = \sqrt{a^2 + b^2}$$

Also

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \quad \Rightarrow \quad \frac{b}{a} = \tan\theta$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{b}{a}$$

So

$$\left(b^2 + a^2\right) I = e^{ax}\left(r\cos\theta \sin bx - r\sin\theta \cos bx\right) + b^2 c_1$$

$$\left(b^2 + a^2\right) I = e^{ax} r\left(\sin bx \cos\theta - \cos bx \sin\theta\right) + b^2 c_1$$

$$\Rightarrow \quad \left(a^2 + b^2\right) I = e^{ax} r \sin\left(bx - \theta\right) + b^2 c_1$$

Putting value of r and θ

$$\left(a^2 + b^2\right) I = e^{ax}\sqrt{a^2 + b^2} \, \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + b^2 c_1$$

$$\Rightarrow I = \frac{\sqrt{a^2 + b^2}}{(a^2 + b^2)} \, e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + \frac{b^2}{a^2 + b^2} c_1$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{a^2 + b^2}} \, e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Where $c = \dfrac{b^2}{a^2 + b^2} c_1$

Question # 4

Evaluate the following indefinite integrals.

(i) $\int \sqrt{a^2 - x^2} \, dx$ (ii) $\int \sqrt{x^2 - a^2} \, dx$

(iii) $\int \sqrt{4 - 5x^2} \, dx$ (iv) $\int \sqrt{3 - 4x^2} \, dx$

(v) $\int \sqrt{x^2 + 4} \, dx$ (vi) $\int x^2 e^{ax} dx$

Solution

(i) Let $I = \int \sqrt{a^2 - x^2} \, dx$ $u = \sqrt{a^2 - x^2}$, $v = 1$

$$= \int \sqrt{a^2 - x^2} \cdot 1 \, dx$$

Integrating by parts

$$I = \sqrt{a^2 - x^2} \cdot x - \int x \cdot \frac{1}{2}\left(a^2 - x^2\right)^{-\frac{1}{2}} \cdot (-2x) \, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}} \, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}} \, dx$$

$$= x\sqrt{a^2-x^2} - \int\left(\frac{a^2-x^2}{\left(a^2-x^2\right)^{\frac{1}{2}}} - \frac{a^2}{\left(a^2-x^2\right)^{\frac{1}{2}}}\right)dx$$

$$= x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + \int\frac{a^2}{\sqrt{a^2-x^2}}dx$$

$$\Rightarrow\ I = x\sqrt{a^2-x^2} - I + a^2\int\frac{1}{\sqrt{a^2-x^2}}dx$$

$$\Rightarrow\ I + I = x\sqrt{a^2-x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow\ 2I = x\sqrt{a^2-x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow\ I = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 Sin^{-1}\frac{x}{a} + \frac{1}{2}c$$

Review

- $\int\frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$
- $\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + c$

(ii)Let $I = \int\sqrt{x^2-a^2}\,dx$

$= \int\sqrt{x^2-a^2}\cdot 1\,dx$

$u = \sqrt{x^2-a^2}$
$v = 1$

Integrating by parts

$$I = \sqrt{x^2-a^2}\cdot x - \int x\cdot\frac{1}{2}\left(x^2-a^2\right)^{-\frac{1}{2}}\cdot(2x)\,dx$$

$$= x\sqrt{x^2-a^2} - \int\frac{x^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}dx$$

$$= x\sqrt{x^2-a^2} - \int\frac{x^2-a^2+a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}dx$$

$$= x\sqrt{x^2-a^2} - \int\left(\frac{x^2-a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}} + \frac{a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}\right)dx$$

$$= x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - \int\frac{a^2}{\sqrt{x^2-a^2}}dx$$

$$\Rightarrow\ I = x\sqrt{x^2-a^2} - I - a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$$

$$\Rightarrow\ I + I = x\sqrt{x^2-a^2} - a^2\ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\because\ \int\frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\Rightarrow\ 2I = x\sqrt{x^2-a^2} - a^2\ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\Rightarrow\ I = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + \frac{1}{2}c$$

