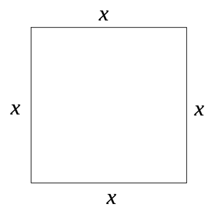
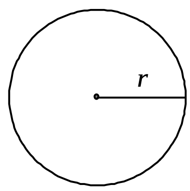
Function and Limits

Concept of Functions:

Historically, the term function was first used by German mathematician Leibnitz (1646-1716) in 1673 to denote the dependence of one quantity on another e.g.



1) The area "A" of a square of side "x" is given by the formula $A=x^2$. As area depends on its side x, so we say that A is a function of x.



2) The area "A" of a circular disc of radius "r" is given by the formula $A=p\,r^2$ As area depends on its radius r, so we say that A is a function of r.

3) The volume "V" of a sphere of radius "r" is given by the formula $V=\frac{4}{3}p\,r^3$. As volume V of a sphere depends on its radius r, so we say that V is a function of r.

The Swiss mathematician, Leonard Euler conceived the idea of denoting function written as $y=f(x)$ and read as y is equal to f of x. $f(x)$ is called the value of f at x or image of x under f.

The variable x is called independent variable and the variable y is called dependent variable of f.

If x and y are real numbers then f is called real valued function of real numbers.

Domain of the function:

If the independent variable of a function is restricted to lie in some set, then this set is called the domain of the function e.g.

Dom of $f = \{0 \le x \le 5\}$

Range of the function:

The set of all possible values of $f(x)$ as x varies over the domain of f is called the range of f e.g. $y = 100 - 4x^2$.

As x varies over the domain [0,5] the values of $y = 100 - 4x^2$ vary between y=0 (when x=5) and y = 100 (when x=0)

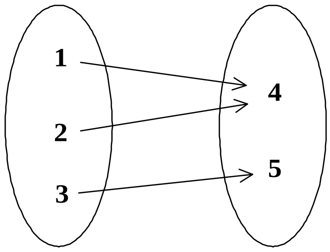
Range of $f = \{0 \le y \le 100\}$

Definition:

A function is a rule by which we relate two sets A and B (say) in such a way that each element of A is assigned with one and only one element of B. For example

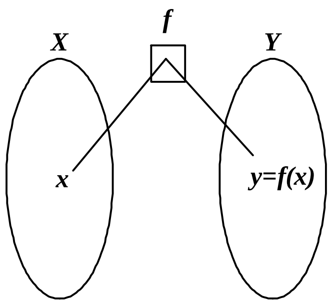
is a function from A to B.

its Domain = {1,2,3} and Range = {4,5}



In general:

A function *f* from a set 'X' to a set 'Y' is a rule that assigns to each element *x* in X one and only one element *y* in Y.(a unique element y in Y)



(*f* is function from X to Y)

If an element "y, of Y is associated with an element "x, of X, then we write y=f (x) &read as y" is equal to f of x. Here f(x) is called image of f at x or value of f at x .

Or if a quantity *y* depends on a quantity *x* in such a way that each value of *x* determines exactly one value of *y*. Then we say that *y* is a function of *x.*

The set x is called Domain of *f* . The set of corresponding elements *y* in y is called Range of *f* . we say that *y* is a function of *x*.

Exercise 1.1

Q1. (a) Given that $f(x) = x^2 - x$

i. $f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

ii. $f(0) = (0)^2 - (0) = 0$

iii. $f(x-1) = (x-1)^2 - (x-1) = x^2 - 2x + 1 - x + 1 = x^2 - 3x + 2$

iv. $f(x^2+4) = (x^2+4)^2 - (x^2+4) = x^4 + 8x^2 + 16 - x^2 - 4 = x^4 + 7x^2 + 12$

