The Solow Growth Model

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The Solow Growth Model: An *Excel*-Based Primer

Mankiw says a goal of macroeconomic analysis is "to explain why our national income grows, and why some economies grow faster than others..." (186). He identifies "the factors of production—capital and labor—and the production technology as the sources of the economy's output and, thus, of its total income. Differences in income, then, must come from differences in capital, labor, and technology" (186). The model that forms the centerpiece of Mankiw's analysis, and the one developed below, is the Solow growth model. Mankiw says of this model, "The Solow growth model shows how saving, population growth, and technological progress affect the level of an economy's output and its growth over time" (186 - 187). The model also identifies some of the reasons that countries vary so widely in their standards of living.

The second claim for the model, that the model identifies reasons for income differences across countries, is stated in a more reserved fashion than the first, that it explains growth over time. This is as it should be. Indeed, some analysts hold that the Solow model developed below should be applied only to modern industrial economies. Hansen and Prescott say:

Until very recently, the literature on economic growth focused on explaining features of modern industrial economies while being inconsistent with the growth facts describing preindustrial economies. This includes both models based on exogenous technical progress…and more recent models with endogenous growth…. But sustained growth has existed for at most the past two centuries, while the millennia prior have been characterized by stagnation with no significant permanent growth in living standards.

With this caveat in mind, we turn to the development of the Solow growth model. This development follows Mankiw's intermediate-level textbook, *Macroeconomics*, but it can be used as a stand alone module. Each step of the development below is accompanied by a figure or a table from an *Excel* workbook. The workbook contains exercises that enable the manipulation of variables and show how changes impact other variables. The ability to manipulate the variables and animate diagrams facilitates the understanding of the model. For a similar development and for some other macroeconomic topics, see Barreto.

The Production Function

The most basic fact of economic life is scarcity. One way of stating this fact of life is via a production function like this one:

$Y = F(K, L)$.

This function specifies that, for a given technology—defined by F(...), only so much output (Y) can be produced for given employment levels of the inputs capital (K) and labor (L). To turn

this observation into a model of economic growth requires some further assumptions. The assumptions regarding production that underlie the Solow growth models are these:

- A single output is produced. Units of this output can be consumed or added to the capital stock.
- A single type of capital and a single type of labor are employed in the production process.
- The production function exhibits constant returns to scale. That is, changing the employment of both L and K by a proportional factor "z" would cause an equi-proportional change in Y. The workbook upon which the illustrations below are based uses a particular constant-returns-to-scale production function, the Cobb-Douglas function, $Y = K^a L^{1-a}$.
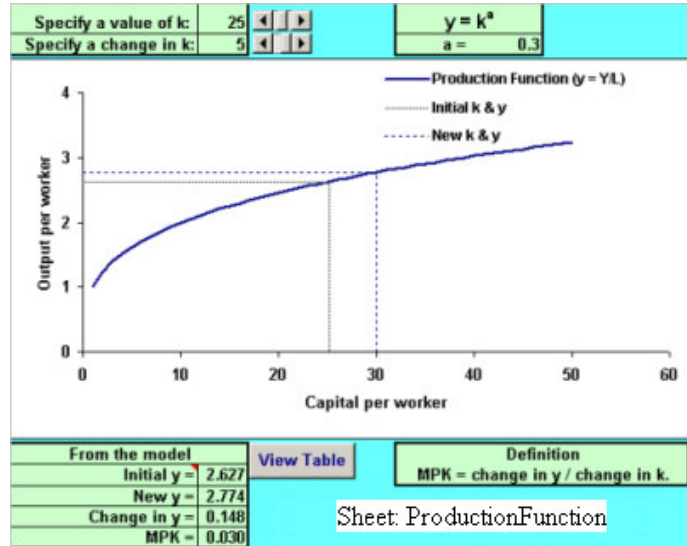
(All graphs are from the workbook. Sheet names appear in the figures.)

A useful characteristic of the constant-returns-to scale production function is that it can be "scaled" by the size of the labor force (assumed to equal, or at least be proportional to, the population). So the per-capita production function is of the form:

$y = Y/L = f(K/L) = f(k)$,

where the lower-case letters indicate per-capita values. In the Cobb-Douglas case,
$Y/L = K^a L^{1-a}/L = K^a L^{-a} = K^a/L^a. = k^a$.
The figure above shows the relationship between y and k.

Two features of the production function stand out: The relationship is positive (more capital per worker implies more output per worker), and the slope decreases as k increases (the marginal product of capital decreases). The marginal product of capital is defined as:
$MPK = (f(k + \Delta k) - f(k))/\Delta k$,
or equivalently
$MPK = f(k_2) - f(k_1)/(k_2 - k_1)$.
In the graph above, $k_1 = 25$ and $k_2 = 30$, so $\Delta k = +5$. In response, y increases from 2.627 units to 2.774 units, so $\Delta y = 0.148$ units. This implies that MPK, over the range considered here is $0.148/5 = 0.0296$, rounded to 0.030. For the Cobb-Douglas production function,
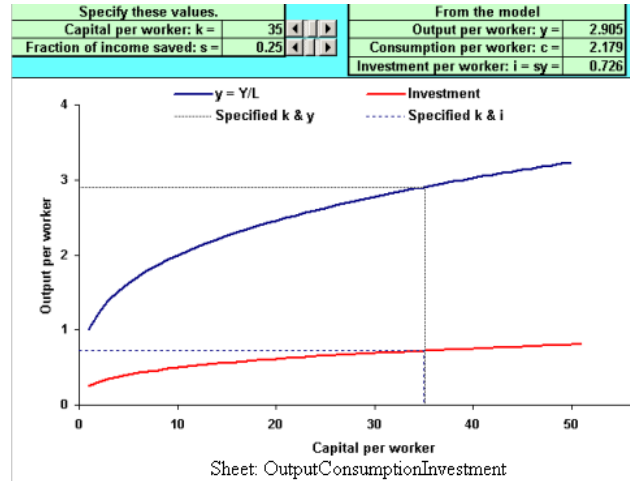$MPL = ak^{a-1}$.
Because $a < 1$, (1 - a) is positive, which implies that $ak^{a-1} = a/k^{1-a}$ decreases as k increases. This fact is reflected in the ever-decreasing slope of the production function in the graph above. See notes on the marginal product.

Output, Income, Consumption, Saving, and Investment

An important lesson of the simple circular flow model is that an economy's output is simultaneously its income; *i. e.*, the means to purchase that output. The next step in developing the Solow model is to trace the implications of this relationship to the allocation of output between consumption and investment. The model's consumption is a simple one:

$$c = (1 - s)y,$$

where c is per-capita consumption and s is the saving rate, the fraction of income not spent for consumption. This simple model is consistent with observed long-run behavior. Friedman cites earlier empirical work by Kuznets that provides evidence of this proportional relationship and develops some of its macroeconomic implications.

