Evaluate the following integrals:

Question # 1

$$\int \frac{-2x}{\sqrt{4-x^2}}\,dx$$

Solution

Let $I = \int \frac{-2x}{\sqrt{4-x^2}}\,dx$

Put $t = 4 - x^2 \quad \Rightarrow \quad dt = -2x\;dx$

So $I = \int \frac{dt}{\sqrt{t}} \quad = \int (t)^{-\frac{1}{2}}\,dt$

$$= \frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \quad = \frac{(t)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{t} + c \quad = 2\sqrt{4-x^2} + c$$

Important Integrals

Since $\frac{d}{dx} Tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$

By Integrating, we have

$$Tan^{-1}\left(\frac{x}{a}\right) = \int \frac{a}{a^2+x^2}\,dx$$

$$= a \cdot \int \frac{1}{a^2+x^2}\,dx$$

$$\Rightarrow \boxed{\int \frac{1}{a^2+x^2}\;dx = \frac{1}{a} Tan^{-1}\left(\frac{x}{a}\right)}$$

Similarly

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\frac{x}{a}}$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} Sec^{-1}\frac{x}{a}}$$

Question # 2

$$\int \frac{dx}{x^2+4x+13}$$

Solution

Let $I = \int \frac{dx}{x^2+4x+13}$

$$= \int \frac{dx}{x^2 + 2(x)(2) + (2)^2 - (2)^2 + 13}$$

$$= \int \frac{dx}{(x+2)^2 - 4 + 13}$$

$$= \int \frac{dx}{(x+2)^2 + 9} \quad = \int \frac{dx}{(x+2)^2 + (3)^2}$$

Put $t = x + 2 \quad \Rightarrow \quad dt = dx$

So $I = \int \frac{dt}{t^2 + 3^2}$

$$= \frac{1}{3} Tan^{-1}\frac{t}{3} + c$$

$$= \frac{1}{3} Tan^{-1}\frac{x+2}{3} + c$$

Question # 3

$$\int \frac{x^2}{4+x^2}\,dx$$

Solution

$$\int \frac{x^2}{4+x^2}\,dx$$

$$\begin{array}{r|l} & 1 \\ 4+x^2 & x^2 \\ & -x^2 - 4 \\ \hline & -4 \end{array}$$

$$= \int \left(1 - \frac{4}{4+x^2}\right) dx$$

$$= \int dx - 4\int \frac{dx}{4+x^2}$$

$$= x - 4\int \frac{dx}{(2)^2 + x^2}$$

$$= x - 4 \cdot \frac{1}{2} Tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= x - 2\;Tan^{-1}\left(\frac{x}{2}\right) + c$$

Question # 4

$$\int \frac{1}{x \ln x}\,dx$$

Solution

Suppose $I = \int \frac{1}{x \ln x}\,dx$

$$= \int \frac{1}{\ln x} \cdot \frac{1}{x}\;dx$$

Put $t = \ln x \quad \Rightarrow \quad dt = \frac{1}{x}dx$

So $I = \int \frac{1}{t}dt \quad = \ln|t| + c$

$$= \ln|\ln x| + c$$

Question # 5

$$\int \frac{e^x}{e^x+3}dx$$

Solution

Suppose $I=\int \frac{e^x}{e^x+3}dx$

Put $t=e^x+3 \Rightarrow dt=e^x dx$

So $I=\int \frac{dt}{t}=\ln|t|+c$

$=\ln|e^x+3|+c$

Question # 6

$$\int \frac{x+b}{\left(x^2+2bx+c\right)^{\frac{1}{2}}}dx$$

Solution

Let $I=\int \frac{x+b}{\left(x^2+2bx+c\right)^{\frac{1}{2}}}dx$

Put $t=x^2+2bx+c$

$\Rightarrow dt=(2x+2b)dx \Rightarrow dt=2(x+b)dx$

$\Rightarrow \frac{1}{2}dt=(x+b)dx$

So $I=\int \frac{\frac{1}{2}dt}{t^{\frac{1}{2}}} = \frac{1}{2}\int t^{-\frac{1}{2}}\,dt$

$=\frac{1}{2}\cdot\frac{t^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)}+c_1 = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{1/2}+c_1$

$=\left(x^2+2bx+c\right)^{\frac{1}{2}}+c_1$

$=\sqrt{x^2+2bx+c}+c_1$

Question # 7

$$\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$$

Solution

Let $I=\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Put $t=\tan x \Rightarrow dt=\sec^2 x\,dx$

So $I=\int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}\,dt$

$=\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c = \frac{t^{\frac{1}{2}}}{1/2}+c$

$=2(\tan x)^{\frac{1}{2}}+c = 2\sqrt{\tan x}+c$

Important Integral

$\int \sec\theta\,d\theta = \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta$

