Mathcity

Merging man and maths

Version: 1.0

EXERCISE 9.4

Solve (problem 1-10)

Question # 1: $(3x^2 + 4xy\ )dx + (2x^2 + 2y)dy = 0$

Solution:-

Given equation is

$$(3x^2 + 4xy\ )dx + (2x^2 + 2y)dy = 0 - - - (1)$$

Here,

$M = 3x^2 + 4xy$	$N = 2x^2 + 2y$
$M_y = \frac{\partial M}{\partial y} = 4x$	$N_x = \frac{\partial N}{\partial x} = 4x$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = 3x^2 + 4xy - - - (a)$$

$$\frac{\partial f}{\partial y} = N = 2x^2 + 2y - - - (b)$$

Integrating (b) w.r.t "y", we have

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$$f(x,y) = 2x^2 \int 1dy + 2\int ydy$$

$$\Longrightarrow f(x,y) = 2x^2y + y^2 + h(x) - - - (c)$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "x", we have

$$\frac{\partial f}{\partial x} = 4xy + \frac{\partial h}{\partial x} - - - (d)$$

Comparing (a) & (d), we have

$$3x^2 + 4xy = 4xy + \frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial x} = 3x^2$$

Integrating both sides $w.r.t$ "x", we have

$$h = x^3$$

Thus equation (c) becomes

$$f(x,y) = 2x^2y + y^2 + x^3$$

Hence the general solution of (1) is

$$2x^2y + y^2 + x^3 = c$$

is required solution.

Question # 2: $\boldsymbol{(2xy + y + \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy}$**=0**

Solution:-

Given equation is

$$(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0 - - - (1)$$

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Here,

$M = 2xy + y - \tan y$	$N = x^2 - x\tan^2 y + \sec^2 y$
$M_y = \frac{\partial}{\partial y}(2xy + y - \tan y)$	$N_x = \frac{\partial}{\partial x}(x^2 - x\tan^2 y + \sec^2 y)$
$M_y = 2x + 1 - sec^2 y$	$N_x = 2x - \tan^2 y$
$M_y = 2x + 1 - sec^2 y$	$N_x = 2x + 1 - \sec^2 y$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = 2xy + y - \tan y \quad ---(a)$$

$$\frac{\partial f}{\partial y} = N = x^2 - x\tan^2 y + \sec^2 y \quad ---(b)$$

Integrating (a) w.r.t "x", we have

$$f(x,y) = \int (2xy + y + \tan y)dx$$

$$\Rightarrow f(x,y) = 2y\frac{x^2}{2} + yx - tan\, y\,.\,x$$

$$\Rightarrow f(x,y) = x^2y + xy - x\tan y + h(y) ---(c)$$

Here $h(y)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "y", we have

$$\frac{\partial f}{\partial y} = x^2 + x - x\sec^2 y + \frac{\partial h}{\partial y} ---(d)$$

Comparing (b) & (d), we have

$$x^2 - x\tan^2 y + \sec^2 y = x^2 + x - x\sec^2 y + \frac{\partial h}{\partial y}$$

$$\Rightarrow -x\tan^2 y + \sec^2 y = x - x - -x\tan^2 y + \frac{\partial h}{\partial y}$$

$$\Rightarrow \frac{\partial h}{\partial y} = \sec^2 y$$

Integrating both sides $w.r.t$ "y", we have

$$h = \tan y$$

Thus equation (c) becomes

$$f(x,y) = x^2y + xy - x\tan y + \tan y$$

Hence the general solution of (1) is

$$x^2y + xy - x\tan y + \tan y = c$$

$$\Rightarrow x^2y + xy + (1-x)\tan y = c$$

is required solution.

Question # 3: $\boldsymbol{\frac{x+y}{y-1}\,dx - \frac{1}{2}\left(\frac{x+1}{y-1}\right)^2 = 0}$

Solution:-

Given equation is

$$\frac{x+y}{y-1}\,dx - \frac{1}{2}\left(\frac{x+1}{y-1}\right)^2 = 0 - - - (1)$$

Here,

$$M = \frac{\boldsymbol{x+y}}{\boldsymbol{y-1}}$$

$$M_y = \frac{\partial}{\partial y}\left(\frac{x+y}{y-1}\right)$$

$$N = -\frac{1}{2}\left(\frac{x+1}{y-1}\right)^2$$

$$N_x = \frac{\partial}{\partial x}\left[\frac{-1}{2}\left(\frac{x+1}{y-1}\right)^2\right]$$