Specify these values.		From the model	
Capital per worker: k =	35 ◄ ►	Output per worker: y =	2.905
Fraction of income saved: s =	0.25 ◄ ►	Consumption per worker: c =	2.179
		Investment per worker: i = sy =	0.726

Sheet: OutputConsumptionInvestment

The model assumes that savings are converted, via the capital market, into investment demand. Thus the level of investment demand is:

$$i = sy.$$

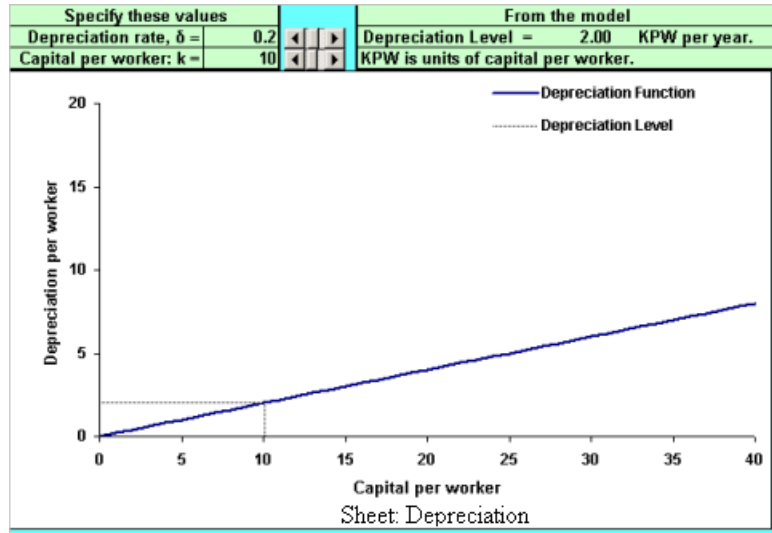This completes the demand for goods and services. The equilibrium condition is that the two are equal:

$$y = c + i .$$

The figure above illustrates these relationships. At any specified value of k (capital per worker), the curve y = Y/L is the total demand for that output. The lower curve (Investment) is the investment demand. And the vertical distance between the two curves is the consumption level.
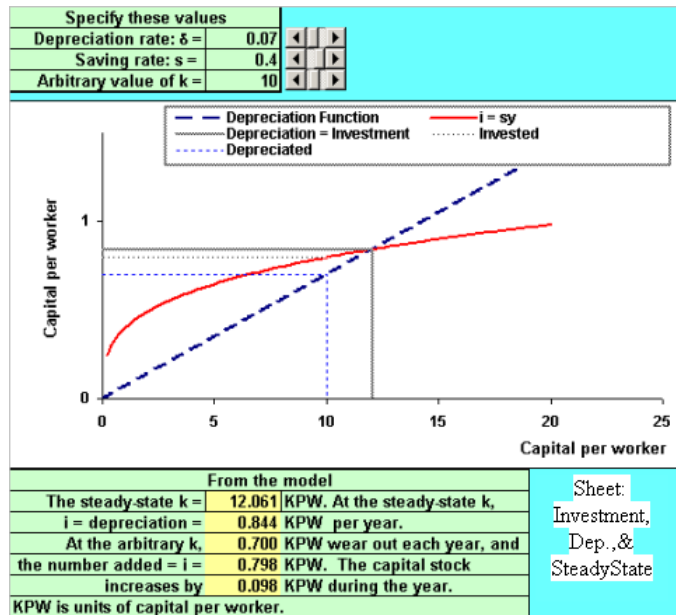
Depreciation

Depreciation is an unfortunate fact of life; capital wears out. The simple model of depreciation used here is that a constant percentage of the total capital stock wears out each year. The example to the right uses a relatively high rate, 20%, so that if the value of k is 10, then 2 units of capital per worker must be replaced each year in order to maintain the capital stock at its beginning-of-the-year level. Any additional investment would result in an increased value of k.

Specify these values				From the model		
Depreciation rate, δ =	0.2	◄	►	Depreciation Level =	2.00	KPW per year.
Capital per worker: k =	10	◄	►	KPW is units of capital per worker.		

Sheet: Depreciation

Steady State, Introduction

Saving is proportional to output (= income), so it increases at a decreasing rate as k increases. For small values of k, saving exceeds depreciation. Since saving equals investment, saving exceeding depreciation implies that the capital stock is growing.

At higher values of k, depreciation exceeds saving (which, to repeat, equals investment). This is so because output rises less than proportionately when k increases while depreciation rises proportionately. Therefore, at higher values of k, depreciation exceeds investment, so the capital stock cannot be maintained.

Specify these values			
Depreciation rate: δ =	0.07	◄	►
Saving rate: s =	0.4	◄	►
Arbitrary value of k =	10	◄	►

From the model			Sheet: Investment, Dep., & SteadyState
The steady-state k =	12.061	KPW. At the steady-state k,	
i = depreciation =	0.844	KPW per year.	
At the arbitrary k,	0.700	KPW wear out each year, and	
the number added = i =	0.798	KPW. The capital stock	
increases by	0.098	KPW during the year.	
KPW is units of capital per worker.			

The illustration at the right shows a case where the initial value of k (10) is below k's steady-state value (12.061). Accordingly, investment equals 0.798 units (40 percent of the income level, which is not shown). Meanwhile, depreciation is 0.07 times k or 0.700. Thus k increases by 0.098 units during this period.
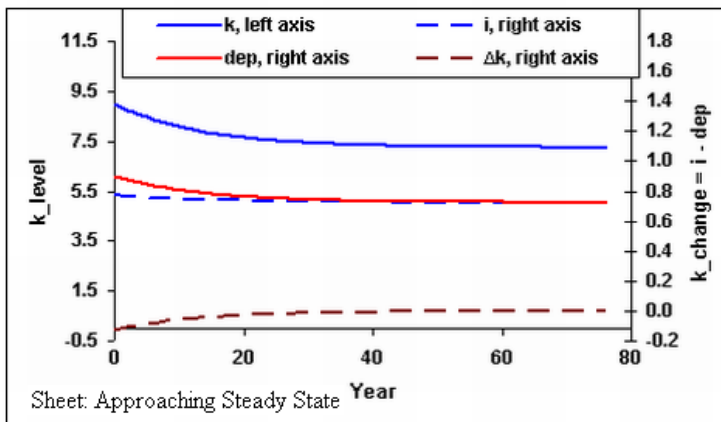
Approaching the Steady State

The graph above shows the adjustment to the steady-state value of k as a function of k itself. Any such adjustment, however, must occur through time. The table at the right provides a view of how the change occurs. The table takes as given the following: the production function ($y = k^{0.3}$), the saving rate (saving = investment = 0.4y), the depreciation function (depreciation = 0.1k), and an initial value of k.

