Solution of Volterra integral equations of first kind

History: Volterra integral equations were introduced by Vito Volterra and then studied by Traian Lalescu in his 1908 thesis, Sur les equations de Volterra, written under the direction of Emile Picard. Volterra integral equations of 2^{nd} were first systematically studied by V. Volterra. A special case of Volterra integral equation of first kind, the Abel integral equation, was first studied by N.H. Abel. In 1911, Lalescu wrote the first book ever on integral equations.

Introduction: Volterra integral equations find application in demography, the study of viscoelastic materials, and in actuarial science through the renewal equation. Volterra integral equation of first kind is a mathematical model of many evolutionary problems with memory arising from biology, chemistry, physics and engineering. In recent years many basic different functions have been used to estimate the solution of integral equations, such as orthogonal bases and wavelets.

The most general form of integral equation in $u(x)$ is

$$h(x)u(x) = f(x) + \int_a^{b(x)} K(x,\mathrm{t})u(t)\,dt \qquad (1)$$

In mathematics there are two types of integral equations, Volterra integral equation and Fredholm integral equation. When $b(x) = x$ in eq. 1 then it is called Volterra integral equation i.e.

$$h(x)u(x) = f(x) + \int_a^{x} K(x,\mathrm{t})u(t)\,dt \qquad (2)$$

Definition: The integral equation in (2) is called Volterra integral equation of first kind when $h(x) = 0$. Mathematically,

$$-f(x) = \int_a^{b(x)} K(x,\mathrm{t})u(t)\,dt \qquad (3)$$

Where $f(x)$ is a given function, $u(x)$ is an unknown function to be solved and $K(x,t)$ is called **kernel** of integral equation.

Eq. 3 is called **homogeneous** Volterra integral equation of first kind if $f(x) \equiv 0$.

Eq. 3 is called a convolution equation if $K(x,t) = K(x-t)$ i.e.

$$-f(x) = \int_a^{b(x)} K(x-\mathrm{t})u(t)\,dt \qquad (4)$$

Where $K(x-t)$ is called **difference kernel**.

Methods to solve Volterra integral equations of first kind:

1-Conversion of Volterra integral equation of 1st kind into 2nd kind:

For the special case of a Volterra equation of the first kind,

$$f(x) = \lambda \int_0^x K(x,t)u(t)\,dt \qquad (5)$$

With kernel $K(x,t)$ such that $K(x,x) \neq 0$, and is differentiable with respect to x. It can be reduced to Volterra equation of 2nd kind whose solution, in general, is much more tractable. If we differentiate both side of (5) with respect to x using Leibnitz rule on the integral, we have

$$\frac{df}{dx} = \lambda \int_0^x \frac{\partial K(x,t)}{\partial x} u(t)\,dt + \lambda K(x,x)u(x) \qquad (6)$$

This can be rewritten as

$$\lambda K(x,x)u(x) = \frac{df}{dx} - \lambda \int_0^x \frac{\partial K(x,t)}{\partial x} u(t)\,dt$$

$$u(x) = \frac{1}{\lambda K(x,x)} \frac{df}{dx} - \int_0^x \frac{1}{K(x,x)} \frac{\partial K(x,t)}{\partial x} u(t)\,dt, \; K(x,x) \neq 0 \qquad (7)$$

Eq.(7) is a Volterra integral equation of 2nd kind with nonhomogeneous term

$$g(x) = \frac{1}{\lambda K(x,x)} \frac{df}{dx}$$

and the new kernel $$H(x,t) = \frac{-1}{K(x,x)} \frac{\partial K(x,t)}{\partial x}$$

Thus $$u(x) = g(x) + \int_0^x H(x,t)u(t)\,dt \qquad (8)$$

So when $K(x,x) \neq 0$ in (5), we can **reduce it to a Volterra equation of 2nd kind** and solve it using one of the methods to solve Volterra equation of 2nd kind.