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$$M_y = \frac{(y-1)(1)-(x+y)(1)}{(y-1)^2} \qquad N_{x=} \frac{-1}{2}\frac{\partial}{\partial x}\ \frac{(x+1)^2}{(y-1)^2}$$

$$M_y = \frac{y-1-x-y}{(y-1)^2} \qquad N_{x=} \frac{-1}{2(y-1)^2}\frac{\partial}{\partial x}(x+1)^2$$

$$M_y = -\frac{(x+1)}{(y-1)^2} \qquad N_{x=} \frac{-1}{2(y-1)^2}\ 2(x+1)$$

$$M_y = -\frac{(x+1)}{(y-1)^2} \qquad N_{x=} = -\frac{(x+1)}{(y-1)^2}$$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = \frac{\boldsymbol{x+y}}{\boldsymbol{y-1}} \quad ---(a)$$

$$\frac{\partial f}{\partial y} = N = -\frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\boldsymbol{x+1}}{\boldsymbol{y-1}}\right)^{\mathbf{2}} ---(b)$$

Integrating (a) w.r.t "x", we have

$$f(x,y) = \frac{1}{(y-1)}\int (x+y)\,dx$$

$$\Longrightarrow f(x,y) = \frac{1}{(y-1)}\left(\frac{x^2}{2} + xy\right) + h(y)$$

$$\Longrightarrow f(x,y) = \frac{x^2+2xy}{2(y-1)} + h(y) ---(c)$$

Here $h(y)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "y", we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2+2xy}{2(y-1)}\right) + \frac{\partial h}{\partial y}$$

$$\frac{\partial f}{\partial y}=\frac{1}{2}\left[\frac{(y-1)(2x)-(x^2+2xy)}{(y-1)^2}\right]+\frac{\partial h}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial y}=\left[\frac{2xy-2x-x^2-2xy}{2(y-1)^2}\right]+\frac{\partial h}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial y}=\frac{1}{2(y-1)^2}(-x^2-2x)+\frac{\partial h}{\partial y}---(d)$$

Comparing (b) & (d), we have

$$\frac{(-x^2-2x)}{2(y-1)^2}+\frac{\partial h}{\partial y}=-\ \frac{1}{2}\left(\frac{x+1}{y-1}\right)^2$$

$$\Rightarrow \frac{-(x^2+2x)}{2(y-1)^2}+\frac{\partial h}{\partial y}=-\ \frac{1}{2}\left(\frac{x^2+2x+1}{(y-1)^2}\right)$$

$$\Rightarrow \frac{-(x^2+2x)}{2(y-1)^2}+\frac{\partial h}{\partial y}=\frac{-(x^2+2x)}{2(y-1)^2}-\frac{1}{2(y-1)^2}$$

$$\Rightarrow \frac{\partial h}{\partial y}=-\ \frac{1}{2(y-1)^2}$$

Integrating both sides $w.r.t$ "y", we have

$$h(y)=-\ \frac{1}{2}\left[\frac{-1}{(y-1)}\right]$$

$$\Rightarrow h(y)=\frac{1}{2(y-1)}$$

Thus equation (c) becomes

$$f(x,y)=\frac{x^2+2xy}{2(y-1)}+\frac{1}{2(y-1)}$$

$$f(x,y)=\frac{x^2+2xy+1}{2(y-1)}$$

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Hence the general solution of (1) is

$$\frac{x^2+2xy+1}{2(y-1)}=c$$

$$\Longrightarrow x^2+2xy+1=2c(y-1)$$

$$\Longrightarrow x^2+2xy+1=c(y-1) \because 2c=c \text{ (a constant)}$$

is required solution.

Question # 4: $\frac{dy}{dx}=\frac{-(ax+hy)}{hx+by}$

Solution:-

Given equation is

$$\frac{dy}{dx}=\frac{-(ax+hy)}{hx+by}---(1)$$

$$\Longrightarrow (hx+by)dy=-(ax+hy)dx$$

$$\Longrightarrow (hx+by)dy+(ax+hy)dx=0$$

$$\Longrightarrow (ax+hy)dx+(hx+by)dy=0$$

Here,

$M=ax+hy$	$N=(hx+by)$
$M_y=\frac{\partial}{\partial y}(ax+hy)$	$N_x=\frac{\partial}{\partial x}(hx+by)$
$M_y=h$	$N_x=h$

$\because M_y=N_x$. Therefore, given equation is exact.

Now

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$$\frac{\partial f}{\partial x} = M = ax + hy \quad ---(a)$$

$$\frac{\partial f}{\partial y} = N = hx + by ---(b)$$

Integrating (a) w.r.t "x", we have

$$f(x,y) = \int (ax + hy)dx$$

$$\Rightarrow f(x,y) = \frac{ax^2}{2} + hxy + h(y) ---(c)$$

Here $h(y)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "y", we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{ax^2}{2} + hxy\right) + \frac{\partial h}{\partial y}$$

$$\frac{\partial f}{\partial y} = hx + \frac{\partial h}{\partial y} ---(d)$$

Comparing (b) & (d), we have

$$hx + \frac{\partial h}{\partial y} = hx + by$$

$$\Rightarrow \frac{\partial h}{\partial y} = by$$

Integrating both sides $w.r.t$ "y", we have

$$h(y) = \frac{by^2}{2}$$

Thus equation (c) becomes

$$f(x,y)=\frac{ax^2}{2}+hxy+\frac{by^2}{2}$$

Hence the general solution of (1) is

$$\frac{ax^2}{2}+hxy+\frac{by^2}{2}=c$$

$$\Longrightarrow ax^2+2hxy+by^2=2c$$

$$\Longrightarrow ax^2+2hxy+by^2=2c \because 2c=c(\text{a constant})$$

is required solution.