Specify these values.			$y = k^a$		
a =	0.3		$i = sy = y - c$		
s =	0.4		$dep = \delta k$		
δ =	0.1		$\Delta k = i - dep$		
initial k =	9		Steady-state k =		7.246

year	k = K/L	y = Y/L	c = C/L	i = (Y-C)/L	dep	Δk
0	9.000	1.933	1.160	0.773	0.900	-0.127
1	8.873	1.925	1.155	0.770	0.887	-0.117
2	8.756	1.917	1.150	0.767	0.876	-0.109
3	8.647	1.910	1.146	0.764	0.865	-0.101
5	8.453	1.897	1.138	0.759	0.845	-0.086
10	8.080	1.872	1.123	0.749	0.808	-0.059
15	7.823	1.854	1.112	0.741	0.782	-0.041
25	7.523	1.832	1.099	0.733	0.752	-0.020
50	7.291	1.815	1.089	0.726	0.729	-0.003
100	7.247	1.812	1.087	0.725	0.725	0.000
200	7.246	1.811	1.087	0.725	0.725	0.000

Sheet: Approaching Steady State

Given these values, during the base year, the following are true: y = 1.933, so saving = 0.773, which is less than the 0.900 units of depreciation, so the capital stock falls from its initial value of 9 to 8.873, the value observed at the beginning of year 1. This process continues, with the decrements to the capital stock decreasing as k approaches its steady-state value of 7.246. During the10th year, the capital stock falls by 0.041 units, and during the 25th year by only 0.020 units—the per-worker capital stock is quite near its steady-state value. (To see what happens during intervening years, see the full table in the workbook.)

The graph shows how this process plays out over the first 76 years, after which the change in k is less than 0.001. The change in k begins at the relatively low level of -0.127 and quickly approaches zero. This reflects the facts that i is below depreciation (the two middle curves) but that the difference is rapidly vanishing. These ever-decreasing decrements to the capital stock imply that k is decreasing over the period

Sheet: Approaching Steady State

(top curve, which refers to the left axis), but an an ever-decreasing rate.
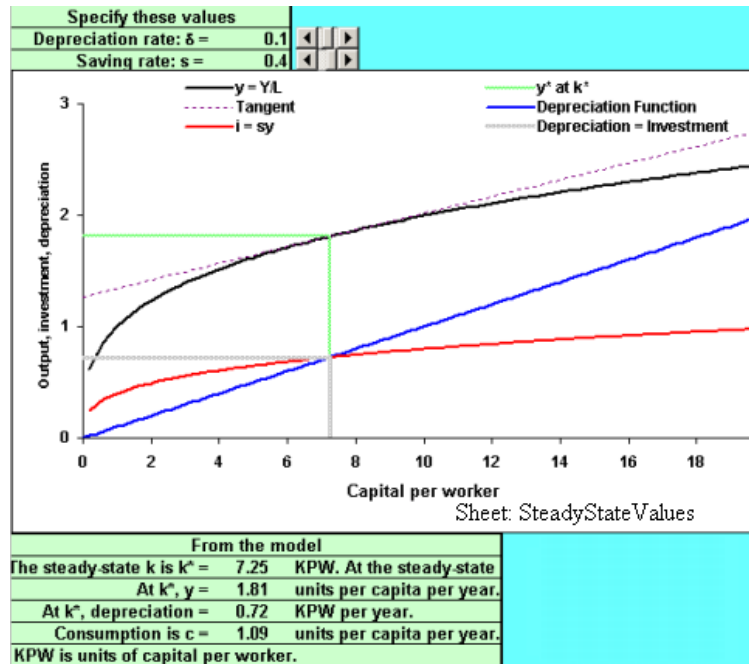
Comparing Steady States

The analysis to this point has been positive, defining how a system works. In that system, the technology and depreciation are given by "nature"—some combination of technological facts of life, institutions, and historical accidents.

The one variable that might be subject to control by policymakers is the saving rate. To some extent that rate is determined by people's preferences regarding future and present consumption, but not entirely. Policies matter. For example, Social Security is a pay-as-you-go transfer program that looks much like a pension plan. Accordingly, its current design can reduce the saving rate. See the note.

Specify these values	
Depreciation rate: δ =	0.1
Saving rate: s =	0.4

Legend:
— y = Y/L
---- Tangent
— i = sy
— y* at k*
— Depreciation Function
···· Depreciation = Investment

Output, investment, depreciation (y-axis)
Capital per worker (x-axis)
Sheet: SteadyStateValues

From the model			
The steady-state k is k* =	7.25	KPW. At the steady-state	
At k*, y =	1.81	units per capita per year.	
At k*, depreciation =	0.72	KPW per year.	
Consumption is c =	1.09	units per capita per year.	
KPW is units of capital per worker.			

Likewise, policies like interest deductions for mortgage interest payments can affect both the level of investment and the sort of capital in which people invest. (The latter, of course, is not addressed in this simple one-good model. See the note.)

If the values of s can be affected by policy and if different values of s lead to different outcomes, then we are faced with a normative issue, to determine a criterion for determining the "best" value of s, and accordingly the "best" steady-state outcome for k, i, and c. We explore a single normative criterion, the maximization of per-capita consumption. That such should be **the** criterion is not self-evident. For example, one might argue for more k, especially if part of k is armament and if one's body politic fears other political entities.

The Golden Rule Steady State

Taking the maximization of per-capita consumption as our goal, we examine the criterion that must be met if the best of the many possible steady-state values of k is to be identified. The graph at the right shows the value of k for which c is maximized.

The value of c does not appear on the graph, but is the difference between y and sy (or, at steady-state, the difference between y and δy). The condition that must be met is that the slope of the y function must equal δ.

Specify these values	
Depreciation rate: δ =	0.1
Saving rate: s =	0.3

From the model			
The steady-state k is k* =	4.80	KPW. At the steady-state k,	
i = depreciation =	0.48	KPW per year.	
At k*, y =	1.60	units per capita per year.	
Consumption is c =	1.12	units per capita per year.	
At k*, MPK =	0.10	KPW during the year.	
KPW is units of capital per worker.			

- To see why, suppose that k is less than this value (which happens to be 4.80 in this case), which would happen if s < 0.3. Then the slope of the y function exceeds that of the depreciation function. so increasing k would cause y to increase by more than depreciation, leaving more for consumption
- Alternatively, suppose that k > 4.8. Then y increases by less than depreciation, so what is left for consumption decreases.
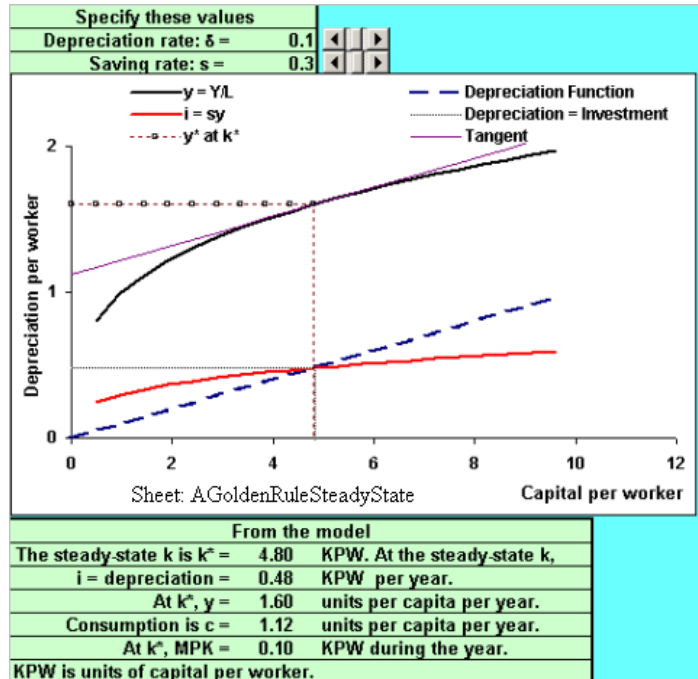
Examination of the graph and of the accompanying tables reveals the optimizing condition—the condition that must be met if the normative criterion is to be satisfied. That condition is that the marginal product of capital (the slope of the y function) must equal the depreciation rate δ. So, if policy is to result in maximum steady-state consumption, then a saving rate must be established such that:

$$MPK = \delta.$$

As Mankiw points out (p. 212), public policy influences national saving in two ways: "The most direct way in which the government affects national saving is through public saving—the difference between what the government receives in tax revenue and what it spends. … The government also affects national saving by influencing private saving—the saving done by households and firms."

