Solution of Volterra integral equation using Resolvent Kernal Method

History

Volterra started working on solution of Integral equations of such type in 1884, which is before the name integral equations was given by Du Bois-Rymond in 1888.

Volterra's serious study began in 1886.

Introduction:-

In mathematics, the volterra integral equations are special type of integral equations. These are divided into two types.

01:- 1st Kind of volterra integral equation

02:- 2nd Kind of volterra integral equation.

⇒ The volterra integral equations were introduced by Vito Volterra and then studied by Traian Lalescu in 1908, in his thesis (Sur les equations de Volterra) written under the direction of Emile Picard.

⇒ In 1911, Lalescu wrote the first book ever on integral equations.

Literature Review:-

Volterra integral equations find application in

⇒ Demography

⇒ Study of Viscoelastic material

⇒ In actual science through Renewal equation.

Number of problems resulted in volterra integral equations. It includes human population problem, mortality of equipment problems and Abel's problems.

Volterra integral equation:-

The linear integral equation in $u(x)$ as follows

$$h(x)u(x) = f(x) + \int_a^{b(x)} K(x,\xi)u(\xi)\,d\xi$$

is called volterra integral equation when $b(x) = x$, i.e:-

Volterra $$h(x)u(x) = f(x) + \int_a^{x} K(x,\xi)u(\xi)\,d\xi \rightarrow (1)$$

Integral equation of 1st kind:-

When $h=0$ put in equation (1) then we obtain volterra equation of 1st kind.

$$(0)\,u(x) = f(x) + \int_a^{x} K(x,\xi)u(\xi)\,d\xi$$

$$\Rightarrow \quad -f(x) = \int_a^{x} K(x,\xi)u(\xi)\,d\xi.$$

Volterra equation of 2nd kind:-

When $h=1$, put in equation (1) then we obtain volterra equation of 2nd kind,

$$(1)\,u(x) = f(x) + \int_a^{x} K(x,\xi)u(\xi)\,d\xi$$

$$\Rightarrow \quad u(x) = f(x) + \int_a^{x} K(x,\xi)u(\xi)\,d\xi.$$

Example:-

Equation for torsion of wire is a volterra equation of second kind, i.e

$$m(t) = h\,\omega(t) + \int_{-\infty}^{t} \varphi(t, \tau)\,\omega(\tau)\,d\tau.$$

Example:-

Abel's equation is volterra equation of first kind, ie:-

$$-\sqrt{2g}\,f(y) = \int_0^y \frac{\varphi(\eta)\,d\eta}{\sqrt{y-\eta}}$$

Resolvent Kernal Method:-
(Neumann Series)

The solution of volterra integral equation of second kind,

$$u(x) = f(x) + \lambda \int_a^x K(x, \xi)\,u(\xi)\,d\xi \rightarrow (1)$$

may often appear as an integral

$$u(x) = f(x) + \lambda \int_a^x \Gamma(x, \xi; \lambda)\,f(\xi)\,d\xi \rightarrow (2)$$

in terms of given function $f(x)$ where $\Gamma(x,\xi;\lambda)$ is called resolvent Kernal of integral equation.

When both $K(x,\xi)$ and $f(x)$ in equat (1) are continuous.

It is very easy to construct the resolvent Kernal $\Gamma(x,\xi,\lambda)$ for equation (1) in terms of following Neumann series.

$$\boxed{\Gamma(x,\xi,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,\xi)} \rightarrow (3)$$

where $K_{n+1}(x,\xi)$ is iterated Kernal and it is evaluated as,

$$\boxed{K_{n+1}(x,\xi) = \int_{\xi}^{x} K(x,y)\, K_n(y,\xi)\, dy} \rightarrow (4)$$

and $K_1(x,y) = K(x,y)$.

