INTEGRO-DIFFERENTIAL EQUATION

HISTORY:-

Integro-differential equation arise frequently as mathematical model in diverse discipline. The origin of the study of integral and integro-differential may be traced to the work of Abel, Lotka, Fredholm Malthus, Verhulst and Voltera on problems in mechanics and mathematical biology. From those beginnings, the theory and application of Voltera integro-differential with bounded and unbounded delays have emerged as new areas. The theory of linear Voltera integro-differential has been developing rapidly in the last three decades. Integro-differential equations are the combination of differential and Fredholm-Voltera equations that have the fractional order with constant co-efficients. of the form

$$u^{(n)}(x) = g(x) + \int_0^x K(x,t)\, u^{(s)}(t)\, dt$$

where $n \geq s$ $K(x,t)$ and $g(x)$ are know functions; and $u(x)$ is unknow function. Integro-differential equations established by Voltera.

In the early, 1900, Vito Voltera studied the phenemenon of population growth, and new types of equations have been developed and term as the integro-differential equations. In this type of equations, the unknown function $u(x)$ appears as the combination of the ordinary derivative and under the integral sign In the electrical engineering problem, the current $I(t)$

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flowing in a closed circuit can be obtained in the form of the following integro-differential equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)\,d\tau = f(t), \quad I(0) = I_0$$

where L is the inductance, R the resistance, C the capacitance, f(t) the applied voltage.

INTRODUCTION:-

The theory and application of integro-differential equations are important subjects in applied mathematics. In recent years there has been a growing interest in the integro-differential equations. The integro-differential equations be an important branch of modern mathematics. It arises frequently in many applied area which include engineering, electrostatis, mechanics, the theory of elasticity, potential, and the mathematical physics. The study of integro-differential equations started in the fifties with the works of Getoor, Blumenthal and Kac, among others. Integro-differential equation is a hybrid of integral and differential equations which have found extensive applications in sciences and engineering since it was established by Voltera.

First order integro-differential equation of the Voltera type is generally of the form:

$$y' = f(t, y(t), z(t)) \quad y(t_0) = y_0$$

where

$$z(t) = \int_{t_0}^{t} K(t, s, y(s))\,ds \quad ; t \in I$$

In solving above equation, we seek the unknown function y(t) given the Kernel K. This Kernal determines the nature of the solutions of integral equation depending on its type. Generally, methods for

solving both integral and differential equations. The Laplace transform method with the Adomian decomposition method to establish exact solutions of the nonlinear Voltera integro differential equations.
Since analytical solutions of these equations are difficult to obtain, much attentions have been invested in the search for effective methods for obtaining approximate or numerical solutions of both linear and non-linear integro-differential equations. Further, the integro-differential equations featuring non-linear terms that are more practical in reality are still difficult to solve numerically or approximately. Therefore, several numerical methods were used for the solutions of these types of equations such the (1) Galerkin method (2) Runge-Kutta method (3) Chebyshev collocation method (4) Taylor collection method (5) rationalized Haar functions method (6) Galerkin methods with hybrid functions (7) Adomian decomposition method (8) and modified homotopy perturbation method among others.

Some numerical approaches in literature include iterative methods, successive approximation method and other methods such as power series method where Chebyshev and Legendre's polynomials are used as basis functions have been applied to obtain solutions of some higher order integro-differential equations of linear type.
Mathematical modeling of real-life problems usually results in functional equations, e.g, partial differential equations, integral and integro-differential equations, stochastic equations and others. In particular, integro-differential equations arise in fluid dynamics, biological models and chemical Kinetics. The analytical solutions of some integro-differential equations cannot be found, thus numerical methods are required and have been extensily studied by many authors.