(iii) Let $I = \int\sqrt{4-5x^2}\,dx$

$= \int\sqrt{4-5x^2}\cdot 1\,dx$

Integrating by parts

$$I = \sqrt{4-5x^2}\cdot x - \int x\cdot\frac{1}{2}(4-5x^2)^{-\frac{1}{2}}\cdot(-10x)\,dx$$

$$= \sqrt{4-5x^2}\cdot x - \int\frac{-5x^2}{(4-5x^2)}dx$$

$$= \sqrt{4-5x^2}\cdot x - \int\frac{4-5x^2-4}{(4-5x^2)}dx$$

$$= \sqrt{4-5x^2}\cdot x - \int\left(\frac{4-5x^2}{(4-5x^2)^{\frac{1}{2}}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}}\right)dx$$

$$= \sqrt{4-5x^2}\cdot x - \int\left((4-5x^2)^{\frac{1}{2}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}}\right)dx$$

$$= \sqrt{4-5x^2}\cdot x - \int\sqrt{4-5x^2}\,dx + 4\int\frac{1}{\sqrt{4-5x^2}}dx$$

$$\Rightarrow\ I = \sqrt{4-5x^2}\cdot x - I + 4\int\frac{1}{\sqrt{5\left(\frac{4}{5}-x^2\right)}}dx$$

$$\Rightarrow\ I + I = \sqrt{4-5x^2}\cdot x + 4\int\frac{1}{\sqrt{5}\sqrt{\frac{4}{5}-x^2}}dx$$

$$\Rightarrow\ 2I = \sqrt{4-5x^2}\cdot x + \frac{4}{\sqrt{5}}\int\frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 - x^2}}dx$$

$$= \sqrt{4-5x^2}\cdot x + \frac{4}{\sqrt{5}}Sin^{-1}\left(\frac{x}{2/\sqrt{5}}\right) + c_1$$

$$\because\ \int\frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\frac{x}{a}$$

$$\Rightarrow\ I = \frac{x}{2}\sqrt{4-5x^2} + \frac{4}{2\sqrt{5}}Sin^{-1}\left(\frac{\sqrt{5}\,x}{2}\right) + \frac{1}{2}c_1$$

$$= \frac{x}{2}\sqrt{4-5x^2} + \frac{2}{\sqrt{5}} Sin^{-1}\left(\frac{\sqrt{5}\,x}{2}\right) + c$$

Where $c = \frac{1}{2}c_1$

(iv) *Same as above.*

(v) *Same as Q #* 4(ii)

Use $\int \frac{dx}{\sqrt{x^2+4}} = \ln\left|x+\sqrt{x^2+4}\right| + c$

(vi) *Do yourself as Question #* 2(v)

Important Formula

Since $\frac{d}{dx}\left(e^{ax} f(x)\right) = e^{ax}\frac{d}{dx}f(x) + f(x)\frac{d}{dx}e^{ax}$

$$= e^{ax} f'(x) + f(x)\cdot e^{ax}(a)$$

$$= e^{ax}\left[\, a\,f(x) + f'(x)\,\right]$$

On integrating

$$\int \frac{d}{dx}\left(e^{ax} f(x)\right)\,dx = \int e^{ax}\left[\, a\,f(x)+f'(x)\,\right]dx$$

$$\Rightarrow\ e^{ax} f(x) = \int e^{ax}\left[\, a\,f(x)+f'(x)\,\right]dx$$

$$\Rightarrow\ \boxed{\int e^{ax}\left[\, a\,f(x)+f'(x)\,\right]dx = e^{ax} f(x) + c}$$

Question # 5

Evaluate the following integrals.