$=\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta$

$=\int \frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}d\theta$

Take $t=\sec\theta+\tan\theta$

$\Rightarrow dt=\left(\sec^2\theta+\sec\theta\tan\theta\right)d\theta$

So $\int \sec\theta\,d\theta=\int \frac{1}{t}dt$

$=\ln|t|+c$

$=\ln|\sec\theta+\tan\theta|+c$

$\Rightarrow \boxed{\int \sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+c}$

Similarly

$\boxed{\int \operatorname{cosec}\theta\,d\theta=\ln|\operatorname{cosec}\theta-\cot\theta|+c}$

See proof at page 133

Question # 8

(a) Show that

$$\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+c$$

(b) Show that

$$\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}Sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$$

Solution

(a) Let $I=\frac{dx}{\sqrt{x^2-a^2}}$

Put $x=a\sec\theta \Rightarrow dx=a\sec\theta\tan\theta\,d\theta$

So $I=\int \frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{(a\sec\theta)^2-a^2}}$

$=\int \frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{a^2\left(\sec^2\theta-1\right)}}$

$=\int \frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{a^2\tan^2\theta}}$

$\because 1+\tan^2\theta=\sec^2\theta$

$=\int \frac{a\sec\theta\tan\theta\,d\theta}{a\tan\theta} = \int \sec\theta\,d\theta$

$=\ln|\sec\theta+\tan\theta|+c_1$

$=\ln\left|\sec\theta+\sqrt{\sec^2\theta-1}\right|+c_1$

$$= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+c_1$$

$x = a\sec\theta$

$\frac{x}{a} = \sec\theta$

$$= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right|+c_1$$

$$= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|-\ln a+c_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|+c$$

where $c = -\ln a + c_1$

(b) Let $I = \sqrt{a^2-x^2}\,dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

So $I = \int\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\ d\theta$

$$= \int\sqrt{a^2\left(1-\sin^2\theta\right)}\cdot a\cos\theta\ d\theta$$

$$= \int\sqrt{a^2\cos^2\theta}\cdot a\cos\theta\ d\theta \qquad \because 1-\sin^2\theta=\cos^2\theta$$

$$= \int a\cos\theta\cdot a\cos\theta\ d\theta$$

$$= a^2\int\cos^2\theta\, d\theta \qquad \because\ \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= a^2\int\frac{1+\cos 2\theta}{2}d\theta \quad = \frac{a^2}{2}\int\left(1+\cos 2\theta\right)d\theta$$

$$= \frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+c$$

$$= \frac{a^2}{2}\left(\theta+\frac{2\sin\theta\cos\theta}{2}\right)+c$$

$$= \frac{a^2}{2}\left(\theta+\sin\theta\sqrt{1-\sin^2\theta}\right)+c$$

$x = a\sin\theta$

$\frac{x}{a} = \sin\theta$

$\sin^{-1}\frac{x}{a} = \theta$

$$= \frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right)+c$$

$$= \frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right)+c$$

$$= \frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)+c$$

$$= \frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a^2}\sqrt{a^2-x^2}\right)+c$$

$$= \frac{a^2}{2}Sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$$

Evaluate the following integrals:

Question # 9

$$\int\frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$$

Solution

Let $I = \int\frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$

Put $x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\, d\theta$

$$I = \int\frac{\sec^2\theta\ d\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$= \int\frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{\frac{3}{2}}} \quad \because\ 1+\tan^2\theta = \sec^2\theta$$

$$= \int\frac{\sec^2\theta\, d\theta}{\sec^3\theta}$$

$$= \int\frac{d\theta}{\sec\theta} = \int\cos\theta\, d\theta \quad = \sin\theta + c$$

$$= \frac{\sin\theta}{\cos\theta}\cdot\cos\theta + c = \tan\theta\cdot\frac{1}{\sec\theta} + c$$

$$= \tan\theta\cdot\frac{1}{\sqrt{1+\tan^2\theta}} + c$$

$$= \frac{x}{\sqrt{1+x^2}}+c \qquad \because\ x = \tan\theta$$

Question # 10

$$\int\frac{1}{(1+x^2)\ Tan^{-1}x}dx$$

Solution

Let $I = \int\frac{1}{(1+x^2)\ Tan^{-1}x}dx$

$$= \int\frac{1}{Tan^{-1}x}\cdot\frac{1}{(1+x^2)}dx$$

Put $t = Tan^{-1}x \quad \Rightarrow \quad dt = \frac{1}{1+x^2}dx$

So $I = \int\frac{1}{t}dt \quad = \ln\left|t\right|+c$

$$= \ln\left|Tan^{-1}x\right|+c$$

Question # 11

$$\int\sqrt{\frac{1+x}{1-x}}\,dx$$

Solution

Let $I=\int\sqrt{\dfrac{1+x}{1-x}}\,dx$

Put $x=\sin\theta \quad\Rightarrow\quad dx=\cos\theta\,d\theta$

So $I=\int\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}\cdot\cos\theta\,d\theta$