Example 1: Solve following Volterra equation of first kind after reducing it to a Volterra equation of second kind:

$$\sin x = \int_0^x e^{x-t} u(t)\,dt \qquad (E.1)$$

Solution: the kernel $K(x,t) = e^{x-t}$ does not vanish when $x = t$ $\left[i.e., K(x,x) = e^0 = 1 \right]$ and hence according to eq. (7) with $f(x) = \sin x, \; \lambda = 1$ we have

$$g(x) = \frac{1}{\lambda K(x,x)} \frac{df}{dx}$$

$$g(x) = 1$$

And $$H(x,t) = \frac{-1}{K(x,x)} \frac{\partial K(x,t)}{\partial x}$$

$$H(x,t) = -\frac{\partial}{\partial x}\left(e^{x-t}\right) = -e^{x-t}$$

Using $g(x)$, $H(x,t)$ and eq. (8) we get

$$u(x) = \cos x - \int_0^x e^{x-t} u(t)\,dt \tag{E.2}$$

This is a Volterra equation of the second kind. We can solve it by **resolvent kernel method**. Here we have

$$K_1(x,t) \equiv K(x,t) = e^{x-t}$$

Using $K(x,\xi) = e^{x-\xi}$ and $K_1(\xi,t) = K(\xi,t) = e^{\xi-t}$ we obtain

$$K_2(x,t) = \int_t^x K(x,\xi) K_1(\xi,t)\,d\xi = \int_t^x e^{x-\xi} e^{\xi-t}\,d\xi$$

$$= \int_t^x e^{x-t} d\xi = e^{x-t} \int_t^x d\xi = (x-t)e^{x-t}$$

Similarly for n=2, we have

$$K_3(x,t) = \int_t^x K(x,\xi) K_2(\xi,t)\,d\xi = \int_t^x e^{x-\xi}(\xi-t)e^{\xi-t} d\xi$$

$$= \int_t^x (\xi-t)e^{x-t} d\xi = e^{x-t} \int_t^x (\xi-t)\,d\xi$$

$$= e^{x-t}\left(\frac{\xi^2}{2} - t\xi\right)\Bigg|_t^x = e^{x-t}\left[\frac{x^2-t^2}{2} - t(x-t)\right] = \frac{(x-t)^2}{2} e^{x-t}$$

These calculations can be continued to find

$$K_{n+1}(x,t) = \frac{(x-t)^n}{n!} e^{x-t}$$

Hence the resolvent kernel would be

$$\Gamma(x,t;\lambda) = K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \ldots + \lambda^n K_{n+1}(x,t) + \ldots$$

$$= e^{x-t} + \lambda(x-t)e^{x-t} + \lambda^2 \frac{(x-t)^2}{2} e^{x-t} + ... + \lambda^n \frac{(x-t)^n}{n!} e^{x-t}$$

$$= e^{x-t}\left[1 + \lambda(x-t) + \lambda^2 \frac{(x-t)^2}{2} + ... + \lambda^n \frac{(x-t)^n}{n!}\right]$$

$$= e^{x-t} e^{\lambda(x-t)} = e^{(1+\lambda)(x-t)}$$

So by using this resolvent kernel the solution to (E.2) is

$$u(x) = \cos x + \lambda \int_0^x e^{(1+\lambda)(x-t)} \cos t \, dt \qquad \text{(E.3)}$$

with $\lambda = -1$ we have

$$u(x) = \cos x - \int_0^x \cos t \, dt$$

$$= \cos x - [\sin t]_0^x = \cos x - \sin x \qquad \text{(E.4)}$$

Remark: When $K(x,x) \equiv 0$, eq. (6) is still a Volterra equation of first kind. However, if $\frac{\partial K}{\partial x}(x,x) \neq 0$ in (6), differentiating again will result in a Volterra equation of the second kind. If these attempts fail, the methods of solution become more involved. An exception to this is the special case when eq. (6) has a difference kernel and hence the method of Laplace transform can be employed.

2-Laplace Transform:

When Volterra integral equation of first kind in eq. 3 has a difference kernel we apply Laplace transform to obtain:

$$L\{f(x)\} = L\left\{\lambda \int_0^x K(x-t)u(t)\,dt\right\}$$

$$F(s) = \lambda L\left\{\int_0^x K(x-t)u(t)\,dt\right\}$$

$$F(s) = \lambda L\{K(x) * u(x)\}$$

$$F(s) = \lambda K(s)U(s)$$

$$U(s) = \frac{1}{\lambda}\frac{F(s)}{K(s)} \qquad (9)$$

where $$K(x) * u(x) = \int_0^x K(x-t)u(t)\,dt$$

and $$F(s) = L\{F(x)\},\ K(s) = L\{K(x)\} \, and \, U(s) = L\{u(x)\}$$

By taking inverse Laplace of eq. 9 we get solution of eq. 4

$$L^{-1}\{U(s)\} = L^{-1}\left\{\frac{1}{\lambda}\frac{F(s)}{K(s)}\right\}$$

$$u(x) = \frac{1}{\lambda}L^{-1}\left\{\frac{F(s)}{K(s)}\right\} \tag{10}$$

Example 2: Solve the following Volterra integral equation of first kind

$$\sin x = \lambda\int_0^x e^{x-t}u(t)\,dt \tag{E.1}$$

by using Laplace transform.