(i) $\int e^x\left(\frac{1}{x}+\ln x\right)dx$ (ii) $\int e^x\left(\cos x + \sin x\right)dx$

(iii) $\int e^{ax}\left[a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right]dx$

(iv) $\int e^{3x}\left(\frac{3\sin x - \cos x}{\sin^2 x}\right)dx$

(v) $\int \frac{xe^x}{(1+x)^2}dx$ (vi) $\int \frac{xe^x}{(1+x)^2}dx$

(vii) $\int e^{-x}\left(\cos x - \sin x\right)dx$

(viii) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$ (ix) $\int \frac{2x}{1-\sin x}dx$

(x) $\int \frac{e^x(1+x)}{(2+x)^2}dx$ (xi) $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

Solution

(i) Let $I = \int e^x\left(\frac{1}{x}+\ln x\right)dx$

$$= \int e^x\left(\ln x + \frac{1}{x}\right)dx$$

Put $f(x) = \ln x \ \Rightarrow\ f'(x) = \frac{1}{x}$

So $I = \int e^x\left(f(x)+f'(x)\right)dx$

$$= e^x f(x) + c \quad = e^x \ln x + c$$

(ii) Let $I = \int e^x\left(\cos x + \sin x\right)dx$

$$= \int e^x\left(\sin x + \cos x\right)dx$$

Put $f(x) = \sin x \ \Rightarrow\ f'(x) = \cos x$

So $I = \int e^x\left(f(x)+f'(x)\right)dx$

$$= e^x f(x) + c$$

$$= e^x \sin x + c$$

(iii) Let $I = \int e^{ax}\left[a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right]dx$

Put $f(x) = \sec^{-1}x \ \Rightarrow\ f'(x) = \frac{1}{x\sqrt{x^2-1}}$

So $I = \int e^{ax}\left[a\,f(x)+f'(x)\right]dx$

$$= e^{ax} f(x) + c$$

$$= e^{ax}\sec^{-1}x + c$$

(iv) Let $I = \int e^{3x}\left(\frac{3\sin x - \cos x}{\sin^2 x}\right)dx$

$$= \int e^{3x}\left(\frac{3\sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)dx$$

$$= \int e^{3x}\left(3\frac{1}{\sin x} - \frac{\cos x}{\sin x\cdot\sin x}\right)dx$$

$$= \int e^{3x}\left(3\csc x - \csc x\cot x\right)dx$$

Put $f(x) = \csc x \ \Rightarrow\ f'(x) = -\csc x\cot x$

$\Rightarrow\ I = \int e^{3x}\left(3f(x)+f'(x)\right)dx$

$$= e^{3x} f(x) + c$$

$$= e^{3x}\csc x + c$$

(v) Let $I = \int e^{2x}\left(-\sin x + 2\cos x\right)dx$

$$= \int e^{2x}\left(2\cos x - \sin x\right)dx$$

Put $f(x) = \cos x \ \Rightarrow\ f'(x) = -\sin x$

So $I = \int e^{2x}\left(2f(x)+f'(x)\right)dx$

$= e^{2x} f(x) + c$

$= e^{2x}\cos x + c$

(vi) Let $I = \int \frac{xe^x}{(1+x)^2}dx$

$= \int \frac{(1+x-1)e^x}{(1+x)^2}\,dx$

$= \int e^x\left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right]dx$

Put $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$\Rightarrow\ f'(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$