$=\int\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}\cdot\dfrac{1+\sin\theta}{1+\sin\theta}}\cdot\cos\theta\,d\theta$

$=\int\sqrt{\dfrac{(1+\sin\theta)^2}{1-\sin^2\theta}}\cdot\cos\theta\,d\theta$

$=\int\sqrt{\dfrac{(1+\sin\theta)^2}{\cos^2\theta}}\cdot\cos\theta\,d\theta$

$=\int\dfrac{1+\sin\theta}{\cos\theta}\cdot\cos\theta\,d\theta \quad =\int(1+\sin\theta)\,d\theta$

$=\theta-\cos\theta+c$

$=\theta-\sqrt{1-\sin^2\theta}+c$

$=Sin^{-1}x-\sqrt{1-x^2}+c$

$\because\ x=\sin\theta$
$\therefore\ \sin^{-1}x=\theta$

Question # 12

$$\int\frac{\sin\theta}{1+\cos^2\theta}\,d\theta$$

Solution

Let $I=\int\dfrac{\sin\theta}{1+\cos^2\theta}\,d\theta$

Put $t=\cos\theta$

$\Rightarrow\ dt=-\sin\theta\,d\theta \quad\Rightarrow\quad -dt=\sin\theta\,d\theta$

So $I=\int\dfrac{-dt}{1+t^2} \quad =-\int\dfrac{dt}{1+t^2}$

$=-\tan^{-1}t+c$

$=-\tan^{-1}(\cos\theta)+c$

Question # 13

$$\int\frac{ax}{\sqrt{a^2-x^4}}\,dx$$

Solution

Let $I=\int\dfrac{ax}{\sqrt{a^2-x^4}}\,dx$

$=a\int\dfrac{x}{\sqrt{a^2-x^4}}\,dx$

Put $t=x^2$ then $t^2=x^4$

$dt=2x\,dx \quad\Rightarrow\quad \dfrac{1}{2}dt=x\cdot dx$

So $I=a\int\dfrac{\frac{1}{2}dt}{\sqrt{a^2-t^2}}$

$=\dfrac{a}{2}\int\dfrac{dt}{\sqrt{a^2-t^2}}$

$=\dfrac{a}{2}Sin^{-1}\left(\dfrac{t}{a}\right)+c \qquad \because\ \int\dfrac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\dfrac{x}{a}$

$=\dfrac{a}{2}Sin^{-1}\left(\dfrac{x^2}{a}\right)+c$

Question # 14

$$\int\frac{dx}{\sqrt{7-6x-x^2}}$$

Solution

Let $I=\int\dfrac{dx}{\sqrt{7-6x-x^2}}$

$=\int\dfrac{dx}{\sqrt{-\left(x^2+6x-7\right)}}$

$=\int\dfrac{dx}{\sqrt{-\left(x^2+2(3)(x)+(3)^2-(3)^2-7\right)}}$

$=\int\dfrac{dx}{\sqrt{-\left((x+3)^2-16\right)}}$

$=\int\dfrac{dx}{\sqrt{16-(x+3)^2}}$

Put $t=x+3 \quad\Rightarrow\quad dx=dt$

So $I=\dfrac{dt}{\sqrt{16-t^2}} \quad =\int\dfrac{dx}{\sqrt{(4)^2-(t)^2}}$

$=Sin^{-1}\left(\dfrac{t}{4}\right)+c$

$=Sin^{-1}\left(\dfrac{x+3}{4}\right)+c$

Question # 15

$$\int\frac{\cos x}{\sin x\cdot\ln\sin x}\,dx$$

Solution

Let $I=\int\dfrac{\cos x}{\sin x\cdot\ln\sin x}\,dx$

$=\int\dfrac{1}{\ln\sin x}\cdot\dfrac{\cos x}{\sin x}\,dx$

Put $t=\ln\sin x \quad\Rightarrow\quad dt=\dfrac{1}{\sin x}\cdot\cos x\,dx$

So $I=\int\dfrac{1}{t}dt \quad =\ln|t|+c$

$=\ln\left|\ln\sin x\right|+c$

Question # 16

$$\int \cos x\left(\frac{\ln \sin x}{\sin x}\right)dx$$

Solution

Let $I = \int \cos x\left(\frac{\ln \sin x}{\sin x}\right)dx$

$= \int \ln \sin x \cdot \frac{\cos x}{\sin x}\, dx$

Put $t = \ln \sin x \quad \Rightarrow \quad dt = \frac{1}{\sin x}\cdot \cos x\ dx$

No do yourself

Question # 17

$$\int \frac{x\,dx}{4+2x+x^2}$$

Solution

Let $I = \int \frac{x\,dx}{4+2x+x^2}$

$= \frac{1}{2}\int \frac{2x\ dx}{x^2+2x+4}$

+ing and −ing 2 in numerator.