Solution: This Volterra equation of first kind (E.1) is with difference kernel $K(x-t) = e^{x-t}$. If we apply Laplace transform on above equation then by Laplace convolution theorem we have

$$\int_0^x e^{x-t}u(t)\,dt = e^x * u(x), \quad L\{e^x * u(x)\} = \frac{1}{s-1}U(s)$$

where $L\{e^x\} = \frac{1}{s-1}$, $L\{\sin x\} = \frac{1}{s^2+1}$ and $L\{u(x)\} = U(s)$

using above results (E.1) becomes

$$\frac{1}{s^2+1} = \lambda\frac{1}{s-1}U(s)$$

$$U(s) = \frac{1}{\lambda}\frac{1/(s^2+1)}{1/(s-1)}$$

$$U(s) = \frac{1}{\lambda}\frac{s-1}{s^2+1} = \frac{1}{\lambda}\left(\frac{s}{s^2+1} - \frac{1}{s^2+1}\right) \tag{E.2}$$

To obtain solution $u(x)$ of (E.1), we will apply inverse Laplace transform of (E.2)

$$L^{-1}\{U(s)\} = \frac{1}{\lambda}L^{-1}\left\{\frac{s}{s^2+1} - \frac{1}{s^2+1}\right\}$$

$$u(x) = \frac{1}{\lambda}(\cos x - \sin x) \tag{E.3}$$

This is the same result as in example 1 with $\lambda = 1$.

Example 3: Solve the following Volterra integral equation of first kind by using Laplace transform.

$$x = \int_0^x \cos(x-t)u(t)\,dt \tag{E.1}$$

Solution: (E.1) has a difference kernel $K(x-t)=\cos(x-t)$ we will apply Laplace transform

$$L\{x\}=L\left\{\int_0^x \cos(x-t)u(t)\,dt\right\}$$

$$\frac{1}{s^2}=L\{\cos x * u(x)\}$$

$$\frac{1}{s^2}=\frac{s}{s^2+1}U(s)$$

$$U(s)=\frac{1}{s^2}\times\frac{s^2+1}{s}=\frac{s^2+1}{s^3}$$

$$U(s)=\frac{1}{s}+\frac{1}{s^3} \tag{E.2}$$

To find solution of (E.1) we will apply inverse Laplace transform on (E.2)

$$L^{-1}\{U(s)\}=L^{-1}\left\{\frac{1}{s}+\frac{1}{s^3}\right\}$$

$$u(x)=1+\frac{x^2}{2} \tag{E.3}$$

Difficulty to solve Volterra equations of first kind:

Above examples seems to go very well for a solution via Laplace transformation. However if we change the problem a little, we will uncover the first main difficulty with solving integral equations of first kind. This is described in that we are not sure that there is a solution $u(x)$ for eq. (5) that corresponds to any given function $f(x)$. We will show here that in the above example 2, the function $f(x)=\sin x$ have been selected very carefully to guarantee the existence of the obtain solution (E.3).

Let us assume that, in example 2 instead of $f(x)=\sin x$ in (E.1) with $\lambda=1$, we were given $f(x)=1$. Then if we don't know about the above difficulty, we can proceed, with the same above Laplace transform method, where we only have the left side of $\mathrm{L}\{\sin x\}=\dfrac{1}{s^2+1}$ changed to $L\{1\}=\dfrac{1}{s}$, and we get,

$$\frac{1}{s}=\frac{1}{s-1}U(s)$$

$$U(s)=\frac{1/s}{1/s-1}=\frac{s-1}{s}=1-\frac{1}{s}$$

However, for such result of $U(s)=1-\frac{1}{s}$, there exist no Laplace transform inverse for the first term 1 i.e. $L^{-1}\{1\}$ doesn't exist. This is an obvious accepted conclusion. Since the important necessary condition for the Laplace transform $F(s)$ is that it must vanish as $S\longrightarrow\infty$, and the above $F(s)=1$ does not.