(b) Given that $f(x) = \sqrt{x+4}$

$i)\ f(-2)=\sqrt{-2+4}=\sqrt{2}$

$ii)\ f(0)=\sqrt{0+4}=\sqrt{4}=2$

$iii)\ f(x-1)=\sqrt{x-1+4}=\sqrt{x+3}$

$iv)\ f(x^2+4)=\sqrt{x^2+4+4}=\sqrt{x^2+8}$

*Q*2. *Given that*

$i)\quad f(x)=6x-9$

$f(a+h)=6(a+h)-9=6a+6h-9$

$f(a)=6a-9$

$$Now \quad \frac{f(a+h)-f(a)}{h}=\frac{(6a+6h-9)-(6a-9)}{h}$$

$$=\frac{6a+6h-9-6a+9}{h}=\frac{6h}{h}=6$$

$ii)\quad f(x)=\sin x \quad given$

$$\because \quad \sin q-\sin j=2\cos\left(\frac{q+j}{2}\right)\sin\left(\frac{q+j}{2}\right)$$

$f(a+h)=\sin(a+h) \quad and \quad f(a)=\sin a$

$$Now \quad \frac{f(a+h)-f(a)}{h}=\frac{\sin(a+h)-\sin a}{h}$$

$$=\frac{1}{h}\left[\sin(a+h)-\sin a\right]$$

$$=\frac{1}{h}\left[2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)\right]=\frac{1}{h}\left[2\cos\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\frac{1}{h}\left[2\cos\left(\frac{2a}{2}+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]=\frac{2}{h}\cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$iii)\quad Given\,that\quad f(x) = x^3 + 2x^2 - 1$

$$f(a+h) = (a+h)^3 + 2(a+h)^2 - 1 = a^3 + h^3 + 3ah(a+h) + 2(a^2 + 2ah + h^2) - 1$$

$$= a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1$$

$$f(a) = a^3 + 2a^2 - 1$$

$Now\quad f(a+h) - f(a)$

$$= \frac{a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1 - (a^3 + 2a^2 - 1)}{h}$$

$$= \frac{1}{h}\left[a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1 - a^3 - 2a^2 + 1\right]$$

$$= \frac{1}{h}\left[h^3 + 3a^2h + 3ah^2 + 4ah + 2h^2\right] = \frac{h}{h}\left[h^2 + 3a^2 + 3ah + 4a + 2h\right]$$

$$= h^2 + 3a^2 + 3ah + 4a + 2h = h^2 + 3ah + 2h + 3a^2 + 4a = h^2 + (3a+2)h + 3a^2 + 4a$$

$iv)\quad Given\,that\quad f(x) = \cos x$

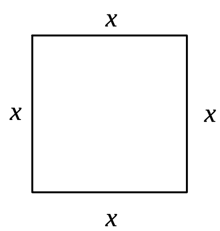
$so\quad f(a+h) = \cos(a+h)$

$and\quad f(a) = \cos a$

$Now\quad \dfrac{f(a+h) - f(a)}{h}$

$$= \frac{\cos(a+h) - \cos a}{h} = \frac{1}{h}\left[-2\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right] = \frac{-2}{h}\sin\left(a + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

Q3. (a) If 'x' unit be the side of square.



Then its perimeter P = $x + x + x + x = 4x$ ……………….. (1)

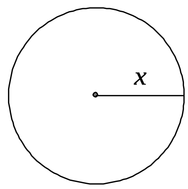
A = Area = $x \cdot x = x^2$ …………… (2)

From (2) $x = \sqrt{A}$ putting in (1)

P = $4\sqrt{A}$

∴ P is expressed as Area

(b) Let x units be the radius of circle



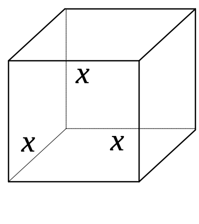
Then Area = A = $p\,x^2$ ……………….. (1)

Circumference = C = $2p\,x$ ……………….. (2)

From (2) $x = \dfrac{C}{2p}$ Putting in (1)

$$A = p\left(\frac{c}{2p}\right)^2 = p\left(\frac{c^2}{4p^2}\right) = \frac{c^2}{4p}$$

$A = \dfrac{c^2}{4p}\quad \because Area\,is\,a\,\,function\,of\,\,Circumference$



(c) Let x unit be each side of cube.