LITERATURE REVIEW:-

The term integro-differential equation in the literature is used in the case when the equation contains unknown function with its derivative and when either unknown or its derivatives, or both appear under an integral. In recent times, many methods have been derived for solving integro-differential equation. Let us recall the general classification of integro-differential equations. If the equation contain derivatives of unknown function of one variable then the integro-differential equation is called ordinary integro differential equation. Then the order of an equation is the same as the highest-order derivative of the unknown function in the equation
The integro-differential equations often encountered in mathematics and physics contain derivatives of various variables, therefore, these equations are called integro-differential equations with partial derivatives or partial integro-differential equations
In the applications very often there are integro-differential equations with partial derivative and multiple integrals as well for example, Boltzmann equation and Kolmogorov-Feller equation.
Voltera is one of the founders of the theory of integral and integro-differential equations. His works, especially, in the integral and integro-differential equations, are often cited till today. The classical book by Voltera is widely quoted in the literature. In 1884 Voltera began his research in to the theory of integral equations devoted to distribution of an electrical charge on a spherical patch.

The work on the theory of electricity became the beginning research of Voltera leading to the theory of partial integro-differential equations. In 1909 Voltera has studied a particular type of such equation is equivalent to a system consisting of three linear integral equations and a second order partial differential equations. The first examples of integro-differential equations with partial derivatives investigated in the beginning of the twentieth century were in Schlesinger's work, where the following equation is investigated

$$\frac{\partial U(x,y)}{\partial x} = \int_a^b f(x,y,s)\, U(x,s)\, ds$$

DEFINITION :-

Integro-differential equation is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function is called an integro-differential equation

For example

$$\frac{d^2u}{dx^2} = f(x) + \int_0^x K(x,t)\,u(t)\,dt$$

We note here that when Kernal $K(x,t)$ is a difference kernal, the right-hand side of above equation is amenable to the Laplace transformation, while the second derivative on the left-hand side needs the initial conditions $u(0)$ and $u'(0)$ for its Laplace transformation

CLASSIFICATION OF INTEGRO DE's :-

Integro-differential equations appears in many scientific applications especially when we convert initial value problems or boundary value problems to integral equations. The integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order. In classifying integro-differential equations.

1) Fredholm Integro-Differential Equations:-

Fredholm integro-differential equations appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$, $n \geq 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contain differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions.

Further, in Fredhom integral equations, the limit of integration are fixed and the unknown function $u(x)$ may appear only inside integral sign. The Fredhom integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$$

where $u^{(n)}$ indicate the n^{th} derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side.

Examples:-

Examples of the Fredholm integro-differential equation are given by

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 x\,u(t)\,dt \quad , \; u(0) = 0 \rightarrow ①$$

and

$$u''(x) + u'(x) = x - \sin x - \int_0^{\pi/2} xt\,u(t)\,dt .$$

$$u(0) = 0, \quad u'(0) = 1 \rightarrow ②$$

whereas equation ① is first kind of Fredholm integro-differential equation and the equation ② is the second kind of the Fredholm integro-differential equation.

2) Voltera Integro Differential Equations:-

Voltera integro-differential equations appear when we convert initial value problem to integral equations. The Voltera integro-differential equation contain the unknown function $u(x)$ and one of its derivatives $u^{(n)}, n \geq 1$ inside and outside of the integral sign. At least one of the limits of integration in this case is a variable as in the Voltera integral equations. The equation is called integro-differential because differential and integral operator are involved in the same equation. It is important to note that initial condition should be given for Voltera integro-differential equations to determine the particular solutions. The

Moreover we know already, in Voltera integral equations, the limit of integration are variable and the unknow function $u(x)$ appears only inside integral sign. The Voltera integro-differential equations appears in the form.

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)\,u(t)\,dt$$

where $u^{(n)}$ indicates the nth derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side.

Examples:-

Examples of the Volterra integro-differential equations are given by

$$u'(x) = -1 + \frac{1}{2}x^2 - xe^x - \int_0^x t\,u(t)\,dt, \quad u(0) = 0 \longrightarrow (3)$$

and

$$u''(x) + u'(x) = 1 - x(\sin x + \cos x) - \int_0^x t\,u(t)\,dt, \quad u(0) = -1, \; u'(0) = 1 \longrightarrow (4)$$

whereas equation ① is the first kind of Volterra integro-differential equation and the equation ② is the second kind of Volterra integro-differential equation.

