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EXPECTATIONS, FORECASTING, AND PERFECT FORESIGHT

A Dynamical Systems Approach

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The paper studies the nature of expectations formation rules for deterministic economic laws with an expectations feedback within the framework of dynamical systems theory. In such systems, the expectations formation rules, called predictors, have a dominant influence. The concept of a perfect predictor, which generates perfect-foresight orbits, is proposed and analyzed. Necessary and sufficient conditions are given for which local as well as global perfect foresight is possible. The concept is illustrated for the general linear model as well as for models of the cobweb type. For the standard overlapping generations model of economic growth, the existence of perfect predictions depends strongly on the savings behavior of the agents and on the technology.

Keywords: Dynamics, Perfect Foresight, Rational Expectations

1. INTRODUCTION

The interplay between realizations of economic variables and expectations concerning the development of these variables constitutes one of the central features of economic systems. On the one hand, actual realizations of a market process depend on agents' expectations about the future prior to the realizations of these variables. On the other hand, these expectations typically are formed on the basis of observed past realizations of the economic variables. Thus, realizations of the past influence the new realizations through the way in which economic agents perceive this interdependence. The interaction of the market process with an expectation formation then determines the actual evolution of the system.

The existing literature usually treats dynamic problems in economic systems by analyzing solutions of a system of implicit difference equations; see, e.g., Grandmont (1985), Grandmont and Laroque (1986), or Chiappori and Guesnerie (1991). Unfortunately, the mathematical techniques available to treat these implicit equations allow, at most, a local analysis in a neighborhood of a stationary solution, i.e., a steady state or a cycle. In this case the implicit function theorem yields an

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implicitly defined function that describes the dynamics of the system in a local neighborhood of this stationary solution. All dynamic properties such as stability or bifurcations are linked to this map. Except for special situations, an explicit solution in the sense of a (local) flow of mappings cannot be computed analytically. In fact, only a linearization of the local dynamics can be computed explicitly. In the case of an unstable steady state this implies that nothing can be said about a solution that eventually leaves the neighborhood of the stationary solution. Also, in cases in which the fixed point is nonhyperbolic, the Hartman-Grobman theorem fails and the linearized dynamics have, in general, nothing to do with the original nonlinear dynamics.

This critique applies in particular for models with rational expectations or perfect foresight. An examination of the existing literature reveals that the research on rational expectations concentrates more or less on the description and characterization of *equilibria* with perfect foresight, or rational expectations; see, e.g., Grandmont (1985), Grandmont and Laroque (1986), or Chiappori and Guesnerie (1991). These equilibria again are defined as implicit solutions of a system of equations near steady states. For the same reasons as above, the resulting perfect-foresight dynamics are therefore of a strictly local nature. Moreover, in nonlinear models it is left unclear which forecasting rules could generate rational expectations equilibria. Taken on its own grounds, in this approach the property of rational expectations is a definitional element of an equilibrium concept and not a description of a particular class of forecasting rules that are perfect or rational in a well-specified sense.

These two drawbacks can be overcome by modeling the recursive structure of economic systems explicitly. The approach taken in this paper is that such modeling is done on the basis of economic reasoning and not by simply applying the implicit function theorem. On the one hand, this means that the basic economic mechanisms have to be described by an explicit map called an *economic law* and not by an implicit equation. On the other hand, for models with an expectations feedback, one has to model how agents form expectations for future realizations of the economy. This can be done by means of a *forecasting rule* as has been done for a purely implicit setup; see, e.g., Grandmont (1988). However, contrary to the earlier work, the combination of an *economic law* in the above sense and a *forecasting rule* then yields an *economic dynamical system* that is defined explicitly and globally on the whole state space. Thus, for arbitrary initial values, the orbits of the system can be generated by simply iterating a well-specified map. The important advantage of such an approach is that the mathematical techniques for analyzing the global dynamics of such a system, i.e., attractors, basins of attraction, bifurcations, stability analysis, etc., can be applied in a straightforward manner.

In this paper, the setup of these explicitly defined economic dynamical systems is the starting point for addressing the question which forecasting rules could generate perfect foresight in the classical sense. This means pointwise coincidence of realizations and forecasts along paths (or orbits) of the system. A structural answer in a general nonlinear setting is provided by developing the formal framework under which perfect forecasting rules (those generating perfect-foresight orbits) can be analyzed. Necessary and sufficient conditions are given for which local as well as global perfect foresight is possible, indicating that for many systems these are quite strong. The concept is illustrated for the multivariate linear model as well as for models of the cobweb type, confirming results known from the literature. It is shown that the standard overlapping generations (OLG) model of economic growth fits into the proposed framework. The analysis reveals that the existence of perfect predictors depends strongly on the structural economic features, i.e., on the savings behavior, the form of the technology, and the size of the depreciation.

The question of learning, i.e., of finding such *perfect* forecasting rules while the dynamical system is evolving, is left to future research. For linear stochastic models, this problem is solved to a large extent; see, e.g., Zenner (1996), Evans and Honkapohja (1997).