Example 4: Abel's integral Equation

Use the Laplace transform to solve Abel's integral equation

$$-\sqrt{2g}f(x)=\int_0^x\frac{\phi(t)dt}{\sqrt{x-t}} \tag{E.1}$$

Solution: Let $F(s)$ *and* $\Phi(s)$ be the Laplace transform of $f(x)$ *and* $\phi(s)$ respectively, and $L\{K(x)\}=L\{1/\sqrt{x}\}=\sqrt{\pi/s}\ with\ \upsilon=-1/2$ noting that $\Gamma(1/2)=\sqrt{\pi}$, using these we obtain

$$-\sqrt{2g}F(s)=\sqrt{\frac{\pi}{s}}\Phi(s) \tag{E.2}$$

$$\Phi(s)=-\sqrt{\frac{2g}{\pi}}\sqrt{s}F(s) \tag{E.3}$$

Hence the solution of Abel's equation is

$$\phi(s)=-\sqrt{\frac{2g}{\pi}}L^{-1}\left\{\sqrt{s}F(s)\right\} \tag{E.4}$$

In trying to use the convolution theorem for $L^{-1}\left\{\sqrt{s}F(s)\right\}$ in (E.4) we have difficulty in finding $L^{-1}\left\{\sqrt{s}\right\}$, which actually does not exist, since an important necessary condition for a Laplace transform $F(s)$ is that it must vanish as $s\to\infty$. Now if we multiply and divide the right hand side of (E.3) by $\sqrt{s}$ we obtain

$$\Phi(s)=-\sqrt{\frac{2g}{\pi}}s\frac{1}{\sqrt{s}}F(s)=-\sqrt{\frac{2g}{\pi}}sH(s) \tag{E.5}$$

Where $H(s)=\frac{F(s)}{\sqrt{s}}$, and we can use convolution theorem to write,

$$L^{-1}\{H(s)\}=L^{-1}\left\{\frac{F(s)}{\sqrt{s}}\right\}$$

$$h(x)=\frac{1}{\sqrt{\pi x}}*f(x)=\frac{1}{\sqrt{\pi}}\int_0^x\frac{f(t)}{\sqrt{x-t}}dt \tag{E.6}$$

Now let $$\frac{dh}{dx} = L^{-1}\{sH(s) - h(0)\} = L^{-1}\{sH(s)\} - L^{-1}\{h(0)\}$$

$$\frac{dh}{dx} = L^{-1}\{sH(s)\} \tag{E.7}$$

Now by applying inverse Laplace and using (E.6) and (E.7) in (E.5) we get

$$\phi(x) = -\sqrt{\frac{2g}{\pi}}\frac{dh}{dx} = -\sqrt{\frac{2g}{\pi}}\frac{d}{dx}\left[\frac{1}{\sqrt{\pi}}\int_0^x \frac{f(t)}{\sqrt{x-t}}\right]$$

$$\phi(x) = -\frac{\sqrt{2g}}{\pi}\frac{d}{dx}\left[\int_0^x \frac{f(t)}{\sqrt{x-t}}\right] \tag{E.8}$$

which is solution of Abel's equation.

Another difficulty to solve integral equations of first kind:

The method to solve Abel's problem as Volterra equation of first kind, seems to flow very well. However, we may still point out to the hidden difficulties in trying to solve integral equations of the first kind. For example, does the integral in (E.8) in above example exist for arbitrary $f(x)$. Abel's first problem of the Tautochrone is for the case of $f(x)$ being constant T, which is very safe, since the integral of (E.8) does exist for $f(x) = 1$,

$$\int_0^x (x-t)^{-1/2}\, dt = -2(x-t)^{1/2}\Big|_{t=0}^{x} = -2\left[0 - x^{1/2}\right] = 2x^{1/2}$$

This result of the integration is a nice continuous function for $x \geq 0$, but what we need to get solution of (E.8) is to differentiate this result of integration. To prepare for this second type of difficulty for equations of first kind, we remind how integration is "smoothing process" while differentiation, as the inverse operation of integration, would uncover the "rough spots" (discontinuities). In our example with $f(x) = T$

$$\phi(x) = -\frac{\sqrt{2g}}{\pi}T\frac{d}{dx}x^{1/2} = -\frac{\sqrt{2g}T}{\pi}\cdot\frac{1}{2}x^{-1/2}$$

which is unbounded at $x = 0$. So, solutions of Volterra integral equations of first kind that involve related differentiation of the given function $f(x)$ would inherit what the differentiation operation may uncover.