3) Voltera-Fredholm Integro-Differential Equations:-

The Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators. The Volterra-Fredholm integro-differential equations appear in the literature in two forms, namely

$$u^{(n)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)\,dt + \lambda_2 \int_a^b K_2(x,t)\,u(t)\,dt, \longrightarrow (5)$$

and

$$u^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))\,d\xi\,d\tau, \quad (x,t) \in \Omega \times [0,T] \longrightarrow (6)$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$ and Ω is a closed subset of $\mathbb{R}^n$, $n = 1,2,3$. It is interesting to note that eq ⑤ contains disjoint Volterra and Fredholm integral equations, whereas equation (6) contain mixed integrals. Other derivatives of less order may appear as well. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral sign, the resulting equations are of first kind. Initial conditions should be given to the determine the particular solution.

Examples:-

Examples of the two types are given by following:

$$u'(x) = 24x + x^4 + 3 - \int_0^x (x-t)u(t)\,dt - \int_0^1 t\,u(t)\,dt, \quad u(0)=0 \rightarrow (7)$$

and

$$u'(x,t) = 1 + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)\,d\xi\,d\tau, \quad u(0,t) = t^3 \rightarrow (8)$$

Whereas eq (7) is a type of disjoint Volterra and Fredholm integral equations. and the eq (8) is a type of mixed Volterra-Fredholm integral equations.

LINEARITY AND HOMOGENEITY:-

Integral equation and integro-differential equations fall into two other types of classifications according to linearity and homogeneity concepts. These two concepts play a major role in the structure of the solutions. In what follows we highlight the definitions of these concepts.

⇒ Linearity Concept:-

If the exponent of the unknown function $u(x)$ inside the integral sign is one then the integro-differential equation is called <u>linear</u>. If the unknown function $u(x)$ has exponent other than one or if the equation contains non linear function of $u(x)$ such as e^u, $\sinh u$, $\cosh u$, $\cos u$, $\ln(1+u)$. then the integro-differential equation is called <u>non-linear</u>. To explain this concept, we consider the equation

$$u'(x) = 1 - \int_0^x (x-t)u(t)\,dt \longrightarrow (9)$$
$$u(0) = 1$$

$$u'(x) = 1 - \int_0^1 (x-t)\,u(t)\,dt \longrightarrow (10)$$
$$u(0) = 0$$

and

$$u'(x) = 1 + \int_0^x (1+x-t)\,u^4(t)\,dt \longrightarrow (11)$$
$$u(0) = 0$$

$$u'(x) = 1 + \int_0^1 xt\,e^{u(t)}\,dt \quad , \; u(0) = 1 \longrightarrow (12)$$

The first two examples are linear Voltera-Fredhdm integro-differential equations and last two equation (11) and (12) are non-linear Voltera-Fredholm integro-differential equation.

It is important to point out that the linear equation solutions are unique instead of non-linear. Non-linear equations usually give more than one solution and it is not easy to handle.

⇒ Homogeneity Concept:-

Integral equations and integro-differential equations of the second kind are classified as homogeneous or inhomogeneous, if the function $f(x)$ in the second kind of Voltera or Fredholm integro-differential equations is identically zero, the equation is called homogeneous. Otherwise it is called inhomogeneous. Notice that this property holds for equations of the second kind only. To clearify this concept we consider the following examples

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(11)

$$u'(x) = \sin x + \int_0^x xt\,u(t)dt \quad , u(0) = 0 \rightarrow (13)$$

$$u'(x) = x + \int_0^1 (x-t)^2 u(t)\,dt \quad , u(0) = 1 \rightarrow (14)$$

and

$$u'(x) = \int_0^x (1+x-t)u^4(t)\,dt \quad , u(0) = 0 \text{ --- } (15)$$

$$u''(x) = \int_0^x xt\,u(t), \quad u(0) = 1, u'(0) = 0, u(0) = 1 \text{ --- } (16)$$

The first two equations are inhomogeneous because $f(x) = \sin x$ and $f(x) = x$ whereas last two equations are homogeneous because $f(x) = 0$ for each equations.