Assume the following infinite series form for the solution of $u(x)$:-

$$\boxed{u(x) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \cdots} \rightarrow (5)$$

putting equation (5) into (1)

$$u(x) = f(x) + \lambda \int_a^x K(x,\xi)\left[u_0(\xi) + u_1(\xi)\lambda + \cdots\right] d\xi$$

$$\Rightarrow \boxed{u(x) = f(x) + \lambda \int_a^x K(x,\xi) u_0(\xi) d\xi + \lambda^2 \int_a^x K(x,\xi) u_1(\xi) d\xi + \cdots} \quad (6)$$

Suppose the infinite series is converging that allows to exchange of it's summation with the integration of equation (1) that led to equation (5).

Now we equate the coefficients of each λ of the same power on both with the assumptions of good enough convergence of infinite series in eq (5)

e.g coefficients of $\lambda^0, \lambda^1, \lambda^2$ in equation (6) are equated to gives the following results,

$$u_0(x) = f(x) \rightarrow (7)$$

$$u_1(x) = \int_a^x K(x,\xi) u_0(\xi) d\xi \rightarrow (8)$$

$$u_2(x) = \int_a^x K(x,\xi) u_1(\xi) d\xi \rightarrow (9)$$

$$\vdots$$

$$u_n(x) = \int_a^x K(x,\xi) u_{n-1}(\xi) d\xi \rightarrow (10)$$

By putting value of $u_0(x)$ in equation (8)

$$\boxed{u_1(x) = \int_a^x K(x,\xi) f(\xi)\, d\xi} \rightarrow (11)$$

and now put $u_1(x)$ in equation (9)

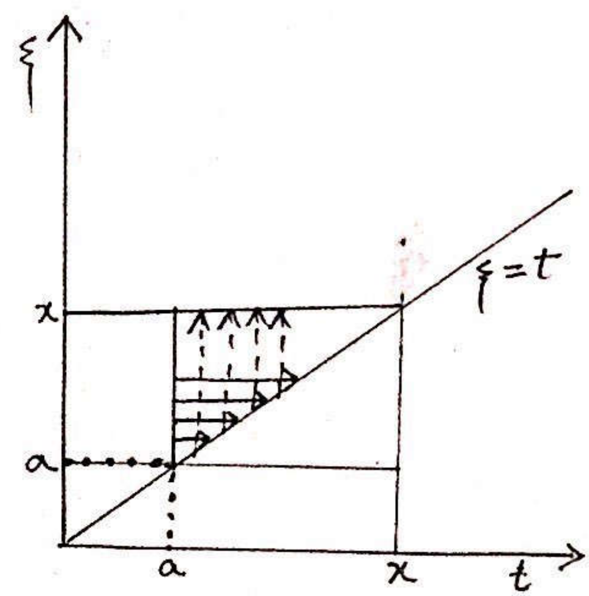
$$\boxed{u_2(x) = \int_a^x K(x,\xi) \int_a^\xi K(\xi,t) f(t)\, dt\, d\xi} \rightarrow (12)$$

Now from (12) we obtain

$$u_2(x) = \int_a^x f(t) \left[\int_t^x K(x,\xi) K(\xi,t)\, d\xi \right] dt \rightarrow (13)$$

$$u_2(x) = \int_a^x f(t) K_2(x,t)\, dt$$

$$\Rightarrow \boxed{u_2(x) = \int_a^x K_2(x,\xi) f(\xi)\, d\xi} \rightarrow (14)$$



"Domain for performing the integration w.r.t 't' first (solid lines) or with respect to 'ξ' first (dashed lines)"

As in equation (11) $K(x,t)$ is taken as $K_1(x,t)$ to give $u_1(x)$, the inside integral in equ (13) defines $K_2(x,t)$, then

iteraterated Kernal,

$$K_2(x,t) = \int_t^x K(x,\xi)K(\xi,t)d\xi$$

$$\Rightarrow \boxed{K_2(x,t) = \int_t^x K(x,\xi)K_1(\xi,t)d\xi} \rightarrow (15)$$

which gives $u_2(x)$ in equation (14).