So $I = \int e^x\left(f(x) + f'(x)\right)dx$

$= e^x f(x) + c$

$= e^x\left(\frac{1}{1+x}\right) + c$

(vii) Let $I = \int e^{-x}\left(\cos x - \sin x\right)dx$

$= \int e^{-x}\left((-1)\sin x + \cos x\right)dx$

Put $f(x) = \sin x \quad \Rightarrow \quad f'(x) = \cos x$

So $I = \int e^{-x}\left((-1)f(x) + f'(x)\right)dx$

$= e^{-x} f(x) + c$

$= e^{-x}\sin x + c$

(viii) Let $I = \int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$

$= \int e^{m\tan^{-1}x}\cdot\frac{1}{1+x^2}\,dx$

Put $t = \tan^{-1}x \quad \Rightarrow \quad dt = \frac{1}{1+x^2}dx$

So $I = \int e^{mt}dt$

$= \frac{e^{mt}}{m} + c$

$= \frac{1}{m}e^{m\tan^{-1}x} + c$

Important Integral

Let $I = \int \tan x\,dx$

$= \int \frac{\sin x}{\cos x}\,dx$

Put $t = \cos x \quad \Rightarrow \quad dt = -\sin x\,dx$

$\Rightarrow \quad -dt = \sin x\,dx$

So $I = \int \frac{-dt}{t} = -\int \frac{dt}{t}$

$= -\ln|t| + c$

$= -\ln|\cos x| + c$

$= \ln|\cos x|^{-1} + c \qquad \because\ m\ln x = \ln x^m$

$= \ln\left|\frac{1}{\cos x}\right| + c \quad = \ln|\sec x| + c$

$\Rightarrow \quad \boxed{\int \tan x\,dx = \ln|\sec x| + c}$

Similarly, we have

$\boxed{\int \cot x\ dx = \ln|\sin x| + c}$

(ix) Let $I = \int \frac{2x}{1-\sin x}dx$

$= \int \frac{2x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}dx$

$= \int \frac{2x(1+\sin x)}{1-\sin^2 x}dx$

$= \int \frac{2x + 2x\sin x)}{\cos^2 x}dx$

$= \int \left(\frac{2x}{\cos^2 x} + \frac{2x\sin x}{\cos^2 x}\right)dx$

$= \int \frac{2x}{\cos^2 x}dx + \int \frac{2x\sin x}{\cos x\cdot\cos x}dx$

$= 2\int x\sec^2 x\ dx + 2\int x\sec x\tan x\,dx$

Integrating by parts

$I = 2\left[x\cdot\tan x - \int \tan x\cdot 1\,dx\right]$
$\qquad + 2\left[x\cdot\sec x - \int \sec x(1)\,dx\right]$

$= 2\left[x\cdot\tan x - \ln|\sec x|\right]$
$\qquad + 2\left[x\cdot\sec x - \ln|\sec x + \tan x|\right] + c$

$= 2x\tan x - 2\ln|\sec x|$
$\qquad + 2x\sec x - 2\ln|\sec x + \tan x| + c$

(x) Let $I = \int \frac{e^x(1+x)}{(2+x)^2}dx$

$$=\int \frac{e^x(2+x-1)}{(2+x)^2}dx$$

$$=\int e^x\left(\frac{2+x}{(2+x)^2}-\frac{1}{(2+x)^2}\right)dx$$

$$=\int e^x\left((2+x)^{-1}-(2+x)^{-2}\right)dx$$

Put $f(x)=(2+x)^{-1} \quad \Rightarrow \quad f'(x)=-(2+x)^{-2}$

So $I=\int e^x\left(f(x)+f'(x)\right)dx$

$$=e^x f(x)+c$$

$$=e^x(2+x)^{-1}+c$$

$$=\frac{e^x}{2+x}+c$$

(xi) Let $I=\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

$$=\int\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)e^x\,dx$$

$$=\int\left(\frac{1}{2\sin^2\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)e^x\,dx$$

$$=\int\left(\frac{1}{2}\text{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right)e^x\,dx$$

$$=\int e^x\left(-\cot\frac{x}{2}+\frac{1}{2}\text{cosec}^2\frac{x}{2}\right)dx$$

Put $f(x)=-\cot\frac{x}{2} \quad \Rightarrow \quad f'(x)=\text{cosec}^2\frac{x}{2}\cdot\frac{1}{2}$

$$\Rightarrow \quad f'(x)=\frac{1}{2}\text{cosec}^2\frac{x}{2}$$

So $I=\int e^x\left(f(x)+f'(x)\right)$

$$=e^x f(x)+c$$

$$=e^x\left(-\cot\frac{x}{2}\right)+c$$

$$=-\,e^x\cot\frac{x}{2}+c.$$

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