FIGURES

⇒ MATHEMATICAL MODELS OF IDE :-

1) The movement Process Linear integro-differential Equation :-

- $N(x,t)$ is the density of a population at position x and at time t.
- At rate D, individuals move to a new location instantaneously
- $K(x-y)$ is the proportion of individuals moving from y to x

$$\partial_t N = D\int_R K(x-y)N(y,t)dy - DN$$

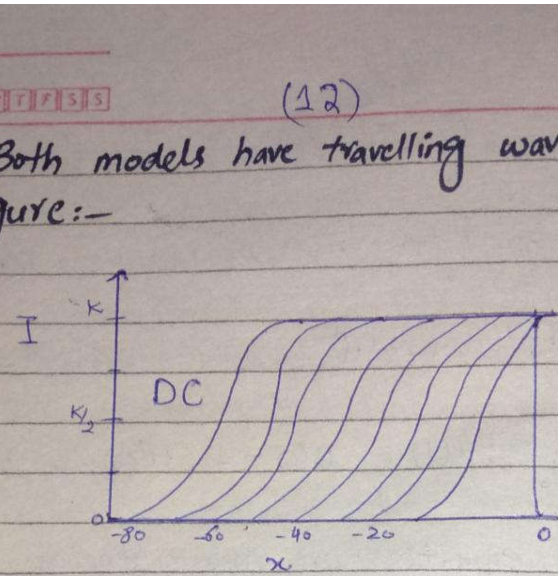
Position Jump Process :-

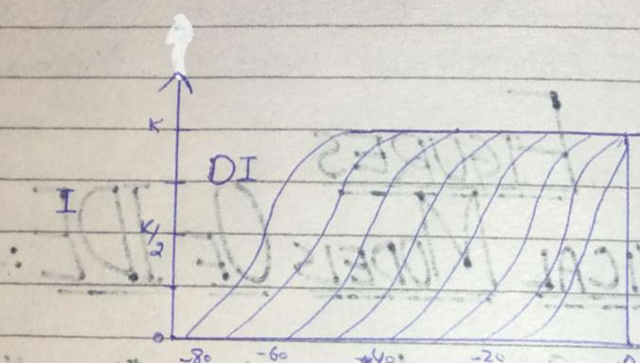
- An individual starts at $x=0$ at $t=0$
- He waits an exponentially-distributed time (with parameter D)
- And then jumps to a new position y that is governed by the distribution $K(x-y)$

$$\partial_t N = D\int_R K(x-y)N(y,t)dy - DN$$

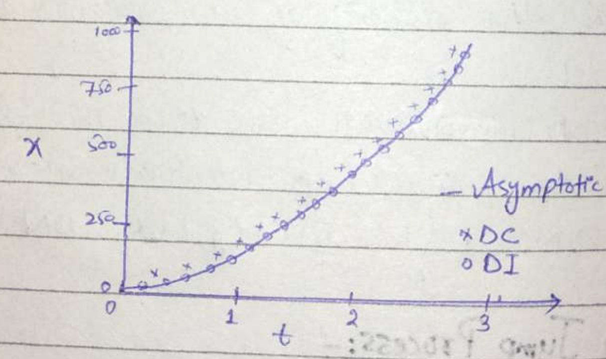
Both models have travelling wave Solution:-

Figure:-



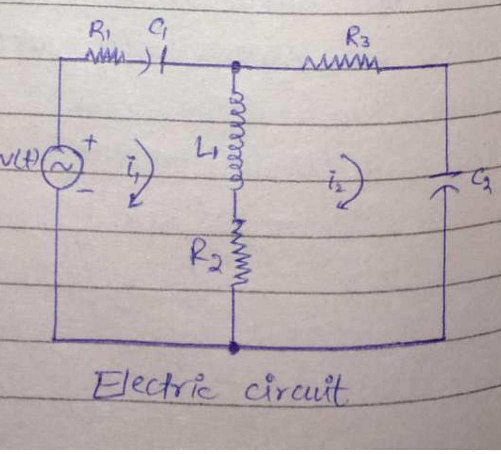


Front position vs. time:-



2)

An electric circuit obtain a set of simultaneous integro-differential equations representing the Network



Electric circuit

SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS:-

1) Laplace Transform Method:-

Methodology:-

Firstly we discuss the stepwise methodology which is used in all examples of integro- differential equation.