Comparing Steady States: Steady-State Consumption as a Function of the Saving Rate

The reasoning above implies that the steady-state equilibrium matters. One question is just how sensitive the outcome, in our case per-capita consumption, is to the steady state. The figure at the right suggests that if the underlying Cobb-Douglas production is a reasonable first approximation to an economy's technology, then the exact value might not be a critical concern.

Steady-state consumption level

a =	0.3
δ =	0.04
$0 \le a \le 0.50$ and $0.01 \le \delta \le 0.10$	

Sheet: SteadyStates
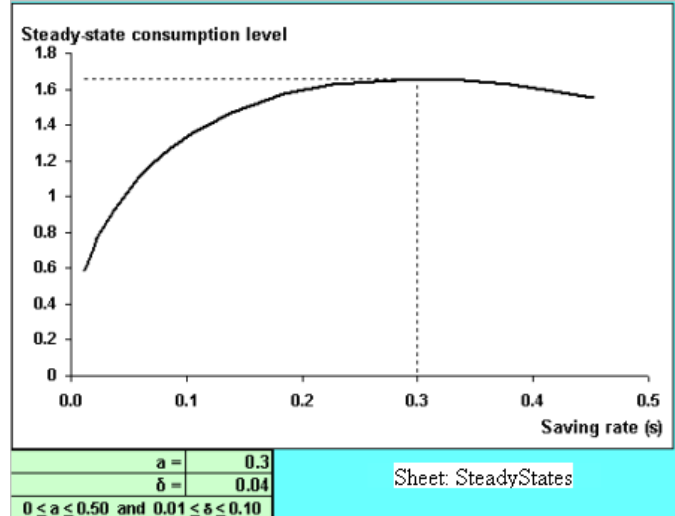
The optimal saving rate is s = 0.3, which results in per-capita consumption of c = 1.660 (see the chart below the graph). If the saving rate falls to just over one-half this level, s = 0.185, the resulting steady-state per-capita consumption falls only to c = 1.571, a decrease of about 5 percent. Likewise, if the saving rate were s = 0.417, more than one-third above the optimal level, c falls only to 1.592, a decrease of about 4 percent.

The table shows the Golden Rule steady-state values for all variables. For the current model, one without population change or technological change, the Golden rule outcome requires that the marginal product of capital equal the depreciation rate, as stated above. With the Cobb-Douglas production function, MPK = ak^{a-1}. Solving for the Golden Rule value of k is straightforward: $k_{GR} = (\delta/a)^{1/(a-1)}$. Given the model's parameters, this implies that the value is k_{GR} = $(0.04/0.3)^{-1/0.7}$ = 17.786. The rest of the values follow from this one as follows:

- y = $k^{0.3}$ = $17.786^{0.3}$ = 2.372,
- δk = 0.04(17.786) = 0.711,
- sy = δk from the nature of steady state,
- s = (sy)/y = 0.711/2.372 = 0.300,
- c = y - s(y) = 2.372 - 0.711 = 1.661 (difference from table value due to rounding errors),
- MPK = $0.3(k^{-0.7})$ = $0.3/(17.786^{0.7})$ = 0.040.

s	k*	y*	δk*	sy	c*	MPK	MPK - δ
"Golden-rule" steady-state values
0.300	17.786	2.372	0.711	0.711	1.660	0.040	0.000
0.012	0.178	0.596	0.007	0.007	0.589		
0.019	0.356	0.733	0.014	0.014	0.719	0.618	0.578
0.023	0.445	0.784	0.018	0.018	0.766	0.529	0.489
0.037	0.889	0.965	0.036	0.036	0.930	0.326	0.286
0.060	1.779	1.189	0.071	0.071	1.117	0.200	0.160
0.064	1.976	1.227	0.079	0.079	1.148	0.186	0.146
0.070	2.223	1.271	0.089	0.089	1.182	0.171	0.131
0.077	2.541	1.323	0.102	0.102	1.221	0.156	0.116
0.086	2.964	1.385	0.119	0.119	1.267	0.140	0.100
0.097	3.557	1.463	0.142	0.142	1.321	0.123	0.083
0.105	3.953	1.510	0.158	0.158	1.352	0.115	0.075
0.114	4.447	1.565	0.178	0.178	1.387	0.106	0.066
0.139	5.929	1.706	0.237	0.237	1.468	0.086	0.046
0.158	7.115	1.802	0.285	0.285	1.517	0.076	0.036
0.185	8.893	1.926	0.356	0.356	1.571	0.065	0.025
0.226	11.858	2.100	0.474	0.474	1.626	0.053	0.013
0.300	17.786	2.372	0.711	0.711	1.660	0.040	0.000
0.341	21.344	2.505	0.854	0.854	1.651	0.035	-0.005
0.380	24.901	2.623	0.996	0.996	1.627	0.032	-0.008
0.417	28.458	2.731	1.138	1.138	1.592	0.029	-0.011
0.453	32.015	2.829	1.281	1.281	1.548	0.027	-0.013

Sheet: SteadyStates

Approaching the Golden Rule Steady State

Suppose that an economy has achieved its steady-state investment rate, but not the one prescribed by the Golden Rule. Then suppose that policy changes occur such that the new saving (= investment) rate results in movement to the Golden Rule levels of k, s, and c. How does this change play out through time? Here we address that case of an economy that has been saving too much, so that its capital stock is too large to generate the maximum flow of consumption. We leave the examination of the other case as an exercise.