2. GENERAL SETTING

Let $X \subset \mathbf{R}^n$ denote the space of endogenous variables of an economic system under investigation. Assume that for each time t, the vector of endogenous variables $\boldsymbol{x}_t \in X$ can be subdivided into $\boldsymbol{x}_t = (\bar{\boldsymbol{x}}_t, \boldsymbol{y}_t) \in \bar{X} \times Y = X$ where $\boldsymbol{y}_t \in Y \subset \mathbf{R}^q$ is the vector of variables for which expectations are formed, $\bar{\boldsymbol{x}}_t \in \bar{X} \subset \mathbf{R}^p$ is the vector of the remaining variables, and $n = p + q$. A function,

$$F : X \times Y \to X, \qquad (1)$$

is called a (discrete time) *economic law* with the interpretation that all states of the economy in time are given by

$$\boldsymbol{x}_{t+1} = F\left(\boldsymbol{x}_t, \boldsymbol{y}^e_{t,t+1}\right), \qquad t \in \mathbf{N},$$

where $\boldsymbol{x}_t$ is the current state of the economy and $\boldsymbol{y}^e_{t,t+1} \in Y$ is the predicted value for $\boldsymbol{y}_{t+1}$ formed at time t; $\boldsymbol{x}_t$ may well be a vector of lagged endogenous variables as well as of past forecasts. Thus the arguments of F are all observable variables up to time t. With this interpretation, F generates the (true) realizations of an economy in one step and not further ahead. The case in which the economic law (1) takes *expectational leads* as arguments, i.e., predictions for values of endogenous variables beyond the step-one realization of F, is not treated here; see Böhm and Wenzelburger (1997a) for this generalization.

The economic law (1) can be split into a pair of maps $F = (\bar{F}, f)$ yielding a system of equations

$$\begin{cases} \bar{\boldsymbol{x}}_{t+1} = & \bar{F}\left(\boldsymbol{x}_t, \boldsymbol{y}^e_{t,t+1}\right) \\ \boldsymbol{y}_{t+1} = & f\left(\boldsymbol{x}_t, \boldsymbol{y}^e_{t,t+1}\right) \end{cases}, \qquad (2)$$

with functions $\bar{F} : X \times Y \to \bar{X}$ and $f : X \times Y \to Y$. In many applications it is assumed that agents form expectations for all relevant variables of the economic process under consideration. Then $X = Y$ and $F : X \times X \to X$.

The law F is only part of the description of what is sometimes called an economic law of motion. F does not describe a dynamical system because, formally, it is not a map from $X \times Y$ into itself as required by dynamical systems theory. However, on intuitive grounds, F describes the evolution of an economic system but not how predictions are formed over time. The structure of (1) covers a large comprehensive class of dynamic economic models in which predictions of agents play a role. These include many macroeconomic disequilibrium models [see, e.g., Böhm et al. (1994)], models of financial markets, many partial equilibrium models, in particular all models of the cobweb type, and all standard models of economic growth. The latter is treated extensively below.

For a complete description of an economic dynamical system, i.e., a dynamical system associated with the economic law (1), it is necessary to specify the way in which the predicted value $\boldsymbol{y}^e_{t,t+1}$ made for $\boldsymbol{y}_{t+1} \in Y$ is determined. By this is meant a function that generates predictions using the information provided in period t. In this setting, it is assumed that all of the information about the economy available at time t is contained in the vector $\boldsymbol{x}_t$. Therefore, the predicted value $\boldsymbol{y}^e_{t,t+1}$ is determined according to a *forecasting rule*, ψ, which is assumed to be a continuous function depending solely on the state $\boldsymbol{x}_t$ of the economy at time t, that is,

$$\psi : X \to Y, \qquad \boldsymbol{y}^e_{t,t+1} = \psi(\boldsymbol{x}_t), \qquad t \in \mathbf{N}. \tag{3}$$

The function ψ also is referred to as a *predictor*. A predictor may or may not be thought of as being derived from a *perceived law of motion* in the sense of Evans and Honkapohja (1986).

Inserting (3) into the economic law (1) yields a discrete-time dynamical system in the sense of Hasselblatt and Katok (1995), defined by

$$\boldsymbol{x}_{t+1} = F_\psi(\boldsymbol{x}_t) := F(\boldsymbol{x}_t, \psi(\boldsymbol{x}_t)), \qquad \boldsymbol{x}_t \in X, \quad t \in \mathbf{N}. \tag{4}$$

Thus, X becomes the *state space* of an economy whose evolution is governed by the time-one map (4). Observe that because F and ψ are defined *explicitly* and *globally* on all of X, the evolution of the economy is defined explicitly for arbitrary states $\boldsymbol{x} \in X$. The fact that the predictor ψ does not depend on the time t implies that the system (4) is autonomous.

3. PERFECT FORESIGHT IN ECONOMIC DYNAMICAL SYSTEMS

The notion of perfect foresight is a common assumption in many macroeconomic models in which the processes of allocations and of prices are conceived of as equilibrium sequences. It is well known for many economies that the two requirements—namely, a dynamic process for an economy that is constantly in equilibrium *and* agents having perfect foresight at all times—may well be inconsistent with each other; see, e.g., Grandmont (1988). Moreover, in this literature the notion of perfect foresight usually is defined in terms of sequences in which the forecasts coincide pointwise with the actual realizations of the economy; see, e.g., Grandmont (1985) or Grandmont and Laroque (1986). Although this concept has been accepted widely to describe economic equilibrium situations over time, it is left unclear what forecasting rules could generate forecasts with the desired property of perfection. This section provides a precise operational definition of a predictor that has these desired properties.

Given the economic law (1) and any predictor (3), the performance of predictors in relation to the dynamical system (4) will be analyzed in the following manner. Define the *error function* associated with the economic law (1) by

$$e_F : X \times Y \to \mathbf{R}^q, \qquad (\boldsymbol{x}, \boldsymbol{y}^e) \mapsto f(\boldsymbol{x}, \boldsymbol{y}^e) - \boldsymbol{y}^e. \tag{5}$$