Question # 5:

$$(\mathbf{1}+\ln \boldsymbol{xy})\boldsymbol{dx}+\left(\mathbf{1}+\frac{\boldsymbol{x}}{\boldsymbol{y}}\right)\boldsymbol{dy}$$

Solution:-

Given equation is

$$(1+\ln xy)dx+\left(1+\frac{x}{y}\right)dy=0---(1)$$

Here,

$$M=1+\ln xy \qquad N=1+\frac{x}{y}$$

$$M_y=\frac{\partial}{\partial y}(1+\ln xy) \qquad N_x=\frac{\partial}{\partial x}\left(1+\frac{x}{y}\right)$$

$$M_y=\frac{1}{xy}(x) \qquad N_x=\frac{1}{y}$$

$\because M_y=N_x$. Therefore, given equation is exact.

Now,

$$\frac{\partial f}{\partial x}=M=1+\ln xy \quad ---(a)$$

$$\frac{\partial f}{\partial y} = N = 1 + \frac{x}{y} - - - (b)$$

Integrating (b) w.r.t "y", we have

$$f(x, y) = y + x\int \frac{dy}{y} + h$$

$$\Rightarrow f(x, y) = y + x \ln y + h(x) - - - (c)$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "x", we have

$$\frac{\partial f}{\partial x} = \ln y + \frac{\partial h}{\partial x} - - - (d)$$

Comparing (a) & (d), we have

$$\ln y + \frac{\partial h}{\partial x} = 1 + \ln xy$$

$$\Rightarrow \ln y + \frac{\partial h}{\partial x} = 1 + \ln x + \ln y$$

$$\Rightarrow \frac{\partial h}{\partial x} = 1 + \ln x$$

$$\Rightarrow \frac{\partial h}{\partial x} = 1 + \ln x$$

Integrating both sides $w.r.t$ "x", we have

$$\int \frac{\partial h}{\partial x} dx = \int (1 + \ln x) dx$$

$$\Rightarrow h(x) = \int 1\, dx + \int \ln x\, dx$$

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$$\Rightarrow h(x) = x + \left[x\ln x - \int x.\frac{1}{x}dx\right]$$

$$\Rightarrow h(x) = x + x\ln x - x$$

$$\Rightarrow h(x) = x\ln x$$

Thus equation (c) becomes

$$f(x,y) = y + x\ln y + x\ln x$$

Hence the general solution of (1) is

$$y + x\ln y + x\ln x = c$$

$$\Rightarrow y + x[\ln x + \ln y] = c$$

$$\Rightarrow y + x\ln xy = c$$

is required solution.

Question # 6: $\frac{ydx + xdy}{1-x^2y^2} + xdx = 0$

Solution:-

Given equation is

$$\frac{ydx + xdy}{1-x^2y^2} + xdx = 0 - - - (1)$$

$$\Rightarrow \frac{ydx}{1-x^2y^2} + \frac{xdy}{1-x^2y^2} + xdx = 0$$

$$\Rightarrow \left(\frac{y}{1-x^2y^2} + x\right)dx + \frac{xdy}{1-x^2y^2} = 0$$

Here,

$M = \frac{y}{1-x^2y^2} + x$ $N = \frac{xdy}{1-x^2y^2}$

$$M_y = \frac{\partial}{\partial y}\left(\frac{y}{1-x^2y^2} + x\right) \qquad N_x = \frac{\partial}{\partial x}\left(\frac{x}{1-x^2y^2}\right)$$

$$M_y = \frac{(1-x^2y^2).1-y(-2x^2y)}{(1-x^2y^2)^2} \qquad N_x = \frac{(1-x^2y^2).1-x(-2xy^2)}{(1-x^2y^2)^2}$$

$$M_y = \frac{1-x^2y^2+2x^2y^2}{(1-x^2y^2)^2} \qquad N_x = \frac{1-x^2y^2+2x^2y^2}{(1-x^2y^2)^2}$$

$$M_y = \frac{1+x^2y^2}{(1-x^2y^2)^2} \qquad N_x = \frac{1+x^2y^2}{(1-x^2y^2)^2}$$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = \frac{y}{1-x^2y^2} + x \quad ---(a)$$

$$\frac{\partial f}{\partial y} = N = \frac{xdy}{1-x^2y^2} \quad ---(b)$$

Integrating (b) w.r.t "y", we have

$$f(x,y) = \int \frac{x}{1-x^2y^2}dy$$

$$\Rightarrow f(x,y) = \frac{x}{x^2}\int \frac{dy}{\left(\frac{1}{x^2}\right)-y^2}$$

$$\Rightarrow f(x,y) = \frac{1}{x}\left[\frac{1}{2\left(\frac{1}{x}\right)}\ln\left(\frac{\frac{1}{x}+y}{\frac{1}{x}-y}\right)\right] + h(x)$$