Sheet: Adjustments
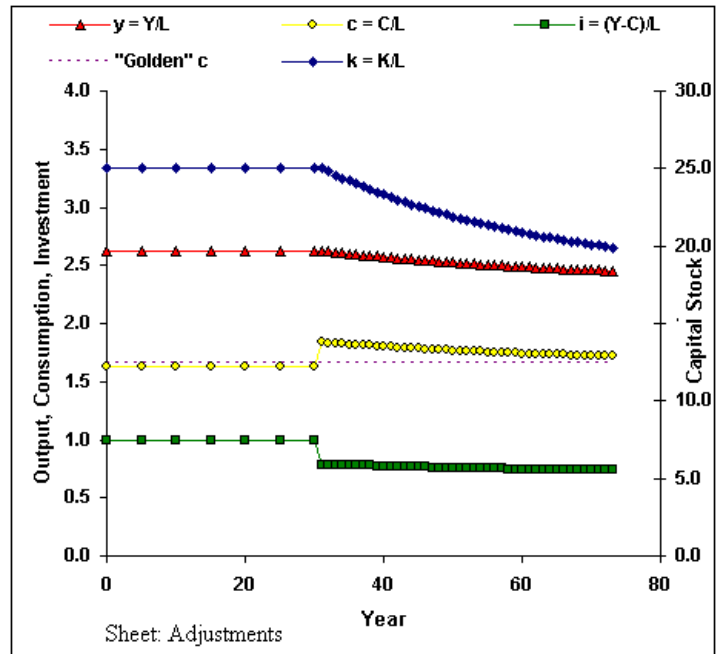
The figure at the right shows that, if the economy were at the Golden Rule steady-state equilibrium, its sustained consumption level would be 1.660. Because the capital stock is above the Golden Rule value, sustaining that capital stock eats into consumption, so the steady-state consumption level is only 1.627 (value read from the spreadsheet), while output is 2.627. This implies that s = (2.627 - 1.627)/2.627 = 0.381. The table above shows that the Golden Rule s is lower, 0.300.

Reducing s to its Golden Rule value, starting in year 31, allows a jump in consumption in that year (from 1.627 to 1.839 (= 0.7 * 2.627). As the capital stock decreases (k's values are on the right axis), so do y and i (= s*y = depreciation, at steady state—values shown on the left axis).

Consumers in each year after 30 have increased consumption, but the model shows a basis for inter-generational tension. The change for s > s$_{GR}$ to s = s$_{GR}$ provides the greatest boon to those in the years immediately after the change. Accordingly, those who institute the policy change in year 30 are appropriating the "free lunch" that those in later years would have enjoyed had s remained at its historically high level of 0.381. The inter-generational tension is, perhaps, more pronounced when the initial s is less than s$_{GR}$. Again, working through that case is left as an exercise.

Population Change and the Steady State

Until now, the population has been held at a constant level, so that k grows whenever K grows (k = K/L, where L is the amount of labor). If L is growing, however, a constant level of K would imply a decreasing level of k.

In this regard, population growth is much like depreciation: both reduce k—depreciation via its effect on the numerator in K/L, and population growth via its effect on the denominator. Mankiw provides the reasoning behind the following equation:
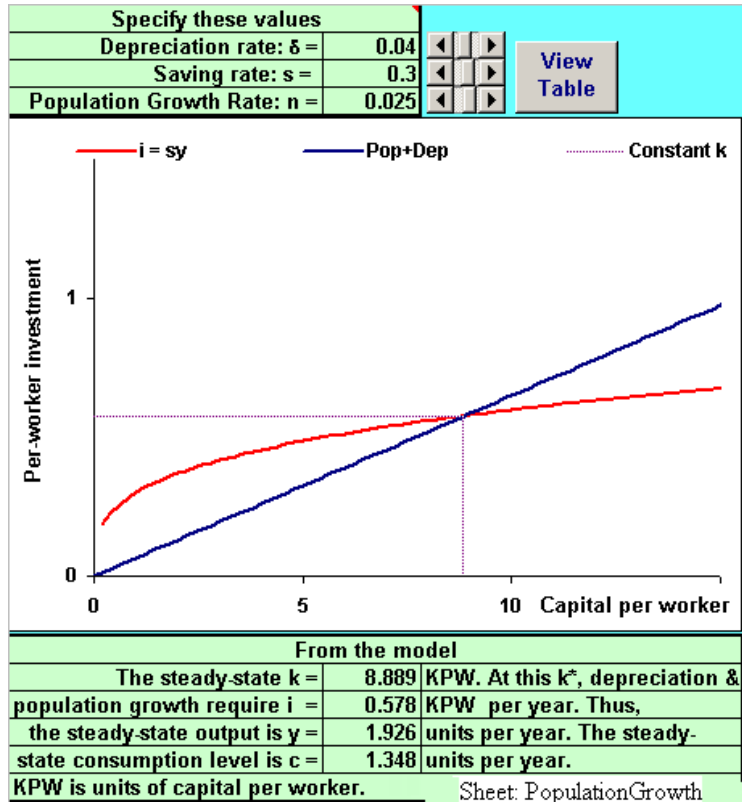
$$\Delta k = i - \delta k - nk$$

or

$$\Delta k = i - (\delta + n)k.$$

The term $(\delta + n)k$ is the amount by which k would decrease in a year's time if no investment were made. This equation is only approximately correct, but the approximation is quite close. See the note. The straight line in the graph shows the amount by which the per-worker capital stock would decrease if no investment were made.

Investment is made, however: sy is invested each time period. Steady-state is attained when sy = $(\delta + n)k$. In the example at the right, a positive population growth rate has been added to the model developed immediately above. When n = 0, a = 0.3 δ = 0.04, and s = 0.30, the resulting steady-state k was 17.786 units of capital per worker. When the population is growing at 2.5 percent per year, however, the same saving function, production function, and depreciation ratio result in a steady-state k of just 8.889 units of capital per unit of labor. Accordingly, per-capita output is 1.926 units, down from 2.372 when n = 0.

Specify these values			
Depreciation rate: δ =	0.04	◄ ►	View Table
Saving rate: s =	0.3	◄ ►	
Population Growth Rate: n =	0.025	◄ ►	

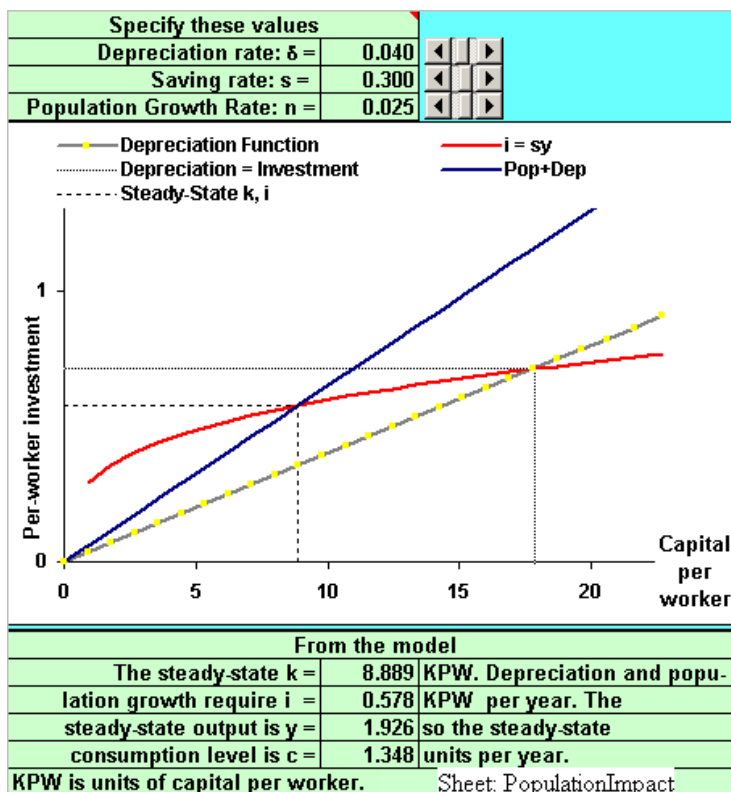| From the model	
The steady-state k =	8.889 KPW. At this k*, depreciation &
population growth require i =	0.578 KPW per year. Thus,
the steady-state output is y =	1.926 units per year. The steady-
state consumption level is c =	1.348 units per year.
KPW is units of capital per worker.	Sheet: PopulationGrowth

Population Change II

The graph at the right recreates the one above, with two exceptions. First, the depreciation-only (n = 0) case is included for comparison. Also, the population growth rate is a bit higher, 3% rather than 2.5%. The result of this increase is that the steady-state k falls from 8.889 to 7.996. Per-capita output and consumption fall as a result of the decreased k.