For arbitrary states $\boldsymbol{x} \in X$ of an economy described by (1) and arbitrary forecasts $\boldsymbol{y}^e \in Y$, the error function (5) yields the (forecast) error $e_F(\boldsymbol{x}, \boldsymbol{y}^e)$ between the forecast $\boldsymbol{y}^e$ and the occurring state $f(\boldsymbol{x}, \boldsymbol{y}^e)$. Notice that the error function is a pointwise measure for arbitrary pairs $(\boldsymbol{x}, \boldsymbol{y}^e)$ in the *extended state space* $X \times Y$ of an economic law F. It therefore supplies structural information on which predictions are better than others. This information is embodied in the economic law F and, in fact, is independent of any predictor ψ. The idea is to measure the deviation of a prediction from the corresponding realization by means of a metric[1] ρ on Y. So, given a state $\boldsymbol{x} \in X$ and some positive number $\epsilon \geq 0$, a prediction $\boldsymbol{y}^e$ is an ϵ-perfect prediction for $f(\boldsymbol{x}, \boldsymbol{y}^e)$ (with respect to ρ), if $\rho(f(\boldsymbol{x}, \boldsymbol{y}^e), \boldsymbol{y}^e) \leq \epsilon$. The set of all pairs $\boldsymbol{x} \in X$ and $\boldsymbol{y}^e \in Y$ that satisfies this criterion is given by

$$\mathcal{W}_F^\epsilon := \{(\boldsymbol{x}, \boldsymbol{y}^e) \in X \times Y \mid \rho(f(\boldsymbol{x}, \boldsymbol{y}^e), \boldsymbol{y}^e) \leq \epsilon\}.$$

The desired criterion for a predictor now is defined as follows.

DEFINITION 1. *Given a law $F = (\bar{F}, f)$, a forecasting rule ψ is called a locally ϵ-perfect predictor for F with respect to a given metric ρ on Y if there exists an open subset $U \subset X$ such that*

$$\{(\boldsymbol{x}, \psi(\boldsymbol{x})) \mid \boldsymbol{x} \in U\} \subset \mathcal{W}_F^\epsilon. \tag{6}$$

For $\epsilon = 0$, locally ϵ-perfect predictors are called locally perfect. *If $U = X$, then ψ is called a (globally) ϵ-perfect predictor.*

Definition 1 is illustrated in Figure 1. Note that the points of $\mathcal{W}_F^\epsilon$ are determined primarily by the f-part of the economic law $F = (\bar{F}, f)$. For $\epsilon = 0$, the set $\mathcal{W}_F^0 \equiv \mathcal{W}_F$ is called the *constraint variety*[2] of $F = (\bar{F}, f)$. Because ρ is a translation-invariant metric, one has

$$\mathcal{W}_F = \{(\boldsymbol{x}, \boldsymbol{y}^e) \in X \times Y \mid f(\boldsymbol{x}, \boldsymbol{y}^e) - \boldsymbol{y}^e = 0\}. \tag{7}$$

Thus, $\mathcal{W}_F$ is precisely the zero-level set of the error function e_F, whereas $\mathcal{W}_F^\epsilon$, $\epsilon > 0$ defines a (closed) ϵ-neighborhood of $\mathcal{W}_F$. The geometric intuition behind Definition 1 then says that the graph of a (globally) ϵ-perfect predictor lies right in between the ϵ^+-level and the ϵ^--level sets of the error function e_F.

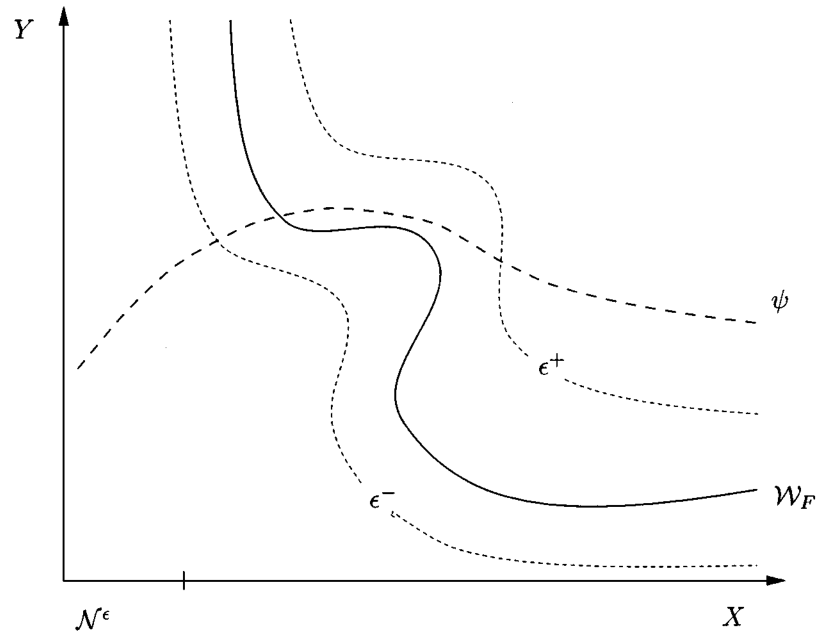

FIGURE 1. A locally ϵ-perfect predictor.

The possibility of finding perfect predictors in the sense of Definition 1 depends strongly on the map F. Using (7), a geometric condition for a predictor ψ to be perfect is provided next.

LEMMA 1. *The predictor ψ is a perfect predictor for an economic law $F = (\bar{F}, f)$ if and only if*

$$\text{graph } \psi := \{(\boldsymbol{x}, \psi(\boldsymbol{x})) \mid \boldsymbol{x} \in X\} \subset \mathcal{W}_F.$$

The predictor ψ is uniquely determined if and only if graph $\psi = \mathcal{W}_F$.