As we know that

$$K_1(\xi,t) = K(\xi,t)$$

Similarly By doing the same procedure from (11) to (15) we derive the general term for iterated Kernal,

$$\boxed{K_{n+1}(x,t) = \int_t^x K(x,\xi)K_n(\xi,t)d\xi} \rightarrow (16)$$

which gives general term of equation (5),

$$\boxed{u_{n+1}(x) = \int_a^x K_{n+1}(x,\xi)f(\xi)d\xi} \rightarrow (17)$$

Th final solution is then from $u_0(x)$ by putting $u_0(x) = f(x)$ in (7) put value of $u_1(x)$ in equation (11), $u_2(x)$ in equation (14) and so on $u_n(x)$ in equ (16).

$$u(x) = f(x) + \lambda \int_a^x K_1(x,\xi) f(\xi) d\xi + \lambda^2 \int_a^x K_2(x,\xi) f(\xi) d\xi$$

$$+ \cdots + \lambda^n \int_a^x K_n(x,\xi) f(\xi) d\xi + \cdots$$

$\Longrightarrow$

$$u(x) = f(x) + \lambda \int_a^x \left[K_1(x,\xi) + \lambda K_2(x,\xi) + \cdots \right.$$

$$\left. + \lambda^n K_n(x,\xi) + \cdots \right] f(\xi) d\xi \longrightarrow (18)$$

$$\Rightarrow u(x) = f(x) + \lambda \int_a^x \left[\sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,\xi) \right] f(\xi) d\xi$$

$$\boxed{u(x) = f(x) + \lambda \int_a^x \Gamma(x,\xi,\lambda) f(\xi) d\xi.}$$

which is wat was sought in equ (2) and (3) to constitute the volterra integral equation (1), via the present method of the iterated Kernals equ (15) for constructing the resalvent Kernal $\Gamma(x,\xi,\lambda)$ in equ (3).

Example:-

The Resalvent Kernal $R(x,t,\lambda)$ for the volterra integral equation,

$$\varphi(x) = x + \lambda\int_a^x \varphi(s)ds$$

Firstly we find iterative Kernel and then try to guess the resolvent kernal,

$$R = \sum_{i=1}^{\infty} K_i(x,t)$$

comparing with general form of volterra equation of 2nd kind,

$$g(x)\varphi(x) = f(x) + \lambda\int_a^x K(x,t)\varphi(t)dt$$

Kernel $K(x,t) = 1$, $K_1(x,t) = K(x,t) = 1$

Now

$$K_2(x,t) = \int_t^x K(x,s)K_1(s,t)ds = \int_t^x ds = x-t$$

$$K_3(x,t) = \int_t^x K(x,s)K_2(s,t)ds.$$

$$= \int_t^x (s-t)ds = \frac{s^2}{2} - ts\Big|_{s=t}^{x}$$

$$K_3(x,t) = \frac{x^2}{2} - tx - \frac{t^2}{2} + t^2$$

$$= \frac{x^2}{2} - tx + \frac{t^2}{2}$$

$$= \frac{1}{\sqrt{2}}(x-t)^2$$

so $(x-t)$ has be found.

It is better to start with definition

$R(x,t,\lambda)$ is the resolvent Kernel for

$$\boxed{\phi(x) = f(x) + \lambda \int_a^x \phi(s)\,ds}$$

If the solution can be written as,

$$\phi(x) = f(x) + \lambda \int_a^x R(x,t,\lambda)\, f(t)\,dt.$$

Taking derivative on Both sides of the equation,

$$\phi(x) = x + \lambda \int_a^x \phi(s)\,ds. \quad \text{we get,}$$

$$\phi'(s) = f'(x) + \lambda\,\phi(x).$$

Therefore

$$\frac{d}{dx} e^{-\lambda x}\phi(x) = e^{-\lambda x} f'(x)$$

And

$$e^{-\lambda x}\phi(x) = e^{-\lambda a}\phi(a) + \int_a^x e^{-\lambda t} f'(t)\,dt$$

$$= e^{-\lambda a}\phi(a) + e^{-\lambda t} f(t)\Big|_x^a - \int_a^x f(t)\,d e^{-\lambda t}$$

$$e^{-\lambda x}\phi(x) = e^{-\lambda x} f(x) + e^{-\lambda a}(\phi(a) - f(a)) + \lambda \int_a^x e^{-\lambda t} f(t)\,dt.$$