Is this particular steady-state outcome the "Golden Rule" outcome, the one that maximizes sustained per-capita consumption? As we shall see, yes.

That outcome requires that MPK = δ + n. See note. Given the production function employed here, the table below reports the Golden Rule values when the population growth rate is 2.5 percent per year.

Specify these values		
Depreciation rate: δ =	0.040	◄ ►
Saving rate: s =	0.300	◄ ►
Population Growth Rate: n =	0.025	◄ ►

Legend: —Depreciation Function —i = sy ⋯Depreciation = Investment —Pop+Dep ------Steady-State k, i

Per-worker investment (vertical axis); Capital per worker (horizontal axis), values 0, 5, 10, 15, 20

From the model		
The steady-state k =	8.889	KPW. Depreciation and popu-
lation growth require i =	0.578	KPW per year. The
steady-state output is y =	1.926	so the steady-state
consumption level is c =	1.348	units per year.
KPW is units of capital per worker.		Sheet: PopulationImpact

**Golden Rule values when
population grows at 2.5% per year**

$$n = 0.025$$
$$k_{GR} = 8.889$$
$$y_{GR} = 1.926$$
$$c_{GR} = 1.348$$
$$i_{GR} = 0.578$$
$$s = 0.300$$

For the Cobb-Douglas production function, the value of s that corresponds to Golden Rule consumption is still the the exponent of k in the production function $y = k^a$.
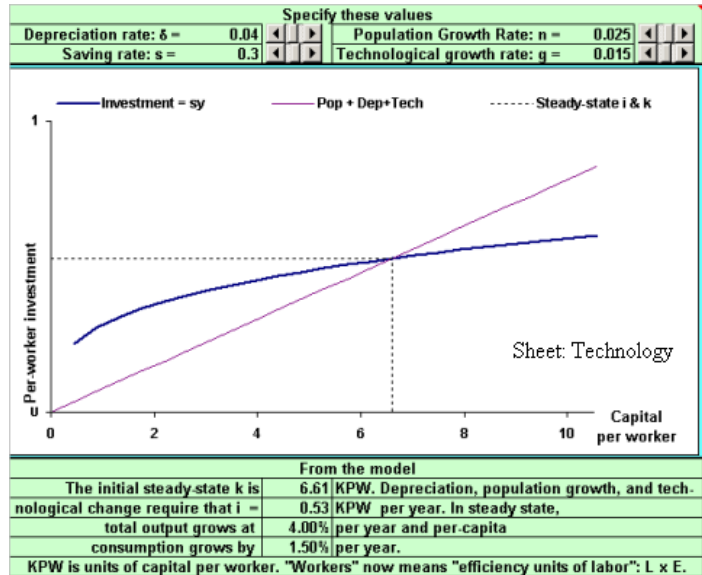
The analysis above treats n as exogenous. Both n and s, however, might be sensitive to policy actions. Now policymakers have two potential tools for affecting the steady-state level of c. They can implement policies to change s, or they can implement policies to affect n. Many modern industrial countries are actively pursuing pro-natalist policies (for reasons unrelated to maximization of c), and some developing countries have implemented policies designed to reduce n, the most notorious being that of China. See Eberstadt.

Technological Change and the Steady State

The Solow growth model treats technology as if more workers were being added. That is, the effective labor force now becomes L times E, where E is a measure of productivity. With this new source of change, the capital per effective unit of labor, the new capital-per-unit-of-labor variable becomes
$$k = K/(L \times E).$$

Now, absent investment, k changes over time for three separate reasons: depreciation of the capital stock, population growth, and productivity growth.

Specify these values				
Depreciation rate: δ =	0.04		Population Growth Rate: n =	0.025
Saving rate: s =	0.3		Technological growth rate: g =	0.015

Sheet: Technology

From the model		
The initial steady-state k is	6.61	KPW. Depreciation, population growth, and tech-
nological change require that i =	0.53	KPW per year. In steady state,
total output grows at	4.00%	per year and per-capita
consumption grows by	1.50%	per year.
KPW is units of capital per worker. "Workers" now means "efficiency units of labor": L x E.		

With this new source of change in k, the change in k becomes the following:
$$\Delta k = i - (\delta + n + g)k,$$

where the first two terms inside the parentheses are as developed earlier, and g is the annual rate at which labor productivity changes. See the note for the derivation of the term $(\delta + n + g)k$.

To see why per-capita consumption grows at the rate g, consider the graph at the right. Steady-state equilibrium now requires that both the amount of capital and the amount of investment per efficiency unit of labor be constant. By the same token y, the output per efficiency unit of labor must be constant. Output must, therefore, grow at a rate (n + g). The number of workers grows at a rate of n, so the difference, g, is the annual rate of increase of per-worker output. Since consumption grows exactly in proportion with output, consumption per worker also grows at a rate equal to g.

The analysis above treats g as exogenous. Both n and s, however, might be sensitive to policy actions. Now policymakers have three potential tools for affecting the steady-state level of c. They can implement policies to change s, they can implement policies to affect n, and they can implement policies to affect g. Such policies include those related to copyright and patents, as well as tax breaks for research and development or subsidies for basic research..

Technological Change II

The graph at the right shows some of the same information as above, but from a different perspective. The worksheet from which this graph is copied focuses on the implications of s, n, and g for per-capita output and consumption.