- Firstly, we apply Laplace transformation on both sides
- Secondly solve Laplace transformation, if Laplace transformation is occuring at differentiation at left hand side of integro-differential equation then solve Laplace transformation by putting Laplace derivatives formula.
- If Laplace occuring at an integral the we solve Laplace by using convolution product them.
- The use given initial conditions and separate the co-efficient values by using partial fraction method.
- Then separate the constant $U(s)$ at the left hand side.
- Lastly, taking the Laplace inverse to get the exact solution of the integro-differential equations.

Now, by following this methodology we solve several problems of integro-differential equation.

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Example #1:- Solve the second-order integro-differential equation

$$y''(x) = e^{x} - x + \int_0^1 xt\, y(t)\, dt \longrightarrow (1)$$

$$y(0) = 1 \; ; \; y'(0) = 1$$

Sol:-

Taking Laplace transform on both sides of eq (1)

$$\mathcal{L}\{y''(x)\}(s) = \mathcal{L}\{e^{x} - x\}(s) + \mathcal{L}\left\{\int_0^1 xt\, y(t)\, dt\right\}(s)$$

So that

$$s^2 Y(s) - sY(0) - Y'(0) = \frac{1}{s-1} - \frac{1}{s^2} + \frac{1}{s^2}\int_0^1 t\, y(t)\, dt$$

or

$$s^2 Y(s) - s(1) - 1 = \frac{1}{s-1} - \frac{1}{s^2} + \frac{1}{s^2}\int_0^1 t\, y(t)\, dt$$

$$\Rightarrow Y(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^2(s-1)} + \frac{1}{s^4}\int_0^1 t\, y(t)\, dt$$

where $\mathcal{L}\{y(x)\}(s) = Y(s)$. Substituting the series assumption for $Y(s)$ as given above equation

$$Y(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^2(s-1)}$$

Now Taking Laplace inverse transform of both sides of above equation, gives

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}$$

By simplifying we get exact solution

$$\boxed{y(x) = e^{x} - \frac{1}{3!}x^3}$$

which is exact solution of integro-differential equation

Example # 2:-

Solve the equation for the response $i(t)$, give that

$$\frac{di}{dt} + 2i + 5\int_0^t i\,dt = u(t)$$

and $i(0) = 0$

Sol:-

Firstly, we find the Laplace transform of each term in above equation.

$$\mathcal{L}\left\{\frac{di}{dt}\right\} + \mathcal{L}\{2i\} + 5\mathcal{L}\left\{\int_0^t i\,dt\right\} = \mathcal{L}\{u(t)\}$$

$$sI - i(0) + 2I + 5\frac{I}{s} = \frac{1}{s}$$

we multiply throughout by s and use the fact that $i(0) = 0$ to obtain

$$s^2 I + 2sI + 5I = 1$$

Solving for I and completing the square on the denominator

$$I = \frac{1}{s^2+2s+5} = \frac{1}{s^2+2s+4+1} = \frac{1}{(s+1)^2+4}$$

multiply & divide by 2

$$I = \frac{1}{2}\frac{2}{(s+1)^2+4}$$

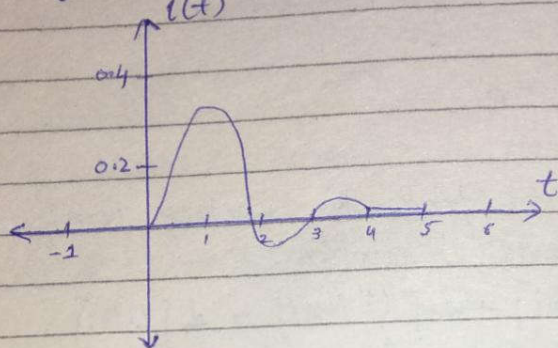
Now, Taking Laplace transform to give us the current as function of time

$$\mathcal{L}^{-1}\{I\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$\boxed{i = \frac{1}{2}e^{-t}\sin 2t}$$

This is the exact solution of integro-differential equation.