The proof of Lemma 1 is immediate. Given an economic law F, Lemma 1 transforms the question whether perfect forecasting in the sense of Definition 1 is possible to the problem whether the constraint variety $\mathcal{W}_F$ admits a predictor ψ whose graph is contained in $\mathcal{W}_F$. From the point of view of differential geometry, this condition is quite restrictive. It amounts to the fact that $\mathcal{W}_F$ admits a coordinate system defined on all of X. For many economic laws, perfect predictions, therefore, may be impossible. Observe that by Definition 1, a perfect predictor ψ must satisfy $f(\boldsymbol{x}, \psi(\boldsymbol{x})) = \psi(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$, which may be interpreted as a fixed-point property on the space of all functions $\{\psi \mid \psi : X \to Y\}$. This fact was noticed earlier; see, e.g., Sargent (1993) or Evans and Honkapohja (1997). Recall that an orbit $\gamma(\boldsymbol{x}_0)$ of the dynamical system F_ψ is defined by $\gamma(\boldsymbol{x}_0) := \{\boldsymbol{x}_t\}_{t \in \mathbf{N}}$, where $\boldsymbol{x}_t = F_\psi^t(\boldsymbol{x}_0)$

and F_ψ^t denotes the tth iterate of the map F_ψ. The notion of ϵ-perfect foresight defined in terms of orbits of the dynamical system (4) is straightforward.

DEFINITION 2. *An orbit $\gamma(\boldsymbol{x}_0) = \{\boldsymbol{x}_t\}_{\in \mathbf{N}}$ of the dynamical system F_ψ is called an ϵ-perfect-foresight orbit if $(\boldsymbol{x}_t, \psi(\boldsymbol{x}_t)) \in \mathcal{W}_F^\epsilon$ for all $t \in \mathbf{N}$.*

For $\epsilon = 0$, an ϵ-perfect-foresight orbit is called a perfect-foresight orbit for short. Observe that an ϵ-perfect predictor generates a sequence of points $\{(\boldsymbol{x}_t, \boldsymbol{y}_{t,t+1}^e)\}_{t\in\mathbf{N}}$ in the (closed) ϵ-neighborhood $\mathcal{W}_F^\epsilon$ of $\mathcal{W}_F$, where $\boldsymbol{x}_t = F_\psi^t(\boldsymbol{x}_0)$ and $\boldsymbol{y}_{t,t+1}^e = \psi(\boldsymbol{x}_t)$ for each $t \in \mathbf{N}$. For $\epsilon = 0$, this sequence lies on the constraint variety $\mathcal{W}_F$. It follows from (2) and (3) that a perfect predictor ψ generates perfect foresight in the classical sense, i.e., $\boldsymbol{y}_{t,t+1}^e = \boldsymbol{y}_{t+1}$ for all times t and for all initial data $x_0 \in X$. In other words, ψ is a perfect predictor if and only if all orbits of F_ψ have identically vanishing forecast errors. This fact is stated in the following slightly more general lemma.

LEMMA 2. *The predictor ψ is an ϵ-perfect predictor for F if and only if all orbits of F_ψ are ϵ-perfect-foresight orbits.*

A main problem with locally perfect foresight in nonlinear dynamical systems is that an orbit of F_ψ with ϵ-perfect foresight in a suitable neighborhood of some initial point $\boldsymbol{x}_0 \in X$ may loose this property over time. Thus, if ψ is a predictor that is locally ϵ-perfect on an open subset $U \subset X$ of $\boldsymbol{x}_0$, then the additional requirement that U is invariant under F_ψ, i.e.,

$$F_\psi(U) = F(U, \psi(U)) \subset U,$$

is needed to obtain ϵ-perfect-foresight orbits. In this case, any orbit $\gamma(\boldsymbol{x})$, $\boldsymbol{x} \in U$ will be an ϵ-perfect-foresight orbit. It turns out (Theorem 1) that, for $\epsilon = 0$, the existence of a set U that is invariant under F_ψ depends solely on the economic law F because then the restriction $\psi|_U$ is uniquely determined by $\mathcal{W}_F$. Hence, the system F_ψ when restricted to the invariant set U is already determined by the economic fundamentals and, in principle, may exhibit any type of dynamic behavior.[3] For $\epsilon > 0$, there is considerable freedom in choosing an ϵ-perfect predictor. For ϵ sufficiently large, this freedom, in principle, could be used to construct a predictor that is locally ϵ-perfect on U such that U is invariant under F_ψ. This observation and the fact that in many economic applications it might be sufficient to have good predictions in the long run lead to the following definition.

DEFINITION 3. *Given a law $F = (\bar{F}, f)$, a predictor ψ is called asymptotically ϵ-perfect if the following conditions hold*:

(i) *There exists an attractor $A \subset X$ of F_ψ.*
(ii) *The predictor ψ is locally ϵ-perfect on A, that is, $\rho(f(\boldsymbol{x}, \psi(\boldsymbol{x})), \psi(\boldsymbol{x})) \le \epsilon$ for all $\boldsymbol{x} \in A$.*

For $\epsilon = 0$, an asymptotically ϵ-perfect predictor is called asymptotically perfect.

4. ON THE EXISTENCE OF PERFECT PREDICTORS

As was seen in Section 3, the constraint variety $\mathcal{W}_F$, i.e., the zero-level set of the error function e_F of a given law F, completely characterizes perfect predictors. This means that the predictor must satisfy the conditions embodied in the constraint variety. It is important to realize that for each fixed state $\boldsymbol{x} \in X$ and each prediction $\boldsymbol{y}^e \in Y$, the error function e_F of an economic law F gives the deviation from next period's realization. Consider the function $\mathcal{E}_F$ induced by e_F, defined by

$$\mathcal{E}_F : X \times Y \to X \times \mathbf{R}^q, \qquad (\boldsymbol{x}, \boldsymbol{y}^e) \mapsto (\boldsymbol{x}, e_F(\boldsymbol{x}, \boldsymbol{y}^e)). \qquad \textbf{(8)}$$

The next theorem shows that the problem of existence of locally as well as globally perfect predictors may be characterized by an invertibility property of the *induced error function* (8).