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$$\Rightarrow f(x,y) = \frac{1}{2}\ln\left(\frac{1+xy}{1-xy}\right) + h(x)$$

$$\Rightarrow f(x,y) = \frac{1}{2}[\ln(1+xy) - \ln(1-xy)] + h(x) - - - (c)$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "x", we have

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{y}{1+xy} - \frac{(-y)}{1-xy}\right] + \frac{\partial h}{\partial x}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{y}{2}\left[\frac{1-xy+1+xy}{(1+xy)(1-xy)}\right] + \frac{\partial h}{\partial x}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{y}{1-x^2y^2} + \frac{\partial h}{\partial x} - - - (d)$$

Comparing (a) & (d), we have

$$\frac{y}{1-x^2y^2} + \frac{\partial h}{\partial x} = \frac{y}{1-x^2y^2} + x$$

$$\Rightarrow \frac{\partial h}{\partial x} = x$$

Integrating both sides $w.r.t$ "x", we have

$$h(x) = \frac{x^2}{2}$$

Thus equation (c) becomes

$$f(x,y) = \frac{1}{2}[\ln(1+xy) - \ln(1-xy)] + \frac{x^2}{2}$$

Hence the general solution of (1) is

$$\frac{1}{2}[\ln(1+xy) - \ln(1-xy)] + \frac{x^2}{2} = c$$

$$\Longrightarrow \ln\left(\frac{1+xy}{1-xy}\right) + x^2 = 2c$$

$$\Longrightarrow \ln\left(\frac{1+xy}{1-xy}\right) + x^2 = c \ \because 2c = c \text{ (a constant)}$$

is required solution.

Question # 7: $(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0$

Solution:-

Given equation is

$$(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0 - - - (1)$$

Here,

$M = (6xy + 2y^2 - 5)$

$M_y = \frac{\partial}{\partial y}(6xy + 2y^2 - 5)$

$M_y = 6x + 4y$

$N = (3x^2 + 4xy - 6)$

$N_x = \frac{\partial}{\partial x}(3x^2 + 4xy - 6)$

$N_x = 6x + 4y$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = 6xy + 2y^2 - 5 - - - (a)$$

$$\frac{\partial f}{\partial y} = N = 3x^2 + 4xy - 6 \ - - - (b)$$

Integrating (b) w.r.t "y", we have

$$f(x,y) = \int (3x^2 + 4xy - 6)dy$$

$$\Rightarrow f(x,y) = 3x^2y + 4x\frac{y^2}{2} - 6y + h(x)$$

$$\Rightarrow f(x,y) = 3x^2y + 2xy^2 - 6y + h(x) - - - (c)$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "x", we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^2y + 2xy^2 - 6y + h)$$

$$\Rightarrow \frac{\partial f}{\partial x} = 3y(2x) + 2y^2(1) + \frac{\partial h}{\partial x}$$

$$\Rightarrow \frac{\partial f}{\partial x} = 6xy + 2y^2 + \frac{\partial h}{\partial x} - - - (d)$$

Comparing (a) & (d), we have

$$6xy + 2y^2 + \frac{\partial h}{\partial x} = 6xy + 2y^2 - 5$$

$$\Rightarrow \frac{\partial h}{\partial x} = -5$$

Integrating both sides $w.r.t$ "x", we have

$$h = -5x$$

Thus equation (c) becomes

$$f(x,y) = 3x^2y + 2xy^2 - 6y - 5x$$

Hence the general solution of (1) is

$$3x^2y + 2xy^2 - 6y - 5x = c$$

or $3x^2y + 2xy^2 - 5x - 6y = c$

is required solution.

Question # 8: $(y\sec^2 x + \sec x\tan x)dx + (\tan x + 2y)dy = 0$

Solution:-

Given equation is

$$(y\sec^2 x + \sec x\tan x)dx + (\tan x + 2y)dy = 0 - - - (1)$$

$$M = (y\sec^2 x + \sec x\tan x)$$

$$M_y = \frac{\partial}{\partial y}(y\sec^2 x + \sec x\tan x)$$

$$M_y = \sec^2 x\,(1)$$

$$M_y = \sec^2 x$$

$$N = (\tan x + 2y)$$

$$N_x = \frac{\partial}{\partial x}(\tan x + 2y)$$

$$N_x = \frac{\partial}{\partial x}(\tan x + 2y)$$

$$N_x = \sec^2 x$$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = y\sec^2 x + \sec x\tan x - - - (a)$$

$$\frac{\partial f}{\partial y} = N = \tan x + 2y - - - (b)$$

Integrating (b) w.r.t "y", we have

$$f(x,y) = \int (\tan x + 2y)dy$$

$$\Rightarrow f(x,y) = \int \tan x\,dy + 2\int y\,dy$$

$$\Rightarrow f(x,y) = y\tan x + \frac{2y^2}{2} + h(x)$$

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$$\Rightarrow f(x,y) = y\tan x + y^2 + h(x) - - - (c)$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "x", we have

$$\frac{\partial f}{\partial x} = y\sec^2 x + \frac{\partial h}{\partial x} - - - (d)$$