Remark: For the above result in (E.8) we have to do numerical integration. This difficulty of the equations of the first kind is considered to be very serious, because even if we know that a solution exists, we may only get useless inaccurate data for it. Such situations are described by "a small change in the input data $f(x)$ may produce a very large change in the sought output (solution) $u(x)$" of the integral equation of first kind, $f(x)=\int K(x,t)u(t)dt$, whereby the solution is termed "unstable". Such difficulties and warnings were, of course, not available to Abel around the year 1826.

3-Simpson's Quadrature rule:

If N is even, the Simpson's Quadrature rule may be applied to each subinterval

$[x_{2i},x_{2i+1},x_{2i+2}];\ \ i=0,1,...,\frac{N}{2}-1$ individually, yielding the approximation,

$$\frac{h}{3}\left[f(x_{2i})+4f(x_{2i+1})+f(x_{2i+2})\right]$$

Summing these $\frac{N}{2}$ approximations results in the composite version of Simpson's quadrature rule,

$$S(h=)\frac{h}{3}\left[f(a)+4f(a+h)+2f(a+2h)+4f(a+3h)+...+2f(b-2h)+4f(b-h)+f(b)\right]$$

for the entire interval.

Step by step method of solution: Consider the following Volterra integral equation of first kind:

$$f(x)=\lambda\int_a^x K(x,\mathrm{t})u(t)dt \quad a\le x\le b \tag{11}$$

Let the interval $[a,b]$ be finite and partitioned by N equally spaced points

$$\begin{cases} x_0=a\ ,\ \ x_N=b \\ x_i=x_0+ih;\ i=0,1,...,\mathrm{N}. \end{cases}$$

The approximation of eq. (11) on the even nodes x_{2m} is given by

$$f(x_{2m})=\int_a^{x_{2m}} K(x_{2m},t)u(t)dt \tag{12}$$

Instead of eq. (12) consider the following equation,

$$f_{2m}=\sum_{l=0}^{m-1}\int_{x_{2l}}^{x_{2l+2}} K(x_{2m},t)u(t)dt \tag{13}$$

By using repeated Simpson's quadrature rule, we have

$$f_{2m}=\sum_{l=0}^{m-1}\frac{h}{3}\left(K_{2m,2l}u_{2l}+4K_{2m,2l+1}u_{2l+1}+K_{2m,2l+2}u_{2l+2}\right)$$

Set $u_{2l+1}\cong\frac{u_{2l}+u_{2l+2}}{2}$, then we have

$$f_{2m} = \sum_{l=0}^{m-1} \frac{h}{3}\left(K_{2m,2l}u_{2l} + 4K_{2m,2l+1}\frac{u_{2l}+u_{2l+2}}{2} + K_{2m,2l+2}u_{2l+2} \right)$$

$$f_{2m} = \sum_{l=0}^{m-1} \frac{h}{3}\left(K_{2m,2l}u_{2l} + 2K_{2m,2l+1}u_{2l} + 2K_{2m,2l+1}u_{2l+2} + K_{2m,2l+2}u_{2l+2} \right)$$

$$f_{2m} = \sum_{l=0}^{m-1} \frac{h}{3}\left(K_{2m,2l} + 2K_{2m,2l+1} \right)u_{2l} + \sum_{l=0}^{m-1} \frac{h}{3}\left(2K_{2m,2l+1} + K_{2m,2l+2} \right)u_{2l+2}$$

$$f_{2m} = \frac{h}{3}\left(K_{2m,0} + 2K_{2m,1} \right)u_0 + \sum_{l=1}^{m-1} \frac{h}{3}\left(K_{2m,2l} + 2K_{2m,2l+1} \right)u_{2l}$$
$$+ \frac{h}{3}\left(2K_{2m,2m-1} + K_{2m,2m} \right)u_{2m} + \sum_{l=0}^{m-2} \frac{h}{3}\left(2K_{2m,2l+1} + K_{2m,2l+2} \right)u_{2l+2}$$

$$f_{2m} = \frac{h}{3}\left(K_{2m,0} + 2K_{2m,1} \right)u_0 + \frac{h}{3}\left(2K_{2m,2m-1} + K_{2m,2m} \right)u_{2m}$$
$$+ \sum_{l=1}^{m-1} \frac{h}{3}\left(K_{2m,2l} + 2K_{2m,2l+1} \right)u_{2l} + \sum_{l=1}^{m-1} \frac{h}{3}\left(2K_{2m,2l-1} + K_{2m,2l} \right)u_{2l+2}$$