Solution graph:-
This is the graph of the solution we just found



Example # 3:-
Use Laplace transform to solve the integro-differential equation in $u(x)$ with the given initial condition

$$\frac{d^2u}{dx^2} - 2\frac{du}{dx} + u(x) = \cos x - 2\int_0^x \cos(x-t)\frac{d^2u}{dt^2}\,dt - 2\int_0^x \sin(x-t)\frac{du}{dt}\,dt$$

$$u(0) = 0, \quad u'(0) = 0$$

Sol:-
Taking Laplace on above equation on both sides

$$\mathcal{L}\left\{\frac{d^2u}{dx^2}\right\} - 2\mathcal{L}\left\{\frac{du}{dx}\right\} + \mathcal{L}\{u(x)\} = \mathcal{L}\{\cos x\} - 2\mathcal{L}\left\{\int_0^x \cos(x-t)\frac{d^2u}{dt^2}\,dt\right\}$$

$$- 2\mathcal{L}\left\{\int_0^x \sin(x-t)\frac{du}{dt}\,dt\right\}$$

$$s^2U(s) - s\cancelto{0}{U(0)} - \cancelto{0}{U'(0)} - 2U(s) + \cancel{2}\cancelto{0}{U(0)} + U(s) = \frac{s}{s^2+1} - 2\mathcal{L}\left\{\cos x * \frac{d^2u}{dt^2}\right\}$$

$$- 2\mathcal{L}\left\{\sin x * \frac{du}{dx}\right\}$$

($\because$ using convolution product theorem)

$$s^2U(s) - 2sU(s) + U(s) = \frac{s}{s^2+1} - 2\left(\frac{s}{s^2+1}\left(s^2U(s) - s\cancelto{0}{U(0)} - \cancelto{0}{U'(0)}\right)\right.$$

$$\left. - 2\left(\frac{1}{s^2+1}\left(sU(s) - \cancelto{0}{U(0)}\right)\right)\right.$$

$$s^2U(s) - 2sU(s) + U(s) = \frac{s}{s^2+1} - \frac{2s^3U(s)}{s^2+1} - \frac{2sU(s)}{s^2+1}$$

$$U(s)\left[s^2 - 2s + 1 + \frac{2s^3}{s^2+1} + \frac{2s}{s^2+1}\right] = \frac{s}{s^2+1}$$

$$U(s)\left[\frac{(s^2+1)(s^2-2s+1) + 2s^3 + 2s}{\cancel{(s^2+1)}}\right] = \frac{s}{\cancel{(s^2+1)}}$$

$$U(s)\left[s^4 - \cancel{2s^3} + s^2 + s^2 - \cancel{2s} + 1 + \cancel{2s^3} + \cancel{2s}\right] = s$$

$$U(s)\left[s^4 + 2s^2 + 1\right] = s$$

$$U(s) = \frac{s}{(s^2+1)^2} \qquad \left(\because \text{By}\left\{\mathcal{L}\{x^n f(t)\} = (-1)^n \frac{d^n}{ds^n} . F(s)\right)\right.$$

$$U(s) = \frac{2s}{2(s^2+1)^2} \qquad (\text{xply } \& \div \text{ by } 2)$$

$$U(s) = \frac{1}{2}\left[\frac{2s}{(s^2+1)^2}\right]$$

Applying Laplace inverse, we get

$$\mathcal{L}^{-1}\{U(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\}$$

$$u(x) = \frac{1}{2} x \sin x$$

$$\boxed{u(x) = \frac{x}{2}\sin x}$$

This is the exact solution of integro-differential equation.