Comparing (a) & (d), we have

$$y\sec^2 x + \frac{\partial h}{\partial x} = y\sec^2 x + \sec x \tan x$$

$$\Rightarrow \frac{\partial \mathrm{h}}{\partial \mathrm{x}} = sec\, x\, tan\, x$$

Integrating both sides $w.r.t$ "x", we have

$$h = \mathrm{secx}$$

Thus equation (c) becomes

$$f(x,y) = y\tan x + y^2 + \mathrm{secx}$$

Hence the general solution of (1) is

$$y\tan x + y^2 + \mathrm{secx} = \mathrm{c}$$

is required solution.

Question # 9: $(\boldsymbol{y}\cos\boldsymbol{x} + \boldsymbol{2xe^y})\boldsymbol{dx} + (\sin\boldsymbol{x} + \boldsymbol{x^2e^y} - \boldsymbol{1})\mathbf{dy} = \mathbf{0}$

Solution:-

Given equation is

$$(y\cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)\mathrm{dy} = 0 - - - (1)$$

Here,

$$M = (y\cos x + 2xe^{y}) \qquad N = (\sin x + x^2 e^{y} - 1)$$

$$M_y = \frac{\partial}{\partial y}(y\cos x + 2xe^{y}) \qquad N_x = \frac{\partial}{\partial x}(\sin x + x^2 e^{y} - 1)$$

$$M_y = \cos x + 2xe^{y}\,(2x) \qquad N_x = \cos x + 2xe^{y}\,(2x)$$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = y\cos x + 2xe^{y} - - - (a)$$

$$\frac{\partial f}{\partial y} = \sin x + x^2 e^{y} - 1 - - - (b)$$

Integrating (a) w.r.t "x", we have

$$f(x,y) = y\sin x + 2e^{y}\frac{x^2}{2} + h(y)$$

$$\Longrightarrow f(x,y) = y\sin x + x^2 e^{y} + h(y) - - - (c)$$

Here $h(y)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "x", we have

$$\frac{\partial f}{\partial y} = \sin x + x^2 e^{y} + \frac{\partial h}{\partial y} - - - (d)$$

Comparing (b) & (d), we have

$$\sin x + x^2 e^{y} + \frac{\partial h}{\partial y} = \sin x + x^2 e^{y} - 1$$

$$\Longrightarrow \frac{\partial h}{\partial y} = -1$$

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Integrating both sides $w.r.t$ "x", we have

$$h = -y$$

Thus equation (c) becomes

$$f(x, y) = y\sin x + x^2e^y - y$$

Hence the general solution of (1) is

$$y\sin x + x^2e^y - y = \text{c}$$

is required solution.

Question # 10: $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\text{cos2x} - 3)dy = 0$

Solution:-

Given equation is

$$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\text{cos2x} - 3)dy = 0 - - - (1)$$

Here,

$M = ye^{xy}\, cos\, 2x - 2e^{xy}\, sin\, 2x + 2x$

$M_y = \dfrac{\partial}{\partial y}(ye^{xy}\, cos\, 2x - 2e^{xy}\, sin\, 2x\; 2x)$

$M_y = cos2x[ye^{xy}(x) + e^{xy}] - 2\, sin\, 2x\, [e^{xy}(x)]$

$M_y = xy\, e^{xy} cos2x + e^{xy} cos2x - 2xe^{xy}\, sin\, 2x$

$M_y = e^{xy}(xy\, cos2x + cos2x - 2x\ sin\, 2x)$

$N = xe^{xy}\text{cos2x} - 3$

$N_x = \dfrac{\partial}{\partial x}(xe^{xy}\text{cos2x} - 3)$

$N_x = e^{xy}\text{cos2x} + xye^{xy}\text{cos2x} - 2xe^{xy}\sin 2x$

$N_x = e^{xy}(\text{cos2x} + xy\text{cos2x} - 2x\sin 2x)$

$N_x = e^{xy}(xy\, cos2x + cos2x - 2x\ sin\, 2x)$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = ye^{xy}\, cos\, 2x - 2e^{xy}\, sin\, 2x + 2x - - - (a)$$