The Volume of Cube = $x \cdot x \cdot x = x^3$ ……………….. (1)

Area of base = A = x^2 ……………….. (2)

From (2) $x = \sqrt{A}$ Putting in (1)

V = $\left(\sqrt{A}\right)^3 = (A)^{3/2}$

$Q5. \quad f(x) = x^3 - ax^2 + bx + 1$

$If \quad f(2) = -3 \qquad and \qquad f(-1) = 0$

$(2)^3 - a(2)^2 + b(2) + 1 = -3$

$8 - 4a + 2b + 1 = -3$

$9 - 4a + 2b = -3$

$12 - 4a + 2b = 0$

$Dividing\ by -2$

$-6 + 2a - b = 0 \ldots\ldots\ldots\ldots\ldots (1)$

$(-1)^3 - a(-1)^2 + (-1) + 1 = 0$

$-1 - a - b + 1 = 0$

$-a - b = 0$

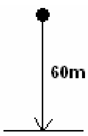
$a + b = 0 \qquad \ldots\ldots\ldots\ldots\ldots (2)$

$Solving\ (1)\ and\ (2)$

$$\begin{array}{rl} 2a - b - 6 & = 0 \\ a + b & = 0 \\ \hline 3a - 6 & = 0 \end{array}$$

$a = 2 \quad and \quad (2) \Rightarrow \ b = -a \qquad \Rightarrow \qquad b = -2$

$Q6. \quad h(x) = 40 - 10x^2$



$(a) \quad x = 1\sec$

$h(1) = 40 - 10(1)^2$

$= 30m$

$(b) \quad x = 1.5\sec$

$h(1.5) = 40 - 10(1.5)^2$

$= 40 - 10(2.25) = 40 - 22.5 = 17.5m$

$(c) \quad x = 1.7\sec$

$h(1.7) = 40 - 10(1.7)^2$

$= 40 - 10(2.89) = 40 - 28.9 = 11.1m$

ii) Does the stone strike the ground = ?

$h(x) = 0$

$40 - 10x^2 = 0$

$-10x^2 = -40 \quad \Rightarrow x^2 = 4$

$x = \pm 2$

Stone strike the ground after 2 sec.

Graphs of Function

Definition:

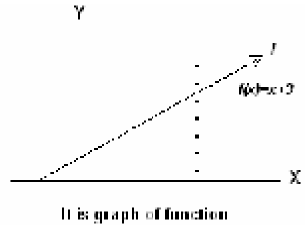
Ex # 1.1 – FSc Part 2

The graph of a function *f* is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane chose co-ordinates (x , y) are input - output pairs for *f* .

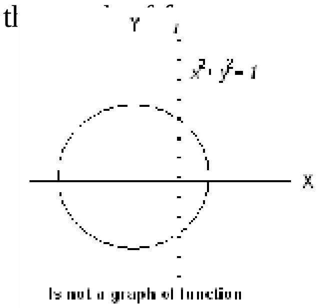
Note that not every curve we draw in the graph of a function. A function *f* can have only one value *f(x)* for each *x* in its domain.

Vertical Line Test

No vertical line can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function. Since some vertical lines intersect the circle Twice. If 'a' is the domain of the function *f* , then the vertical line *x* = a will intersect tl in the single point (a , *f(a)*).



It is graph of function



Is not a graph of function

Types of Function

ALGEBRAIC FUNCTIONS

Those functions which are defined by algebraic expressions.

1) Polynomial Functions:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ **Is a Polynomial Function for all x where** $a_0, a_1, a_2 .... a_n$ **are real numbers, and exponents are non-negative integer .** a_n **is called leading coefft of p(x) of degree n, Where** $a_n \neq 0$

$\Rightarrow$ *Degree of polynomial function is the* max*imum power of x in equation*

$P(x) = 2x^4 - 3x^3 + 2x - 1$ deg*ree* $= 4$

2) Linear Function: if the degree of polynomial fn is '1, is called linear function .i.e. p(x)=ax+b

or $\Rightarrow$ Degree of polynomial function is one.