Using the fact $\phi(a) = f(a)$

By evaluating integral equation at $x = a$

i.e:- $\phi(x) = f(x) + \lambda \int_a^x e^{\lambda(x-t)} f(t)\,dt.$

Therefore

Resolvent Kernal can be given as

$$\boxed{R(x,t,\lambda) = e^{\lambda(x-t)}.}$$

By using mathematical induction,

$$K_n(x,t) = (x-t)^{n-1}/(n-1)!$$

Resolvent Kernel is that

$$R(x,t,\lambda) = K_1(x,t) + \lambda K_2(x,t)$$

$$+ \lambda^2 K_3(x,t) + \dots$$

$$R(x,t,\lambda) = e^{\lambda(x-t)}$$

Analytic solution of integral equation

$$\phi(x) = x + \lambda \int_a^x \phi(s)\,ds \rightarrow (A)$$

is given as

$$\frac{d\phi}{dx} = 1 + \lambda\phi(x)$$

The general solution of linear ODE is,

$$\phi(x) = -\frac{1}{\lambda} + c e^{\lambda x} \rightarrow (B)$$

put in equation (A)

$$-\frac{1}{\lambda} + c e^{\lambda x} = x + \lambda \int_a^x +\left(-\frac{1}{\lambda} + c e^{\lambda s}\right) ds$$

$$= a + c e^{\lambda x} - c e^{\lambda a}$$

$$\boxed{c = \left(a + \frac{1}{\lambda}\right) e^{-\lambda a}}$$

put in (B)

$$\boxed{\phi(x) = -\frac{1}{\lambda} + \left(a + \frac{1}{\lambda}\right) e^{\lambda(x-a)}}$$

Instead of getting $\frac{(x-t)^2}{\sqrt{2}}$ we can get $\frac{(x-t)^2}{2}$.

Similarly K_4 can be find in same ways. and answer will be series of $\exp(x-t)$

we find

$$K_3(x,t) = \frac{(x-t)^2}{2!}$$

$$K_4(x,t) = \frac{(x-t)^3}{3!}$$

And about final series to get Resolvent Kernal it should be

$$R = \sum_{i=1}^{\infty} \lambda^{i-1} K_i(x,t)$$

So

$$R = 1 + \lambda(x-t) + \lambda^2 \frac{(x-t)^2}{2!} + \lambda^3 \frac{(x-t)^3}{3!}$$

$$+ \cdots$$

$$\boxed{R = e^{\lambda(x-t)}}$$ Resolvent Kernal.

Example:-

Given $K(x,t) = K_1(x,t) = \dfrac{2+\cos x}{2+\cos t} \rightarrow (1)$

Solve by Resolvent Kernal Method:-

$$K_2(x,t) = \int_t^x K(x,z)\, K_1(z,t)\, dz$$

$$= \int_t^x \cancel{K}\ \frac{2+\cos x}{2+\cos z} \cdot \frac{2+\cos z}{2+\cos t}\, dz$$

$$= \frac{2+\cos x}{2+\cos t} \int_t^x dz$$

$$= \frac{2+\cos x}{2+\cos t} \left(z \right|_t^x$$

$$\boxed{K_2(x,t) = \frac{2+\cos x}{2+\cos t}(x-t)} \rightarrow (2)$$

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz$$

$$= \int_t^x \frac{2+\cos x}{2+\cos z} \cdot \frac{2+\cos z}{2+\cos t}(z-t)\, dz$$