THEOREM 1. *Let $F = (\bar{F}, f)$ be an economic law. If the induced error function $\mathcal{E}_F$ is locally invertible at $(\boldsymbol{x}_0, \boldsymbol{y}_0) \in X \times Y$ with $e_F(\boldsymbol{x}_0, \boldsymbol{y}_0) = 0$, then the law F admits a locally perfect predictor ψ. If $\mathcal{E}_F$ is globally invertible, then the law F admits a unique perfect predictor $\psi_\star$. Moreover, $\mathcal{E}_F$ is globally invertible, if the following conditions hold:*

(i) *$f \in C^1(X \times Y, Y)$ and $D_2 f(\boldsymbol{x}, \boldsymbol{y}) - Id$ is invertible for all $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$;*
(ii) *X and Y are contractible spaces;*
(iii) *there exist positive constants α and β such that $\| D\mathcal{E}_F(\boldsymbol{x}, \boldsymbol{y})^{-1} \| \leq \alpha \|(\boldsymbol{x}, \boldsymbol{y})\| + \beta$ on $X \times Y$.*

Proof. Consider the global case first. If $\mathcal{E}_F$ is globally invertible, then $\mathcal{E}_F$ has an inverse $\mathcal{G}$ which is of the form

$$\mathcal{G} : X \times \mathbf{R}^q \to X \times Y, \qquad (\boldsymbol{x}, z) \mapsto (\boldsymbol{x}, G(\boldsymbol{x}, z)). \qquad \textbf{(9)}$$

In particular, $e_F(\boldsymbol{x}, G(\boldsymbol{x}, 0)) = 0$ for all $\boldsymbol{x} \in X$. Setting $\psi_\star(\boldsymbol{x}) := G(\boldsymbol{x}, 0)$, $\boldsymbol{x} \in X$, this implies

$$f(\boldsymbol{x}, \psi_\star(\boldsymbol{x})) = \psi_\star(\boldsymbol{x}), \qquad \boldsymbol{x} \in X.$$

Thus $\psi_\star$ is a perfect predictor in the sense of Definition 1. Suppose $\psi_\star$ is not unique. Then there exist points $\boldsymbol{x}_0 \in X$ and $\boldsymbol{y}_0 \neq \boldsymbol{y}_1 \in Y$ such that $e_F(\boldsymbol{x}_0, \boldsymbol{y}_0) = e_F(\boldsymbol{x}_0, \boldsymbol{y}_1) = 0$. This implies that $\mathcal{E}_F$ is not invertible in $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ which is a contradiction to the initial assumption.

If $\mathcal{E}_F$ is only locally invertible around $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, then the inverse $\mathcal{G}$ is defined only locally around $(\boldsymbol{x}_0, 0) \in X \times \mathbf{R}^q$. Hence, the function G appearing in (9) is defined only locally on some open neighborhood $U \subset X$ of $\boldsymbol{x}_0$. Consequently, any predictor ψ satisfying $\psi(\boldsymbol{x}) = G(\boldsymbol{x}, 0)$ on U is locally perfect.

For the last statement of the theorem, observe that $D\mathcal{E}_F(\boldsymbol{x}, \boldsymbol{y})$ is invertible, if and only if $D_2 f(\boldsymbol{x}, \boldsymbol{y}) - id$ is invertible; cf. Lang (1968). The rest of the statement then follows from a slight variant of a global inverse function theorem; see Deimling (1980, Theorem 15.4). ∎

Theorem 1 reduces the problem of existence and uniqueness of a perfect predictor to the problem of global invertibility of the error function. Condition (i) in Theorem 1 provides a local invertibility criterion for the induced error function (8), whereas (ii) is a topological criterion and (iii) is an estimate sufficient for the existence of a unique global inverse. By the well-known theorem on Neumann series, a sufficient condition for $D_2 f(\boldsymbol{x}, \boldsymbol{y}) - Id$ to be invertible is $\|D_2 f(\boldsymbol{x}, \boldsymbol{y})\| < 1$, where $\|\cdot\|$ denotes a matrix norm; cf. Lang (1968). Notice that, for the existence of a perfect predictor, it suffices to have invertibility of (8) on an open neighborhood of $X \times \{0\}$ in the image set of $\mathcal{E}_F$. By virtue of the inverse function theorem, locally as well as globally perfect predictors are uniquely determined by $\mathcal{W}_F$ and hence by the fundamentals of the economy. This observation implies in particular that perfect predictors depend exclusively on the current state $\boldsymbol{x}$. In the local case, a locally perfect predictor may be changed arbitrarily outside the region U of perfection. However, even if all conditions of Theorem 1 hold, for economic laws $F = (\overline{F}, f)$ with nonlinear f-parts, it generally will be impossible to construct a perfect predictor $\psi_\star$ explicitly. The same argument applies for locally perfect predictors.

If the induced error function (8) is not globally invertible, it may happen that the constraint variety $\mathcal{W}_F$ will not coincide with the graph of some predictor ψ. In particular, perfect predictors need not be unique. For models of the cobweb type (see below), the corresponding constraint variety may consist of different hyperplanes reflecting the nonuniqueness of different perfect predictors. Observe that Theorem 1 can be generalized easily to the case of ϵ-perfect predictors. A necessary geometric condition for an ϵ-perfect predictor to exist is that the natural projection of $\mathcal{W}_F^\epsilon$ on X,

$$pr : \mathcal{W}_F^\epsilon \to X, \qquad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x},$$

is a surjective mapping. If pr is not surjective, then the state space X may be written as the disjoint union of two sets $X = \text{range } pr \cup \mathcal{N}^\epsilon$, where range pr denotes the range of pr; see Figure 1. In this case, ϵ-perfect foresight is impossible if the state of the economy is contained in the set $\mathcal{N}^\epsilon$. This may happen for noninvertible error functions; see Section 5.4 for an example.