$\Rightarrow \quad I = \frac{1}{2}\int \frac{(2x+2)-2}{x^2+2x+4}\, dx$

$= \frac{1}{2}\int\left(\frac{2x+2}{x^2+2x+4} - \frac{2}{x^2+2x+4}\right) dx$

$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}\, dx - \frac{1}{2}\int \frac{2}{x^2+2x+4}\, dx$

$= \frac{1}{2}\int \frac{\frac{d}{dx}\left(x^2+2x+4\right)}{x^2+2x+4}\, dx - \frac{2}{2}\int \frac{dx}{x^2+2x+1+3}$

$= \frac{1}{2}\ln\left| x^2+2x+4 \right| - \int \frac{dx}{(x+1)^2+\left(\sqrt{3}\right)^2}$

$= \frac{1}{2}\ln\left| x^2+2x+4 \right| - \frac{1}{\sqrt{3}}Tan^{-1}\frac{x+1}{\sqrt{3}} + c$

Question # 18

$$\int \frac{x}{x^4+2x^2+5}\, dx$$

Solution

Let $I = \int \frac{x}{x^4+2x^2+5}\, dx$

Put $t = x^2$ then $t^2 = x^4$

$dt = 2x\ dx \quad \Rightarrow \quad \frac{1}{2}dt = x\,dx$

So $I = \int \frac{\frac{1}{2}dt}{t^2+2t+5} \quad = \frac{1}{2}\int \frac{dt}{t^2+2t+1+4}$

$= \frac{1}{2}\int \frac{dt}{(t+1)^2+(2)^2}$

$= \frac{1}{2}\cdot\frac{1}{2}Tan^{-1}\left(\frac{t+1}{2}\right) + c$

$= \frac{1}{4}Tan^{-1}\left(\frac{x^2+1}{2}\right) + c$

Question # 19

$$\int \left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)\ dx$$

Solution

Let $I = \int \left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)\ dx$

Put $t = \sqrt{x} - \frac{x}{2}$

$\Rightarrow \quad dt = \left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}\right)dx \ \Rightarrow \ dt = \frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right)dx$

$\Rightarrow \quad 2\,dt = \left(\frac{1}{\sqrt{x}}-1\right)dx$

So $I = \int \cos t \cdot 2\,dt$

$= 2\int \cos t\ dt$

$= 2\sin t + c$

Question # 20

$$\int \frac{x+2}{\sqrt{x+3}}\, dx$$

Solution

Let $I = \int \frac{x+2}{\sqrt{x+3}}\, dx$

Put $t = x+3$ then $t-3 = x$

$\Rightarrow \quad dt = dx$

So $I = \int \frac{t-3+2}{\sqrt{t}}\,dx$

$= \int \frac{t-1}{(t)^{\frac{1}{2}}}\, dx \quad = \int \left(\frac{t}{(t)^{\frac{1}{2}}} - \frac{1}{(t)^{\frac{1}{2}}}\right) dx$

$= \int \left((t)^{\frac{1}{2}} - (t)^{-\frac{1}{2}}\right) dx$

$= \frac{(t)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{(t)^{\frac{3}{2}}}{3/2} - \frac{(t)^{\frac{1}{2}}}{1/2} + c$

$= \frac{2(x+3)^{\frac{3}{2}}}{3} - 2(x+3)^{\frac{1}{2}} + c$

$$= \frac{2(x+3)^{\frac{3}{2}}}{3} - 2\sqrt{x+3} + c$$

Question # 21

$$\int \frac{\sqrt{2}}{\sin x + \cos x} dx$$

Solution

Let $I = \int \frac{\sqrt{2}}{\sin x + \cos x} dx$

$$= \int \frac{1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} dx$$

$$= \int \frac{1}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x} dx$$

Put $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

So $I = \int \frac{1}{\sin\frac{\pi}{4}\cdot\sin x + \cos\frac{\pi}{4}\cdot\cos x} dx$

$$= \int \frac{1}{\cos\left(x - \frac{\pi}{4}\right)} dx \quad = \int \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \ln\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right| + c$$

Question # 22

$$\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$$

Solution

Let $I = \int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$

$\because \cos\frac{\pi}{3} = \frac{1}{2}$ & $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore I = \int \frac{dx}{\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x}$

$$= \int \frac{dx}{\sin\left(x + \frac{\pi}{3}\right)} = \int \operatorname{cosec}\left(x + \frac{\pi}{3}\right) dx$$

$$= \ln\left|\operatorname{cosec}\left(x + \frac{\pi}{3}\right) - \cot\left(x + \frac{\pi}{3}\right)\right| + c$$

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