$$\frac{\partial f}{\partial y} = xe^{xy}\cos 2x - 3 - - - (b)$$

Integrating (b) w.r.t "y", we have

$$f(x,y) = \int (\, xe^{xy}cos2x - 3)dy$$

$$\Longrightarrow f(x,y) = \int xe^{xy}cos2xdy - 3\int dy$$

$$\Longrightarrow f(x,y) = xcos2x\left(\frac{e^{xy}}{x}\right) - 3y + h(x)$$

$$\Longrightarrow f(x,y) = e^{xy}cos2x - 3y + h(x) - - - (c)$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "x", we have

$$\frac{\partial f}{\partial x} = \cos 2x.\, ye^{xy} + e^{xy}(-2\sin 2x) + \frac{\partial h}{\partial x}$$

$$\Longrightarrow \frac{\partial f}{\partial x} = ye^{xy}\cos 2x - 2e^{xy}.\sin 2x + \frac{\partial h}{\partial x} - - - (d)$$

Comparing (a) & (d), we have

$$ye^{xy}\cos 2x - 2e^{xy}.\sin 2x + \frac{\partial h}{\partial x} = ye^{xy}\, cos\, 2x - 2e^{xy}\, sin\, 2x + 2x$$

$$\Longrightarrow \frac{\partial h}{\partial x} = 2x$$

Integrating both sides $w.r.t$ "x", we have

$$h(x) = x^2$$

Thus equation (c) becomes

$$f(x,y) = e^{xy}\cos 2x - 3y + x^2$$

Hence the general solution of (1) is

$$e^{xy}\cos 2x - 3y + x^2 = c$$

is required solution.

❖ Solve the initial value problem.

Question # 11: $(2x-3)dx + (x^2+4y)dy = 0, \quad y(1) = 2$

Solution:-

Given equation is

$$(2x-3)dx + (x^2+4y)dy = 0 ---(1)$$

Here,

$$M = 2x - 3 \qquad N = x^2 + 4y$$

$$M_y = \frac{\partial}{\partial y}(2x-3) \qquad N_x = \frac{\partial}{\partial x}(x^2+4y)$$

$$M_y = 2x \qquad N_x = 2x$$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = 2x - 3 ---(a)$$

$$\frac{\partial f}{\partial y} = x^2 + 4y ---(b)$$

Integrating (a) w.r.t "x", we have

$$f(x,y) = 2y\frac{x^2}{2} - 3x + h(y)$$

$$\Rightarrow f(x,y) = x^2y - 3x + h(y) - - - (c)$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "y", we have

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial h}{\partial y}$$

Comparing (b) & (d), we have

$$x^2 + \frac{\partial h}{\partial y} = x^2 + 4y$$

$$\Rightarrow \frac{\partial h}{\partial y} = 4y$$

Integrating both sides $w.r.t$ "y", we have

$$h = 2y^2$$

Thus equation (c) becomes

$$f(x,y) = x^2y - 3x + 2y^2$$

Hence the general solution of (1) is

$$x^2y - 3x + 2y^2 = \text{c} - - - (\text{e})$$

Applying the condition $y(1) = 2$ on (e), we have

$$(1)^2.2 - 3 + 2(2)^2 = c$$

$$\Rightarrow c = 7$$

Thus equation (e) becomes

$$x^2y - 3x + 2y^2 = 7$$

is required solution.

Question # 12:

$$\boldsymbol{(2x\cos y+3x^2y)dx+(x^3-x^2\sin y-y)dy=0,\qquad y(0)=2}$$

Solution:-

Given equation is

$$(2x\cos y+3x^2y)dx+(x^3-x^2\sin y-y)dy=0---(1)$$

Here,

$M=2x\cos y+3x^2y$	$N=x^3-x^2\sin y-y$
$M_y=\frac{\partial}{\partial y}(2x\cos y+3x^2y)$	$N_x=\frac{\partial}{\partial x}(x^3-x^2\sin y-y)$
$M_y=2x(-\sin y)+3x^2$	$N_x=3x^2-\sin y\,(2x)$
$M_y=-2x\sin y+3x^2$	$N_x=-2x\sin y+3x^2$

$\because M_y=N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x}=M=2x\cos y+3x^2y---(a)$$

$$\frac{\partial f}{\partial y}=N=x^3-x^2\sin y-y---(b)$$

Integrating (a) w.r.t "x", we have

$$f(x,y)=\int(2x\cos y+3x^2y)dx$$

$$\Rightarrow f(x,y)=\int 2x\cos y\,dx+\int 3x^2y dx$$

$$\Rightarrow f(x,y)=2\cos y\frac{x^2}{2}+3y\frac{x^3}{3}+h(y)$$

$$f(x,y)=x^2\cos y+x^3y+h(y)---(c)$$

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Here $h(y)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "y", we have

$$\frac{\partial f}{\partial y} = x^2(-\sin y) + x^3 + \frac{\partial h}{\partial y}$$

$$\frac{\partial f}{\partial y} = -x^2 \sin y + x^3 + \frac{\partial h}{\partial y} - - - (d)$$