5. APPLICATIONS

This section is to relate the concepts introduced in the preceding sections to more established results from the literature. First, the relation to the temporary equilibrium theory is discussed. Then, two simple standard setups are treated: the linear (affine) case and models of the cobweb type. Finally, the general nonlinear OLG model of economic growth is analyzed.

5.1. EQUILIBRIUM DYNAMICS AND PERFECT FORESIGHT

Most models in temporary equilibrium theory do not describe orbits in the explicit sense of dynamical systems theory. They are defined as sequences of solutions of an implicit equation defined by the so-called *temporary equilibrium map* (Grandmont, 1988). In a general nonlinear setup, such implicit equilibrium dynamics can be written as an explicit dynamical system at best locally around some known solution, where the equation is locally invertible. This known solution may be, for instance, a steady state or a cycle. Moreover, the existence and uniqueness of these solutions well may be at question, unless the system is globally invertible.

To be more precise, let $X = Y \subset \mathbf{R}^n$, the state space of an economy, and consider the implicit difference equation defined by a temporary equilibrium map T of the form

$$T : X \times X \times X \to \mathbf{R}^n, \qquad T\left(\boldsymbol{x}_t, \boldsymbol{x}_{t+1}, \boldsymbol{x}^e_{t,t+1}\right) = 0. \tag{10}$$

Assume that there exists a steady state $\bar{\boldsymbol{x}} \in X$ such that $T(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}) = 0$ and that the partial derivative $D_2T(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})$ is invertible. Then, by the implicit function theorem, there exist an open neighborhood $U \times V$ of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}) \in X \times X$ and a map $F_{\text{loc}} : U \times V \to X$ such that $F_{\text{loc}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}}$ and

$$T(\boldsymbol{x}, F_{\text{loc}}(\boldsymbol{x}, \boldsymbol{x}^e), \boldsymbol{x}^e) = 0 \quad \text{for all} \quad (\boldsymbol{x}, \boldsymbol{x}^e) \in U \times V.$$

The map F_{loc} may be interpreted to define a local economic law in the sense of Section 2. Given a predictor $\psi : U \to V$, the local dynamics of the implicit difference equation (10) is generated by

$$\boldsymbol{x}_{t+1} = F_{\text{loc}}(\boldsymbol{x}_t, \psi(\boldsymbol{x}_t)) \tag{11}$$

for all $t \in \mathbf{N}$ such that $\boldsymbol{x}_t \in U$. By Theorem 1, a locally perfect predictor $\psi_\star$ in the sense of Definition 1 exists, if $Id - D_2F_{\text{loc}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}) = Id + D_2T(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})^{-1}D_3T(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})$ is invertible. The orbits of the local perfect-foresight dynamics generated by F_{loc} and $\psi_\star$ do coincide, as long as they exist with the solutions of the local implicit perfect-foresight dynamics defined by

$$T(\boldsymbol{x}_t, \boldsymbol{x}_{t+1}, \boldsymbol{x}_{t+1}) = 0, \tag{12}$$

which is obtained by setting $\boldsymbol{x}_{t+1} = \boldsymbol{x}^e_{t,t+1}$ for $t \in \mathbf{N}$. This shows that, for economic laws in the sense of (1), the local dynamics of locally perfect predictors coincide with the local implicit perfect-foresight equilibrium dynamics.[4] Observe, however, that in most cases F_{loc} will not be computable in an explicit sense. From a dynamical systems point of view, all one gets is linearized dynamics around $\bar{\boldsymbol{x}}$ induced by the Jacobian matrix of F_{loc}, if $\bar{\boldsymbol{x}}$ is a hyperbolic fixed point. If $\bar{\boldsymbol{x}}$ is unstable, nothing can be said about a solution that eventually leaves the neighborhood of the stationary solution. In cases in which $\bar{\boldsymbol{x}}$ is nonhyperbolic, the Hartman-Grobman theorem fails and the linearized dynamics generally have nothing to do with the original nonlinear dynamics. In summary, this implies that, contrary to the global setup presented in Section 2, the analysis outlined in this paragraph is of a strictly local nature.

5.2. LINEAR CASE

Let $X = Y = \mathbf{R}^n$ or $X = Y = \mathbf{R}^n_+$ and consider an economic law (1),

$$F : X \times X \to X, \qquad (\boldsymbol{x}, \boldsymbol{x}^e) \mapsto A\boldsymbol{x} + B\boldsymbol{x}^e + b, \tag{13}$$

where A and B both are $n \times n$ matrices and $\boldsymbol{b} \in \mathbf{R}^n$ is a fixed vector. Clearly, F itself then is an affine map. Many linear economic models with rational expectations [see, e.g., Evans and Honkapohja (1986), Marcet and Sargent (1989), or Zenner (1996)] are of the form (13). Lemma 3 shows that, under mild technical assumptions, a unique perfect predictor for the law (13) can be constructed explicitly.