The graph at the right is based on the assumption that the economy is on its Golden Rule steady-state path. The following exercise is instructive: Set the saving rate very low (what happens if s = 0?) and raise it toward the Golden Rule value and then above that value. Observe that, as s increases so does the Y/L trend for all values of s. In contrast,

Specify these values		
Depreciation rate: δ =	0.030 ◄ ►	Technological
Saving rate: s =	0.300 ◄ ►	growth rate: g = 0.015 ◄ ►
Population Growth Rate: n =	0.025 ◄ ►	a = 0.300 ◄ ►

Output, Consumption per capita — Sheet: Technology&Growth

— Y/L
— — C/L

From the model	
The steady-state k =	8.00 KPW. Depreciation, population growth, &
technological change require that i =	0.56 KPW per year.
In steady-state, output grows at	4.00% per year.
Per-capita consumption grows by	-1.00% per year.
KPW is units of capital per worker. "Workers" now means "efficiency units of labor": L x E.	

however, the C/L trend shifts upward only until s = a (the Golden Rule value) and then shifts downward, with the ever-increasing difference between Y/L and C/L being the depreciation of the ever-larger capital stock.
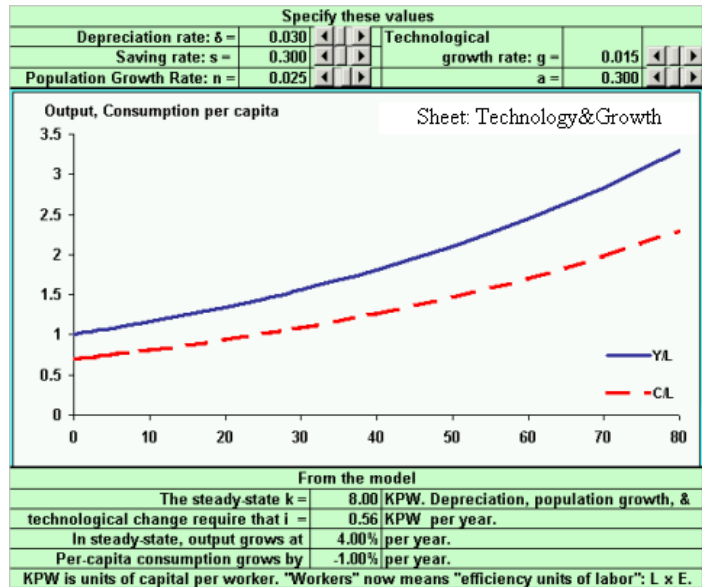
This worksheet is normalized so that L = 1 and Y = 1 in the first year. Thus, the initial value of E is determined in a way that makes this normalization "work." As a result, the sheet does not not directly

Year	L	E	K	Y	Y/L	C/L
0	1.000	0.502	5.000	1.000	1.000	0.700
1	1.015	0.509	5.151	1.030	1.015	0.711
2	1.030	0.517	5.307	1.061	1.030	0.721
3	1.046	Sheet: Technology&Growth			1.046	0.732

show the negative impact of population growth on per-capita consumption, but the negative effect can be can be deduced. The two inserts at the right show part of the table that gives rise to the graph above. In one case, the population growth rate is n = 1.5% and in the other it is n = 2.5%. In both cases, the growth of per-capita consumption is the same--it

Year	L	E	K	Y	Y/L	C/L
0	1.000	0.536	4.286	1.000	1.000	0.700
1	1.025	0.544	4.459	1.040	1.015	0.711
2	1.051	0.552	4.639	1.082	1.030	0.721
3	1.077	0.Sheet: Technology&Growth6				0.732

grows at a rate equal to g, the rate of technological change. What differs, however, is the initial level of efficiency necessary to sustain these identical paths. When the population is growing at 1.5 percent per year (L = 1, 1.015, 1.046, ...), an initial efficiency index of 0.552 is sufficient to generate observed income stream. When population is growing at 2.5 percent per year (L = 1, 1.025, 1.077, ...) the necessary efficiency index is 0.582 in the initial period. This means that a given group of laborers with a given level of efficiency must have lower consumption if the population is growing faster.

An Empirical Note

While reservations about the adequacy of the simple Solow model for explaining differences in economic growth are warranted, the model's predictions are consistent with observed outcomes. The estimated equation below is based on a data set developed by Mankiw, Romer, and Weil. Based on cross-section data from 121 countries, the following estimates are derived:

$$gdp\_growth\_rate = 2.246 - 1.344*OECD + 0.115*investment\_rate.$$
$$(2.934) \qquad (5.036)$$

The coefficient of determination is $R^2 = 0.185$. The dependent variable is the average annual growth rate between 1960 and 1985. OECD is a binary variable that equals 1 if the country is a member of the Organization for Economic Cooperation and Development; the point estimate indicates that the growth rates averaged about 1.3 percentage points less for these countries than for others. The growth rate increased by an estimated 0.115 percentage points per one-percentage-point increase in the fraction of a country's income that was invested. The associated t-statistic is quite large, indicating strong evidence that investment affects output. While investment is an important part of the story, it is far from being the whole story: The R^2 of 0.185 indicates that either variables other than the two included above or random effects account for 81.5 percent of the variation among growth rates.

Notes

The decreasing importance of land

[Hansen and Prescott](#) argue that in a pre-industrial economy, the fact that land is a fixed factor has serious implications for the relevance of the Solow model, in which no factor is in fixed supply. They point out that the implication of land's being a fixed factor becomes increasingly unimportant as economies progress toward the industrial (and post-industrial) stage. Their Table 2 (page 1209) shows this progression for the United States.

TABLE 2—U.S. FARMLAND VALUE RELATIVE TO GNP

| Year	Percentage
1870	88
1900	78
1929	37
1950	20
1990	9

Deriving the marginal product function from the production function

At any point on the production function the marginal product is the derivative of the function with respect to the independent variable. For the Cobb-Douglas production function, $y = k^a$, MPL $= ak^{a-1}$. Defining the MPL in terms of per-capita values might appear inappropriate. After all, MPL is typically defined as the change in total output per one-unit change in the variable input (capital here) given the employment level of the fixed input (labor here). Are the two definitions compatible? To see that they are return to the original production function:

$$Y = K^a L^{1-a}.$$

| Specify a value of k: | 25 | ◄ | ► | View Table |
| Specify a change in k: | 5 | ◄ | ► | |

From the model	
Initial y =	2.627
New y =	2.774
Change in y =	0.148
MPK =	0.030
Slope of "Tangent" =	0.032

Definition
MPK = change in y / change in k.
Slope of "Tangent" is MPK at a point.
Calculated MPK is MPK over the range.

The marginal product of capital is

$$\begin{aligned}
\text{MPK} &= aK^{a-1}L^{1-a} \\
&= aK^{a-1}/L^{-(1-a)} \\
&= aK^{a-1}/L^{a-1} \\
&= (K/L)^{a-1} \\
&= k^{a-1}.
\end{aligned}$$
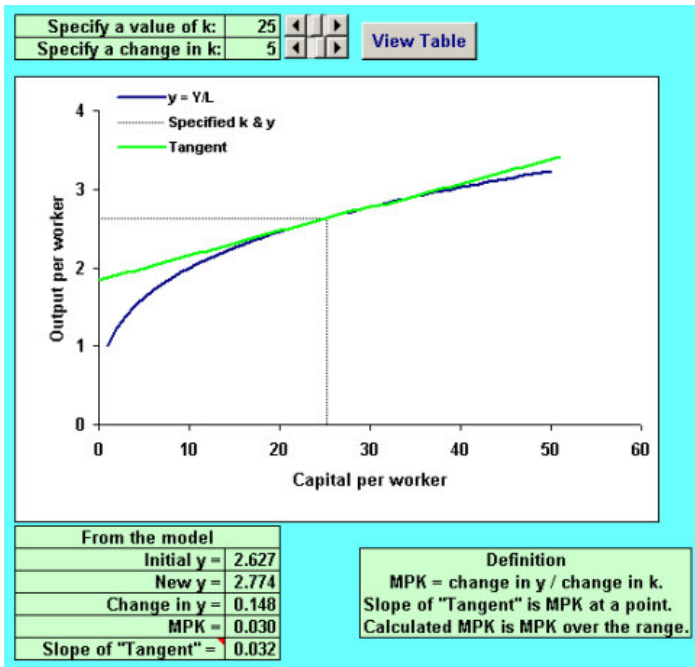
This graph shows how the MPL as derivative compares to MPL in terms of discrete changes. In this case, $\Delta k = 5$. The true MPL for this size change is 0.030, the value derived in the text. The derivative is a short-hand way of defining the MPL for the entire function. At the initial value of k, the MPL is 0.032, which overstates the change when $\Delta k = 5$ rather than an arbitrarily small value, but the overstatement is slight.