Example # 4:-

Consider the following second order problem

$$u'(x) + 2u(x) + 5\int_0^x u(t)\,dt = \theta(x)$$

where

$$\theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

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is the heavy side step function. The Laplace transform is defined by

$$U(s) = \mathcal{L}\{u(x)\} = \int_0^{\infty} e^{-sx} u(x)\, dx$$

Upon taking term by term Laplace transforms and utilising the rules for derivatives and integrals, the integro-differential equation is converted into the following algebraic equation.

$$sU(s) - u(0) + 2U(s) + \frac{5}{s} U(s) = \frac{1}{s}$$

Thus

$$U(s) = \frac{1}{s^2 + 2s + 5}$$

Taking Laplace inverse transformation, we get

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+2s+5} \right\}$$

$$\boxed{u(x) = \frac{1}{2} e^{-x} \sin(2x)\, \theta(x)}$$

This is the exact solution of integro-differential equation.

Example # 5:-

Solve the following initial value problem associated with the integro-differential equation.

$$\frac{d^2u}{dx^2} = e^{2x} - \int_0^x e^{2(x-t)} \frac{du}{dt}\, dt$$

$$u(0) = 0, \quad u'(0) = 0$$

Sol:-

Taking Laplace transform on both sides of above equation, we get

$$\mathcal{L}\left\{\frac{d^2u}{dx^2}\right\} = \mathcal{L}\left\{e^{2x}\right\} - \mathcal{L}\left\{\int_0^x e^{2(x-t)}\frac{du}{dt}\,dt\right\}$$

$$s^2U(s) - s\cancel{u(0)} - \cancel{u'(0)} = \frac{1}{s-2} - \mathcal{L}\left\{e^{2u} * \frac{dx}{dt}\right\}$$

(Using convolution theorem)

$$s^2U(s) = \frac{1}{s-2} - \frac{1}{s-2}\left[sU(s) - \cancel{u(0)}\right]$$

$$s^2U(s) = \frac{1}{s-2} - \frac{sU(s)}{s-2}$$

$$U(s) = \frac{1}{s(s-1)^2}$$

Using partial fraction on the right hand side of and finding the values of co-efficients, we get

$$\frac{1}{s(s-1)^2} = \frac{1}{(s-1)^2} - \frac{1}{(s-1)} + \frac{1}{s}$$

Using inverting Laplace transform, we get

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\boxed{u(x) = xe^x - e^x + 1}$$

This is the exact solution of the integro-differential equation.

Numerical Methods:-

There are several numerical methods were used for the solutions of these type of equations.

⇒ Galerkin Method:-

Galerkin method is a powerful tool for solving many kinds of equation in various fields of Mathematics and engineering. It is one of the most important weighted residual methods invented by the Russian mathematician Boris Grigoryevich Galerkin. Various Galerkin algorithm applied in numerical solution of integral and integro-differential equations.

⇒ Taylor Collocation Method:-

Taylor collocation method is presented for numerically solving the linear integro-differential equations by truncated Taylor series, this method transforms the integro-differential equation to a matrix equation with corresponding the system of linear algebraic equations.

⇒ Rationalized Haar Functions method:-

Rationalized Haar function method is used to approximate the solutions of the integro differential equations. It is used to reduce the computation of integro-differential equations to some algebraic equations.

⇒ **Variational Iterative method:-**

Variational iterative method is used to solve the system of non-linear Vollera's integro-differential equations. This method provides a straightforward and powerful mathematical tool for solving various integro-differential equations.

⇒ **Chebyshev Collocation Method:-**

Chebyshev collocation method used to transforms the integro-differential equation to a matrix equation which corresponds to a system of linear algebraic equations

⇒ **Adomian decomposition method:-**

The adomian decomposition method is applied to solve both linear and non-linear boundary value problems for fourth-order integro-differential equation. The numerical result obtained with minimum amount of computation or mathematics compare reasonably well with exact solutions

⇒ **Homotopy Perturbation method:-**

Homotopy Perturbation method is used for an analytic and approximate treatment of non-linear integral and integro-differential equation. This method continuously deforms the difficult problem under study into a simple problem which is easy to solve.

⇒ **Modified Laplace adomian method:-**

This method is used to solve the integro-differential equations. This method based on numerically techniques basically illustrate how the laplace transform may be used to approximate the solution of the non-linear partial differential equations by multiplying decomposition method

COMPARISON WITH OTHER EXAMPLES:-

Here we see comparison of integro-differential equation by solving examples with some other method other than Laplace transform method and then we see that which method is best and easy to handle.
Here we discuss the example of variation iterative method.