Comparing (b) & (d), we have

$$-x^2 \sin y + x^3 + \frac{\partial h}{\partial y} = x^3 - x^2 \sin y - y$$

$$\Rightarrow \frac{\partial h}{\partial y} = -y$$

Integrating both sides $w.r.t$ "y", we have

$$h(y) = -\frac{y^2}{2}$$

Thus equation (c) becomes

$$f(x,y) = x^2\text{cosy} + x^3 y - \frac{y^2}{2}$$

Hence the general solution of (1) is

$$x^2\text{cosy} + x^3 y - \frac{y^2}{2} = \text{c} - - - (\text{e})$$

Applying the condition $y(0) = 2$ on (e), we have

$$0 + 0 - \frac{(2)^2}{2} = c$$

$$\Rightarrow c = -2$$

Thus equation (e) becomes

$$x^2\text{cosy} + x^3y - \frac{y^2}{2} = -2$$

is required solution.

Question # 13:

$$\mathbf{(3x^2y^2 + 2x)dx + (2x^3y - 2xy^2 + 1) = 0, \quad y(-2) = 1}$$

Solution:-

Given equation is

$$(3x^2y^2 + 2x)dx + (2x^3y - 2xy^2 + 1) = 0 - - - (1)$$

Here,

$$M = 3x^2y^2 + 2x$$

$$M_y = \frac{\partial}{\partial y}(3x^2y^2 + 2x)$$

$$M_y = 3x^2(2y)\text{-}3y^2$$

$$M_y = 6x^2y - 3y^2$$

$$N = 2x^3y - 2xy^2 + 1$$

$$N_x = \frac{\partial}{\partial x}(2x^3y - 2xy^2 + 1)$$

$$N_x = 2y(3x^2) - 3y^2(1)$$

$$N_x = 6x^2y - 3y^2$$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = 3x^2y^2 + 2x - - - (a)$$

$$\frac{\partial f}{\partial y} = N = 2x^3y - 2xy^2 + 1 - - - (b)$$

Integrating (b) w.r.t "y", we have

$$f(x,y) = \int (2x^3y - 2xy^2 + 1)dy$$

$$\Longrightarrow f(x,y) = 2x^3\frac{y^2}{2} - 3x\frac{y^3}{3} + y + h(x)$$

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$$\Rightarrow f(x,y) = x^3y^2 - xy^3 + y + h(x)$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "x", we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y^2 - xy^3 + y + h)$$

$$\frac{\partial f}{\partial x} = y^2(3x^2) - y^3 + \frac{\partial h}{\partial x}$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - y^3 + \frac{\partial h}{\partial x} - - - (d)$$

Comparing (a) & (d), we have

$$3x^2y^2 - y^3 + \frac{\partial h}{\partial x} = 3x^2y^2 + 2x$$

$$\Rightarrow \frac{\partial h}{\partial x} = 2x$$

Integrating both sides $w.r.t$ "y", we have

$$h = x^2$$

Thus equation (c) becomes

$$f(x,y) = x^3y^2 - xy^3 + y + x^2$$

Hence the general solution of (1) is

$$x^3y^2 - xy^3 + y + x^2 = \text{c} - - - (\text{e})$$

Applying the condition $y(-2) = 1$ on (e), we have

$$(-2)^3(1)^2 - 2(1) + 1 + (-2)^2 = c$$

$$\Rightarrow -8 + 2 + 1 + 4 = c$$

$$\Rightarrow c = -1$$

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Thus equation (e) becomes

$$x^3y^2 - xy^3 + y + x^2 = -1$$

$$\Longrightarrow x^3y^2 - xy^3 + y + x^2 + 1 = 0$$

is required solution.

Question # 14:

$$\left(\frac{3-y}{x^2}\right)dx - \left(\frac{y^2-2x}{xy^2}\right)dy = 0 \quad , y(-1) = 2$$

Solution:-

Given equation is

$$\left(\frac{3-y}{x^2}\right)dx - \left(\frac{y^2-2x}{xy^2}\right)dy = 0 \;---(1)$$

***h**ere,*

$M = \frac{3-y}{x^2}$	$N = \frac{y^2-2x}{xy^2}$
$M_y = \frac{\partial}{\partial y}\left(\frac{3-y}{x^2}\right)$	$N_x = \frac{\partial}{\partial x}\left(\frac{y^2-2x}{xy^2}\right)$
$M_y = \frac{1}{x^2}(-1)$	$N_x = \frac{-1}{x^2}$
$M_y = \frac{-1}{x^2}$	$N_x = \frac{-1}{x^2}$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = \frac{3-y}{x^2} \;---(a)$$

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$$\frac{\partial f}{\partial y} = N = \frac{y^2 - 2x}{xy^2} - - - (b)$$