Downloading the workbook

- We recommend saving the workbook file to a disk and then opening it.
- The workbook contains macros. Activating the macros requires that *Excel*'s security level be set at medium or lower <u>before the workbook is opened</u>. The default is high, and will not allow the opening of macros. Under "Tools/ Macro.. / Security" set the security level to <u>medium</u>. To repeat: <u>Failing to set the security level below its default level will cause Excel to load the workbook but to strip it of all macros.</u>

Social Security and savings

Most observers see the potential of Social Security for reducing saving as a weakness. It was not always so. According to Feldstein (page 10), "Keynesian economists in the 1940s … praised the unfunded character of the new Social Security program for its ability to depress national saving and stimulate aggregate demand."

Differing production functions

One reason that the "aggregate production function" that represents an economy might differ from one economy to another is the degree to which funds are allocated to those investments with the highest rates of return. If capital markets were perfect and if property rights were perfectly defined and enforced, then a system of markets would ensure equal marginal rates of return for all investments. As noted above, however, subsidies might favor one type of investment (in residential real estate in this case) over others. Mankiw (227) points out that three major types of investment can be identified: infrastructure (roads, bridges, sewer systems, *etc.*), human capital, and investments in non-infrastructure physical capital. Significant barriers to equalizing rates of returns across these broad categories can be identified. Furthermore, equalization within the categories is unlikely. Such is even more so in many pre-industrial economies. DeSoto argues that an important reason for failure of many third-world countries to develop is insecure property rights. He observes that squatters build up a considerable stock of capital, in the form of housing, without any clear title. They cannot, however, use any equity in this housing to underwrite small businesses, no matter how high the rates of returns from such investments might be.

We are examining the rate at which k would decrease if it were not replaced. Now, two factors lead to reduced k: depreciation of the capital stock and dispersion of the capital stock among increasingly more workers. Consider two adjoining periods, 0 and 1:

$K_1 = K_0(1 - \delta)$ (Depreciation)

$L_1 = L_0(1 + n)$ (Population growth)

so

$k_1 = k_0(1 - \delta)/(1 + n)$

Some algebra shows that

$k_0 - k_1 = k_0(\delta + n)/(1 + n)$

Here is the algebra.

$k_0 - k_1$ is the amount by which the capital stock declines, absent offsetting investment.

$k_0 = K_0/L_0$ and $k_1 = K_1/L_1 = (K_0/L_0)(1 + \delta)/(1 + n)$, so

$k_0 - k_1 = k_0 - k_0(1 - \delta)/(1 + n)$

$k_0 - k_1 = [k_0(1 + n) - k_0(1 - \delta)]/(1 + n)$

$k_0 - k_1 = k_0(\delta + n)/(1 + n)$

This is the amount that the per-capita stock of capital decreases if no investment is made. This equation differs slightly from the equation in Mankiw and the equation used in the workbook. For simplicity, the denominator $(1 + n)$ is ignored. Leaving out this term simplifies the exposition at the cost of introducing an error of $1/(1 + n)$. For reasonable values of n, this error is about 2 or 3 percent.

To repeat, the simpler equation that very closely approximates the actual decrease in k in the absence of any investment is this:

$\Delta k = k(\delta + n)$

To illustrate, suppose that $k = 10$ and that depreciation and population growth are as indicated above. For concreteness, suppose that $K = 10,000$ and $L = 1000$ in the base year. Then a year later, $K = 10,000(0.96) = 9600$ and $L = 1000(1.025) = 1025$. Therefore, in the next year $k = 9600/1025 = 9.366$. Except for rounding, this values equals $10,000(1 - 0.04)/1.025$, which is 9365.854. The decrease in k from 10 to 9.366 implies that $10 - 9.366$ or 0.634 units of output per worker must be set aside for maintenance of the per-worker capital stock, roughly 0.4 to maintain the necessary 10,000 units of capital and 0.234 to provide the 25 additional workers with as much capital as the initial 1000 workers had.

The numbers shown are exact. Compare them to the results of the simpler equation: $10(0.04 + 0.025) = 0.650$. The discrepancy is $(0.650 - 0.634)/0.634$ or about 2.5 percent.

Consumption, saving, and investment

Consumption is $c = y - i$ because $i = s$. For any steady-state to occur $i = (\delta + n)k$. Therefore, we seek the value of k that maximizes $y - (\delta + n)k$. But y is f(k). To find the maximum value of c, find the k for which the slope of the $y = f(k)$ function equals $(\delta + n)$. That is, find the k for which $MPK = (\delta + n)$.

Factors changing the value of k

When technology is taken into account k is defined as follows: $k = K/(L * E)$. L changes at a rate of n, and E changes at a rate of g. The capital stock depreciates at a rate δ. Consider the implication for Δk for two adjoining years. Year 1 values are as follows:

$L_1 = L_0(1 + n)$ and $E_1 = E_0(1 + g)$.

Accordingly $k_1 = K_1/(L_1 \times E_1) = \{(K_0 - \delta K_0)/[L_0(1 + n)E_0(1 + g)]\} = k_0(1 - \delta)/[(1 + n)(1 + g)]$. Absent investment, $k_1 < k_0$ for three reasons: depreciation, increase in the population, and increase in the number of efficiency units of labor per worker.

We now use this information to determine the exact relationship between k_0 and the decrease in k, absent investment.

$k_0 - k_0(1 - \delta)/[(1 + n)(1 + g)] =$

$[k_0(1 + n + g + ng)] - k_0(1 - \delta)]/(1 + n + g + ng) =$

$[k_0(\delta + n + g + ng)]/(1 + n + g + ng)$

This differs slightly from the approximation used in the text above, and used by Mankiw. There the decrease in k per time period, absent investment, is $k(\delta + n + g)$. To see how much the two differ, suppose that n = 2.5 percent and g = 2.0 percent, both fairly large values. Let $\delta = 4$ percent. Mankiw's approximation is that k falls by $(0.04 + 0.02 + 0.025)k$ or by 0.085k. In fact, over a year's time, the exact decrease is

$[(0.04 + 0.02 + 0.025 + .0005)/(1.02*1.025)]k = 0.081779k$, for an error of less than 4 percent.

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