$$\frac{h}{3}\left(2K_{2m,2m-1} + K_{2m,2m} \right)u_{2m} = f_{2m} - \frac{h}{3}\left(K_{2m,0} + 2K_{2m,1} \right)u_0 + \frac{2h}{3}\sum_{l=1}^{m-1}\left(K_{2m,2l-1} + K_{2m,2l} + K_{2m,2l+1} \right)u_{2l}$$

$$u_{2m} = \frac{f_{2m} - \frac{h}{3}\left(K_{2m,0} + 2K_{2m,1} \right)u_0 + \frac{2h}{3}\sum_{l=1}^{m-1}\left(K_{2m,2l-1} + K_{2m,2l} + K_{2m,2l+1} \right)u_{2l}}{\frac{h}{3}\left(2K_{2m,2m-1} + K_{2m,2m} \right)} \qquad (14)$$

where $m = 1, 2, ..., \frac{N}{2}$.

By taking derivative of eq (11) we get

$$f'(x) = \int_a^x \frac{\partial K(x,\mathrm{t})}{\partial x} u(t)\,dt + K(x,x)u(x)$$

At $x = a$, $$f'(a) = K(a,a)u(a)$$

$$u_0 = u(a) = \frac{f'(a)}{K(a,a)} \qquad (15)$$

The computation of u_{2m} for $m = 1, 2, ..., \frac{N}{2}$ will be facilitated by recurrence relation (14) and above mentioned relation (15).

Example 5: Consider Volterra integral equation of first kind

$$x=\int_0^x e^{t-x}u(t)\,dt, \qquad 0\le x\le 1$$

and exact solution $u(x)=x+1$. Find numerical results by using repeated Simpson's quadrature rule.

Solution: Numerical solution and exact solutions on different notes are shown in following table:

Nodes x	Approximation for h=0.1	Approximation for h=0.01	Exact solution
0.0	1	1	1
0.2	1.199994521631	1.199999999971	1.2
0.4	1.399999431759	1.399999999943	1.4
0.6	1.599994623000	1.599999999916	1.6
0.8	1.79998906759	1.799999999891	1.8
1.0	1.999994685149	1.999999999868	2.0

Conclusion: In above example we applied an application of repeated Simpson's quadrature rule for solving the Volterra integral equation of first kind. According to the numerical results which we obtained from above example, we conclude that for sufficiently small **h** we gat a good accuracy, since by reducing step size length the least square error will be reduced. The same approach can be used to solve other problems. The computation of this approach is simple and effectively executed using symbolic computing codes on any personal computer.

Comparison:

General form of Volterra integral equation of first kind is

$$f(x)=\lambda\int_0^x K(x,t)u(t)\,dt$$

When kernel $K(x,t)$ is such that $K(x,x)=0$, also it is differentiable with respect to x, then we **reduce it to Volterra integral equation of second kind.** And we can solve it by using one of the methods to solve Volterra integral equations of second kind. Further if $\frac{\partial K}{\partial x}(x,x)\neq 0$ we can differentiate it again and resulting equation again will be a Volterra equation of second kind. If these attempts fail, the methods of solution become more involved.

An exceptional case of Volterra integral equations of first kind is when it has a difference kernel. Such equations can be solved by **Laplace transform method**. But using Laplace transformation we uncover two main difficulties, which we have already mentioned in above discussion. First difficulty is that the function $f(x)$ does not guarantee the existence of $u(x)$. Other difficulty is that the Volterra integral equations of first kind that involve related differentiation of the given $f(x)$ as we saw in Abel's equation, would uncover the rough spots (discontinuities), on differentiating.

For more general types of problems, we often resort to **approximate methods** where the integral equation is replaced or approximated by another one which is closely related and can be handled by the usual methods, with solutions close to those of the original equations.

When such methods are not feasible, we have to resort to **numerical methods** which are also approximate methods and where the integral in the equation is approximated by a sum of N terms. As a result integral equation may be reduced to a set of N unknowns $u(x_i),\ i=1,2,...,N-1$, the samples of the approximate solution. In above discussion, we have used a numerical method which is named as **Simpson's quadrature rule.**

References:

- Abdul. J. Jerri, 1932, Introduction to integral equations with applications, Second ed.
- A first course in integral equations; Abdul Majid Wazwaz saint Xavier University, USA