LEMMA 3. *Let F be an affine economic law* (13). *Then, the induced error function $\mathcal{E}_F$ is globally invertible iff $Id - B$ is invertible. In this case, F admits a unique perfect predictor, given by*

$$\psi_\star(\boldsymbol{x}) = (Id - B)^{-1}[A\boldsymbol{x} + b]. \tag{14}$$

Proof. For given $\boldsymbol{x}, \boldsymbol{x}^e \in X$, the error function e_F associated with F reads

$$e_F(\boldsymbol{x}, \boldsymbol{x}^e) = Ax + [B - Id]\boldsymbol{x}^e + b. \tag{15}$$

It follows from (8) that $\mathcal{E}_F(\boldsymbol{x}, \boldsymbol{x}^e) = (\boldsymbol{x}, e_F(\boldsymbol{x}, \boldsymbol{x}^e))$ is globally invertible iff $Id - B$ is invertible. Uniqueness and existence then follow from Theorem 1. Formula (14) is easily derived from the inverse of $\mathcal{E}_F$, which can be computed explicitly. ■

It follows from Lemma 3 that predictors for a linear economic law (13) should be affine functions of the form

$$\psi(\boldsymbol{x}) := C\boldsymbol{x} + c, \qquad \boldsymbol{x} \in X, \tag{16}$$

where C is an $n \times n$ matrix and $c \in \mathbf{R}^n$. With the help of the error function, agents believing in a linear world with a linear forecast feedback are, in principle, able to find out the correct specification of (13). To see this, observe that an agent at time T has observed the time series $\{\boldsymbol{x}_t\}_{t=0}^T$ and knows all predictions $\{\boldsymbol{x}^e_{t,t+1}\}_{t=0}^T$. Moreover, he knows all past forecast errors $\zeta_t := \boldsymbol{x}_t - \boldsymbol{x}^e_{t-1,t}$, $t = 0, \ldots, T$. A possible candidate of an error function of type (15), e.g., $\hat{e}_F$, therefore has to satisfy

$$\hat{A}\boldsymbol{x}_t + [\hat{B} - Id]\boldsymbol{x}^e_{t,t+1} + \hat{b} = \zeta_{t+1}, \qquad t = 0, \ldots, T-1. \tag{17}$$

This is a system of T vectorial equations for the two unknown $n \times n$ matrices $\hat{A}$, $\hat{B}$ and the unknown vector $\hat{\boldsymbol{b}} \in \mathbf{R}^n$. More precisely, (17) is a system of $T \cdot n$ scalar equations for $2n^2 + n$ unknown coefficients of $\hat{e}_F$. Thus, at time $T = 2n + 1$, an agent is able to compute the unique solution $(\hat{A}, \hat{B}, \hat{b})$ of (17), provided all $2n^2 + n$ equations are linearly independent. Because this solution is unique, it must coincide with the true specification (A, B, b) of (13). Hence, the problem of learning the linear feedback is transformed to the task of generating enough linearly independent equations.

If $\psi_\star$ is the perfect predictor (14), the perfect-foresight dynamics generated by $F_{\psi_\star}$ may exhibit any type of dynamic behavior known from the theory of linear (affine) dynamical systems.[5] It can be a sink, a source, or a saddle point. In particular, the linear perfect-foresight dynamics may be unstable. In the latter case, asymptotically ϵ-perfect predictors may be used to stabilize a fixed point or a cycle, where ϵ-perfect foresight is possible. For an affine predictor of the form (16) and a linear economic law F, the resulting dynamical system is driven by the affine map

$$F_\psi(\boldsymbol{x}) = (A + B \circ C)\boldsymbol{x} + (Bc + b). \tag{18}$$

Choose the standard metric ρ on X, given by $\rho(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|$ with the Euclidean norm $\|\cdot\|$ on X. If C and c can be chosen such that $\boldsymbol{x}_\star$ is a (globally) asymptotically stable fixed point of F_ψ which, in addition, satisfies $\rho(\boldsymbol{x}_\star, \psi(\boldsymbol{x}_\star)) < \epsilon$, then ψ is an asymptotically ϵ-perfect predictor in the sense of Definition 3. For stable ϵ-perfect cycles, similar arguments apply. It seems to be evident that, for a given linear law F, there will be a trade-off between the stability of a fixed point or a cycle and the size of ϵ.

5.3. MODELS OF THE COBWEB TYPE

The so-called cobweb model is one of the most widely used dynamical economic systems to demonstrate the role of expectations in determining the dynamic behavior of a market economy. Several authors have shown that such systems may demonstrate almost any degree of dynamic complexity, depending on expectations formation procedures [see, e.g., Chiarella (1992) or Brock and Hommes (1997) and references therein]. For this setup, the essential feature of *models of the cobweb type* is the fact that the f-part of the economic law $F = (\bar{F}, f)$ depends exclusively on the current prediction and of no other state variables. Hence, the resulting dynamics for $\bar{\boldsymbol{x}}$ and $\boldsymbol{y}$ are coupled only via predictors.

If $f(\boldsymbol{x}, \boldsymbol{y}^e) \equiv f(\boldsymbol{y}^e)$ for all $(\boldsymbol{x}, \boldsymbol{y}^e) \in X \times Y$, then the constraint variety $\mathcal{W}_F$ corresponds to the set of all fixed points of the map f; i.e., it has the simple product structure

$$\mathcal{W}_F = X \times \{\boldsymbol{y} \in Y \mid \boldsymbol{y} = f(\boldsymbol{y})\}. \tag{19}$$

$\mathcal{W}_F$ may be visualized as an assembly of hyperplanes parallel to the state space X. With this observation, the following Lemma becomes obvious:

LEMMA 4. *Under the assumptions made above, the only (continuous) perfect predictors of $F = (\bar{F}, f)$ are the constant predictors $\psi_\star$ defined by*

$$\psi_\star(\boldsymbol{x}) \equiv \boldsymbol{y}_\star \quad \forall \quad \boldsymbol{x} \in X, \qquad i = 1, \ldots, k, \tag{20}$$

where the $\boldsymbol{y}_\star$ denotes a fixed point of f. The law F has a unique perfect predictor iff the fixed point of f is unique.