$$= \frac{2+\cos x}{2+\cos t} \int_t^x (z-t)\, dz$$

$$K_3(x,t) = \frac{2+\cos x}{2+\cos t}\left[\frac{(z-t)^2}{2!}\right|_t^x$$

$$K_3(x,t) = \frac{2+\cos x}{2+\cos t}\left[0 + \frac{(x-t)^2}{2!} + 0\right]$$

$$\boxed{K_3(x,t) = \frac{2+\cos x}{2+\cos t}\,\frac{(x-t)^2}{2!}} \longrightarrow (3)$$

$$\vdots$$

$$K_n(x,t) = \frac{2+\cos x}{2+\cos t}\,\frac{(x-t)^{n-1}}{(n-1)!}$$

Resolvent Kernel:-

$$R(x,t,\lambda) = ?$$

$$R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t)$$

$$= K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \cdots$$

$$= \frac{2+\cos x}{2+\cos t} + \frac{2+\cos x}{2+\cos t}\cdot\frac{\lambda(x-t)}{1!} +$$

$$+ \frac{2+\cos x}{2+\cos t}\cdot\frac{\lambda^2(x-t)^2}{2!} + \cdots$$

$$R(x,t,\lambda) = \frac{2+\cos x}{2+\cos t}\left[1+\lambda\frac{(x-t)}{1!}+\lambda^2\frac{(x-t)^2}{2!}+\cdots\right]$$

$$\boxed{R(x,t,\lambda) = \frac{2+\cos x}{2+\cos t}\, e^{\lambda(x-t)}}$$

solution will be

Example:-

Given $u(x) = \lambda \int_0^1 x e^t u(t)\, dt + x \longrightarrow (1)$

Solution:-

Here $K(x,t) = xe^t$

$$K_1(x,t) = K(x,t) = xe^t$$

$$K_2(x,t) = \int_0^1 K(x,z)\cdot K_1(z,t)\, dz$$

$$= \int_0^1 xe^z \cdot ze^t\, dz$$

$$= xe^t \int_0^1 ze^z\, dz$$

$$= xe^t\left[e^z\cdot z - e^z\right]_0^1 = xe^t$$

$$\boxed{K_2(x,t) = xe^t}$$

$$K_3(x,t) = \int_0^1 K(x,z)\, K_1(z,t)\, dz$$

$$= \int_0^1 x e^{z} \cdot z e^{t}\, dz$$

$$= x e^{t} \int_0^1 z e^{z}\, dz.$$

$$\boxed{K_3(x,t) = x e^{t}}$$

$$\vdots$$

$$K_n(x,t) = x e^{t}$$

Resolvent Kernal,

$$R(x,t,\lambda) = K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \cdots$$

$$= x e^{t} + \lambda x e^{t} + \lambda^2 x e^{t} + \cdots$$

$$= x e^{t} \left[1 + \lambda + \lambda^2 + \lambda^3 + \cdots \right]$$

$$= x e^{t} \left[\frac{1}{1-\lambda} \right] = x e^{t} (1-\lambda)^{-1}$$

$$\boxed{R(x,t,\lambda) = \frac{x e^{t}}{(1-\lambda)}}$$

Integral solution,

$$y(x) = f(x) + \lambda \int_0^1 R(x,t,\lambda)\, f(t)\, dt.$$

$$= x + \lambda \int_0^1 \frac{x e^{t}}{(1-\lambda)} \cdot t\, dt$$

$$y = x + \frac{\lambda x}{(1-\lambda)} \int_0^1 te^t\,dt$$

$$= x + \frac{\lambda x}{1-\lambda}\left[te^t - e^t\right]_0^1$$

$$= x + \frac{\lambda x}{1-\lambda}$$

$$= \frac{(1-\lambda)x + \lambda x}{(1-\lambda)}$$

$$\boxed{y = \frac{x}{1-\lambda x}}$$