$f(x) = ax + b$ $a \neq 0$

$\because$ $y = 5x + b$

3) Identity Function: For any set X, a function I: $X \rightarrow x$ of the form y = *x* or *f(x)* = *x*. Domain and range of I is *x*. Note. I (x)= ax +b be a linear fn if a=1,b=0 then I(x)=x or y=x is called identity fn

4) Constant Function: **or**

$C : X \rightarrow y$ *defined by* | $f : X \rightarrow y$ If f (x)=c, (const) then f is called constant fn

$C(x) = a$ $\forall x \in X$ *and* $a \in y$

e.g. $C : R \rightarrow R$

eg y=5

$C(x) = 2$ *or* $y = 2$ $\forall x \in R$

5) Rational Function:

$$R(x)=\frac{P(x)}{Q(x)}$$

Both $P(x)$ *and* $Q(x)$ *are polynomial and* $Q(x)\neq 0$

e.g. $R(x)=\frac{3x^2+4x+1}{5x^3+2x^2+1}$

Domain of rational function is the set of all real numbers for which Q(x)$\neq 0$

6) Exponential Function:

A function in which the variable appears as exponent (power) is called an exponential function.

$i)$ $y=a^x \;\therefore\; x\in R$ $\quad a>0$

$ii)$ $y=e^x \;\therefore\; x\in R$ *and* $e=2.178$

$iii)$ $y=2^x$ *or* $y=e^{xh}$

are some exp*onential functions.*

7) Logarithmic Function:

If $x=a^y$ *then* $y=\log_a{}^x$ x >0

$\because$ $a>0 \quad a\neq 1$

'a' is called the base of Logarithemic function

Then $y=\log_a{}^x$ *is Logarithmic function of base 'a'*

$i)$ *If base* $=10$ *then* $y=\log_{10}{}^x$

is called common Logarithm of x

$ii)$ *If base* $=e=2.718$

$y=\log_e{}^x=\ln x$ *is called natural* log

8) Hyperbolic Function:

We define as

$i)$ $y=\sinh(x)=\frac{e^x-e^{-x}}{2}$ Sine hyperbolic function or hyperbolic sine function

$Dom=\{x/x\in R\}$ *and* $Range=\{y/y\in R\}$

$ii)$ $y=\cosh(x)=\frac{e^x+e^{-x}}{2}$ *is called hyperbolic* cos*ine function* $\Rightarrow$ $x\in R, y\in[1,\infty)$

$iii)$ $y=Tanhx=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{\sinh x}{\cosh x}$ $\quad iv)$ $y=\coth x=\frac{\cosh x}{\sinh x}$

$v)$ $y=\sec hx=\frac{1}{\cosh x}=\frac{2}{e^x+e^{-x}}$ $\quad x\in R$

$vi)$ $y=\cos echx=\frac{1}{\sinh x}=\frac{2}{e^x-e^{-x}}$ $\quad Dom=\{x\neq 0: x\in R$

9) Inverse Hyperbolic Function: (Study in B.Sc level)

$i)\quad y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad for \quad \forall x \in R$

$ii)\quad y = \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad for \quad \forall x \in R \quad and \quad x > 1$

$iii)\quad y = Tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \quad x \neq 1 \quad and \quad |x| < 1$

$iv)\quad y = \sec h^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right) \quad 0 < x \leq 1$

$v)\quad y = \coth^{-1} x = \frac{1}{2}\left|\frac{x+1}{x-1}\right| \quad \because \quad |x| > 1$

$vi)\quad y = \cos ech^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) \quad x \neq 0$

10) Trigonometric Function:

Functions	Domain(x)	Range(y)
$i)\ y = \sin x$	*All real numbers* $\because -\infty < x + \infty$	$-1 \leq y \leq 1$
$ii)\ y = \cos x$	*All real numbers* $\because -\infty < x < \infty$	$-1 \leq y \leq 1$
$iii)\ y = \tan x$	$x \in R - (2k+1)\frac{p}{2}$ $k \in Z$	$\because$ 'R' *all real numbers*
$iv)\ y = \cot x$	$x \in R - kp$ $k \in Z$	R
$v)\ y = \sec x$	$x \in R - (2k+1)\frac{p}{2}$ $k \in Z$	$R - (-1, 1)$ *or* $R - (-1 < y < 1)$
$vi)\ y = \cos ecx$	$x \in R - (kp)$ $k \in Z$	$R - (-1 < y < 1)$

11) Inverse Trigonometric Functions:

Function	Dom(x)	Range(y)
$y = \sin^{-1} x \Leftrightarrow x = \sin y$	$-1 \leq x \leq 1$	$-\frac{p}{2} \leq y \leq \frac{p}{2}$
$y = \cos^{-1} x \Leftrightarrow x = \cos y$	$-1 \leq x \leq 1$	$0 \leq y \leq p$

$y = Tan^{-1}x \Leftrightarrow x = Tany$	$x \in R$ $or\ -\infty < x < \infty$	$-\frac{p}{2} \le y \le \frac{p}{2}$
$y = Sec^{-1}x \Leftrightarrow x = \sec y$	$x \in R-(-1,1)$	$y \in [0, p] - \left\{\frac{p}{2}\right\}$
$y = Co\sec^{-1}x \Leftrightarrow x = \cos ecy$	$x \in R-(-1,1)$	$y \in \left[-\frac{p}{2}, \frac{p}{2}\right] - \{0\}$
$y = Cot^{-1}x \Leftrightarrow x = \cot y$	$x \in R$	$0 < y < p$

12) **<u>Explicit Function:</u>**

If *y* is easily expressed in terms of *x*, then *y* is called an explicit function of *x*.

$\Rightarrow y = f(x)$ *e.g.* $y = x^3 + x + 1$ *etc.*

13) **<u>Implicit Function:</u>**

If *x* and *y* are so mixed up and *y* cannot be expressed in term of the independent variable *x*, Then *y* is called an implicit function of *x*. It can be written as. $f(x, y) = 0$

e.g. $x^2 + xy + y^2 = 2$ etc.

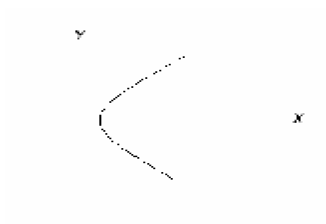
14) **<u>Parametric Function:</u>**

For a function y $= f(x)$ if both *x*& *y* are expressed in another variable say 't' or *q* which is called a parameter of the given curve.

Such as:

i) $x = at^2$ Parametric parabola

$y = 2at$



$y^2 = 4a$

ii) $x = a\cos t$ *Parametric equation of circle*

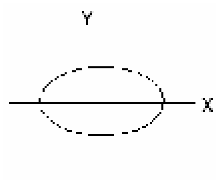
$y = a\sin t$

$x^2 + y^2 = a^2$

iii) $x = a\cos q$ *Parametric equation of Ellipse*

$y = b\sin q$

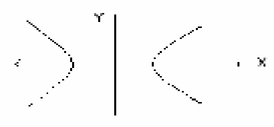
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



vi) $x = a\sec q$ *Parametric equation of hyperbola*

$y = b\tan q$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Exercise 1.1

$Q7.$ *Parabola* $\Rightarrow \quad y^2 = 4ax$ ....(1)

$x = at^2$ ....(i)

$y = 2at$ ....(ii)

To eliminating 't' from (ii) $\quad t = \dfrac{y}{2a}$ *putting* (i)

$$x = a\left(\frac{y}{2a}\right)^2 \Rightarrow x = a\left(\frac{y^2}{4a^2}\right) \Rightarrow x = \frac{y^2}{4a}$$

$\Rightarrow \quad y^2 = 4ax$ *which is same as* (1)

which is equation of parabola.