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Example #1:- (Variational iterative Method)

Consider the following second-order Fredholm non-linear integro-differential equation

$$u''(x) = e^x - x + \int_0^1 x, t, u(t)\,dt$$

with initial conditions

$$u(0) = 1,\ u''(0) = 1$$

for $x \in [0,1]$ with the exact solution $u(x) = e^x$.

Sol:-

Making $u_{n+1}(x)$ stationary with respect to $u_n(x)$ we can identify the Lagrange multiplier, which reads $\lambda = s - x$. So we can construct a variational iteration form in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[u_n''(s) - e^s + s - \int_0^1 s p\, u(p)\,dp\right] ds$$

we begin with $u_0(x) = e^x(a+bx)$ where a & b are unknown constants to be further determined. By the iteration formulation, we have

$$u_1(x) = (a-1) + (a+b-1)x + \left(-\frac{1}{6} + \frac{1}{6}a - \frac{1}{3}b + \frac{1}{6}be\right)x^3 + e^x$$

If the first order approximation solution is enough, by the aid of the initial conditions we can identify the unknown constants as $a = 1$ and $b = 0$, so we obtain the following first order approximation solution

$$\boxed{u(x) = e^x}$$

which is the exact solution of the problem.

Example #2:- (Laplace Adomian Decomposition Method)

Consider the second-order non-linear integro-differential equation

$$u''(x) = \sinh(x) + x - \int_0^1 x\left(\cosh^2(t) - u^2(t)\right)dt,\quad u(0) = 0,\ u'(0) = 1$$

Sol:-

Applying the Laplace transform and by using the initial conditions we obtain

$$s^2U(s) - 1 = \frac{1}{s^2-1} + \frac{1}{s^2} - \mathcal{L}\left[\int_0^1 x(\cosh^2(t) - u^2(t))dt\right]$$

or

$$U(s) = \frac{1}{s^2} + \frac{1}{s^2(s^2-1)} + \frac{1}{s^4} - \frac{1}{s^2}\mathcal{L}\left[\int_0^1 x(\cosh^2(t) - u^2(t))dt\right]$$

Applying the inverse Laplace transform we get

$$u(x) = \sinh(x) + \frac{x^3}{6} - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left(x\int_0^1 (\cosh^2(t) - u^2(t))dt\right)\right]$$

Applying the same procedure

We decompose the solution as an infinite sum

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

using this relation in above equation we get modified recursive relation given below

$$u_0(x) = \sinh(x)$$

$$u_1(x) = \frac{x^3}{6} - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left(x\int_0^1 (\cosh^2(t) - u_0^2(t)\,dt\right)\right] = 0$$

$$u_{n+1}(x) = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left(x\int_0^1 (\cosh^2(t) - A_n(t))\,dt\right\} = 0,\ n \geq 1$$

in which $A_n = \sum_{j=0}^{n} u_j u_{n-j}$, where for every $n \geq 1$, $A_n = 0$

Thus the exact solution is

$$\boxed{u(x) = \sinh(x)}$$

* **Comparison:-**

We see in above both method of integro-differential equation that these method are effective in finding the different kinds of integro-differential equation but we observe that these method are difficult to understand as compared to the

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Laplace transformation method. We see in previous Laplace transformation examples, it is very easy to understand and easy to handle. We solve any kind of integro-differential equation in an easy way by performing the steps of Laplace transformation method.

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CONCLUSION:-

Integro-differential Equation plays an important role in Mathematics field. Enormous efforts have been conducted to obtain the numerical solution of integro-differential equations

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We combined our method with Laplace Transform method that is advantageous to solve any kind of integral-differential Equations. Finally, we have analyzed the solution of each illustrative example at different numerical methods of integro-differential Equation. But we see that some numerical methods solutions are difficult to obtain, much attension have been invested.. In a general view Integro-differential equation is a hybrid of integral and differential equations which have found extensive applications in science and engineering. Integro-differential method play a major role in a affective solutions of Integral Equation. and also reduces the number of errors in accuracy.

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