Integrating (a) w.r.t "x", we have

$$f(x,y) = (3-y)\int \frac{1}{x^2}dx$$

$$\Rightarrow f(x,y) = -\frac{3-y}{x} + h(y) - - - (c)$$

Here $h(y)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "y", we have

$$\frac{\partial f}{\partial y} = (-1)\frac{-1}{x} + \frac{\partial h}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{1}{x} + \frac{\partial h}{\partial y} - - - (d)$$

Comparing (b) & (d), we have

$$\frac{1}{x} + \frac{\partial h}{\partial y} = \frac{y^2 - 2x}{xy^2}$$

$$\Rightarrow \frac{\partial h}{\partial y} = -\frac{2}{y^2}$$

Integrating both sides $w.r.t$ "y", we have

$$h(y) = (-2)\left(\frac{-1}{y}\right)$$

$$\Rightarrow h(y) = \frac{2}{y}$$

Thus equation (c) becomes

$$f(x,y) = (3-y)\frac{-1}{x} + \frac{2}{y}$$

$$\Rightarrow f(x,y) = \frac{y-3}{x} + \frac{2}{y}$$

Hence the general solution of (1) is

$$\frac{y-3}{x} + \frac{2}{y} = \mathrm{c} - - - (\mathrm{e})$$

Applying the condition $y(-1) = 2$ on (e), we have

$$\Rightarrow c = 2$$

Thus equation (e) becomes

$$\frac{y-3}{x} + \frac{2}{y} = 2$$

$$\Rightarrow y(y-3) + 2x = 2xy$$

$$\Rightarrow y^2 - 3y + 2x = 2xy$$

$$\Rightarrow 2x - 3y + y^2 = 2xy$$

is required solution.

Question # 15:

$\boldsymbol{(4x^3 e^{x+y} + x^4 e^{x+y} + 2x)dx + (x^4 e^{x+y} + 2y)dy = 0,} \qquad \boldsymbol{y(0) = 1}$

Solution:-

Given equation is

$$(4x^3 e^{x+y} + x^4 e^{x+y} + 2x)dx + (x^4 e^{x+y} + 2y)dy = 0, - - -(1)$$

Here,

$M = \boldsymbol{4x^3 e^{x+y} + x^4 e^{x+y} + 2x}$

$M_y = \frac{\partial}{\partial y}(4x^3 e^{x+y} + x^4 e^{x+y} + 2x)$

$\boldsymbol{N = x^4 e^{x+y} + 2y}$

$N_x = \frac{\partial}{\partial x}(x^4 e^{x+y} + 2y)$

$M_y = 4x^3e^{x+y} + x^4e^{x+y}$ $\quad\quad N_x = e^{x+y}(4x^3) + x^4e^{x+y}$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\frac{\partial f}{\partial x} = M = \boldsymbol{4x^3e^{x+y} + x^4e^{x+y} + 2x} --- (a)$$

$$\frac{\partial f}{\partial y} = N = \boldsymbol{x^4e^{x+y} + 2y} --- (b)$$

Integrating (b) w.r.t "y", we have

$$f(x,y) = \int (\boldsymbol{x^4e^{x+y} + 2y})\boldsymbol{dy}$$

$$f(x,y) = \int \boldsymbol{x^4e^{x+y}dy} + \int \boldsymbol{2y\,dy}$$

$$f(x,y) = \boldsymbol{x^4e^{x+y} + 2.\frac{y^2}{2} + h(x)}$$

$$f(x,y) = \boldsymbol{x^4e^{x+y} + y^2 + h(x)} --- (\boldsymbol{c})$$

Here $h(x)$ is the constant of integration.

Differentiating partially (c) $w.r.t$ "y", we have

$$\frac{\partial f}{\partial x} = \boldsymbol{x^4e^{x+y}} + 4x^3e^{x+y} + \frac{\partial h}{\partial y} --- (d)$$

Comparing (a) & (d), we have

$$\boldsymbol{x^4e^{x+y}} + 4x^3e^{x+y} + \frac{\partial h}{\partial y} = \boldsymbol{4x^3e^{x+y} + x^4e^{x+y} + 2x}$$

$$\Longrightarrow \frac{\partial h}{\partial y} = 2x$$

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Integrating both sides $w.r.t$ "y", we have

$$h = x^2$$

Thus equation (c) becomes

$$f(x,y) = \boldsymbol{x^4 e^{x+y}} + \boldsymbol{y^2} + x^2$$

Hence the general solution of (1) is

$$\boldsymbol{x^4 e^{x+y}} + \boldsymbol{y^2} + x^2 = \mathrm{c} - - - (\mathrm{e})$$

Applying the condition $y(0) = 1$ on (e), we have

$$\mathbf{0} + \mathbf{1} + 0 = \mathrm{c}$$

$$\Rightarrow c = 1$$

Thus equation (e) becomes

$$\boldsymbol{x^4 e^{x+y}} + \boldsymbol{y^2} + x^2 = 1$$

is required solution.

THE END.

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