$ii)$ $\quad x = a\cos q, \quad y = b\sin q$

$\Rightarrow \quad \dfrac{x}{a} = \cos q$...........$(i)$ *and* $\dfrac{y}{b} = \sin q$...............(ii) *To eliminating* q *from* (i) *and* (ii)

Squaring and adding (i) *and* (ii)

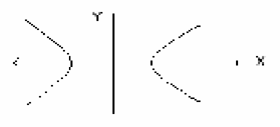
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y^2}{b}\right) = 1$$ *represent a Ellipse*

$iii)$ $\quad x = a\sec q, \quad y = b\tan q$

$\dfrac{x}{a} = \sec q$................(i) $\quad \dfrac{y}{b} = \tan q$..................$(ii)$

Squaring and Subtracting (i) *and* (ii)

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \sec^2 q - \tan^2 q \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \tan^2 q - \tan^2 q \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Which is equation of hyperbola

$Q8.$ (i) $\quad \sinh 2x = 2\sinh x\cosh x$

$$R.H.S = 2\sinh x\cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = 2\left(\frac{e^{2x} - e^{-2x}}{4}\right) = \frac{e^{2x} - e^{-2x}}{2}$$

$= \sinh 2x = L.H.S$

$ii)$ $\quad \sec^2 hx = 1 - \tan^2 hx$

$$R.H.S. = 1 - \tan^2 hx = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1 - \left(\frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2}\right)$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{e^{2x} + e^{-2x} + 2} = \frac{4}{\left(e^x + e^{-x}\right)^2} = \frac{1}{\left(e^x + e^{-x}/2\right)^2}$$

$= \dfrac{1}{\cosh^2 x} = \sec h^2 x = L.H.S$

$iii)$ $\cos eh^2 x = \coth^2 x - 1$

$$R.H.S = \coth^2 x - 1 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1 = \frac{\left(e^x + e^{-x}\right)^2 - \left(e^x - e^{-x}\right)^2}{\left(e^x - e^{-x}\right)^2} = \frac{\left(e^{2x} + e^{-2x} + 2\right) - \left(e^{2x} + e^{-2x} - 2\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{\left(e^x - e^{-x}\right)^2} = \frac{4}{\left(e^x - e^{-x}\right)^2} = \frac{1}{\left(e^x - e^{-x}/2\right)^2} = \frac{1}{\sinh^2 x} = \cos ech 2x = L.H.S$$

$Q9.$ $f(x) = x^3 + x$

replace x by − x

$f(-x) = (-x)^3 + (-x) = -x^3 - x = -\left[x^3 + x\right] = -f(x)$

$\Rightarrow$ $f(x) = x^3 + x$ *is odd function*

$ii)$ $f(x) = (x+2)^2$

replace x by − x

$f(-x) = (-x+2)^2 \neq \pm f(x)$

$f(x) = (x+2)^2$ *is neither even nor odd*

$iii)$ $f(x) = x\sqrt{x^2+5}$

replace x by − x

$f(-x) = (-x)\sqrt{(-x)^2+5} = -\left[x\sqrt{x^2+5}\right] = -f(x)$ $f(x)$ *is odd function.*

$iv)$ $f(x) = \dfrac{x-1}{x+1}$

replace x by − x

$f(-x) = \dfrac{-x-1}{-x+1} = \dfrac{-(x+1)}{-(x-1)} = \dfrac{x+1}{x-1} \neq \pm f(x)$

$f(x)$ *is neither even nor odd function.*

$v)$ $f(x) = x^{\frac{2}{3}} + 6$

replace x by − x

$f(-x) = (-x)^{\frac{2}{3}} + 6 = \left[(-x)^2\right]^{\frac{1}{3}} + 6 = x^{\frac{2}{3}} + 6 = f(x)$

$f(x)$ *is an even function.*