5.4. EQUILIBRIUM GROWTH WITH PERFECT FORESIGHT

Consider the standard version of the OLG model of economic growth, as, for example, of Blanchard and Fischer (1989) or Azariadis (1995). Consumers consist of two-period-lived OLGs with preferences over consumption in both periods. Each member of a generation supplies one unit of labor to the market inelastically in the first period of his life, receives wage income only in the first, and saves to consume in the second period of his life. His intertemporal consumption/savings decision is made given an expected real rate of return $R^e = 1 + r^e$ on savings, where r^e is the expected interest rate. The real wage is determined by the marginal product of labor at full employment in each period. Old consumers receive all profit income from production, which determines the actual rate of return on their savings. For each period t, denote by L_t the number of young consumers, by w_t the real wage of a young consumer, and by $R^e_{t,t+1} = 1 + r^e_{t,t+1}$ the expected rate of return for his savings held at date t.

Aggregate output Y_t in each period is produced from the total amount of labor L_t (number of young consumers) and capital K_t by use of a standard atemporal neoclassical production function which is assumed to be homogeneous of degree 1 in both inputs. Capital depreciates at a rate $0 \leq d \leq 1$ and the generations of workers grow at a constant rate $n \geq 0$. Define $k_t := K_t/L_t$ and let $g : \mathbf{R}_+ \to \mathbf{R}_+$ be a strictly concave, strictly monotonically increasing function with $g(0) \geq 0$. Suppose that output Y_t in each period t is given by the production function

$$Y_t = L_t g(k_t).$$

Then, the economic process is defined by the following set of equations:

$$\begin{aligned} k_{t+1} &= \frac{1}{1+n} S\big(w_t, R^e_{t,t+1}\big), \\ R_{t+1} &= \frac{y_{t+1} - w_{t+1}L_{t+1} + (1-d)K_{t+1}}{S\big(w_t, R^e_{t,t+1}\big)L_t}, \\ w_t &= g(k_t) - k_t g'(k_t), \end{aligned} \tag{21}$$

where S denotes the savings function of a young consumer and w_t the wage rate. Setting

$$s\big(k_t, R^e_{t,t+1}\big) := S\big(g(k_t) - k_t g'(k_t), R^e_{t,t+1}\big)$$

and inserting the first and the third equations in (21) into the second equation, one obtains an economic law in the sense of Section 2, namely

$$\begin{cases} k_{t+1} = \bar{F}\big(k_t, R^e_{t,t+1}\big) := \dfrac{1}{1+n} s\big(k_t, R^e_{t,t+1}\big) \\ R_{t+1} = f\big(k_t, R^e_{t,t+1}\big) := g'\big(\bar{F}\big(k_t, R^e_{t,t+1}\big)\big) + (1-d). \end{cases} \tag{22}$$

Two features are worth noting. First, expectations matter only if the savings behavior depends on the expected rate of return. Second, both functions in (22) are independent of the actual rate of return R_t, implying that the two equations in (22) are only coupled via a forecasting rule. Thus, if the predictor ψ is independent of the actual rate of return, then the dynamics of the system are one-dimensional.

Because the law F given in (22) is independent of R, the error function e_F has the form

$$e_F(k, R^e) = g'\left(\frac{1}{1+n}s(k, R^e)\right) + (1-d) - R^e. \tag{23}$$

The corresponding constraint variety is a cylindric set $\mathcal{W}_F = \widehat{\mathcal{W}}_F \times \mathbf{R}_+$ in $\mathbf{R}^3_+$, where $R \in \mathbf{R}_+$ and $(k, R^e) \in \widehat{\mathcal{W}}_F$ if and only if $e_F(k, R^e) = 0$. It therefore suffices to look at $\widehat{\mathcal{W}}_F$ instead of $\mathcal{W}_F$. The major implication of this observation is that globally or locally perfect predictors, if they exist, depend exclusively on k_t. Also, predictors that depend on R_t in a nontrivial manner cannot be perfect. It follows from the discussion above that perfect-foresight dynamics of the system, if it exists, will be one-dimensional.

The form of the error function (23) shows that the interaction between the marginal product function g' and the expectations effect of savings will determine whether perfect prediction is possible and unique for every value of the capital intensity k. It is a remarkable fact that for each of the two functions involved, the production function and the savings function, there exists a large class for which unique perfect predictors exist under mild assumptions on the respective other function. However, if they are violated, at least uniqueness need not hold. Propositions 1 and 2 provide the two positive results of existence and uniqueness. A succeeding example shows how perfect prediction fails in general. For saving functions that are nondecreasing in R^e, unique perfect predictors exist for all technologies satisfying the Inada conditions. This rather strong result is stated in the following proposition.

PROPOSITION 1. *Consider the standard growth model as introduced above. Let the production function g satisfy the Inada conditions and assume the savings function s to be nondecreasing in* R^e; *i.e.*, $\partial_2 s(k, R^e) \geq 0$ *for all* (k, R^e). *Then there exists a unique perfect predictor* $\psi_\star$. *If both goods are normal*, *then* $\psi_\star$ *is a strictly decreasing function of the capital intensity.*

All proofs are given in the Appendix. As a consequence of Proposition 1, $\overline{F}(k, \psi_\star(k))$ is strictly monotonically increasing in k. Therefore, all orbits with perfect foresight are monotonic with a possibility of multiple steady states. Proposition 1 covers all preferences that generate nonincreasing offer curves. Assume, for example, that consumers have homothetic preferences. The savings function is then of the form

$$s(k, R^e) = [g(k) - kg'(k)]\tilde{s}(R^e),$$

where $0 \leq \tilde{s}(R^e) \leq 1$ for all R^e. In particular, consider savings propensities

$$\tilde{s}(R^e) = \frac{1}{1 + (\delta R^e)^{\frac{\rho}{\rho-1}}}, \qquad 0 < \delta, \quad \rho < 1 \tag{24}$$

derived from CES utility functions. Equation (24) is nondecreasing for $0 \leq \rho < 1$ and strictly monotonically decreasing for $\rho < 0$.

COROLLARY 1. *Let the production function g satisfy the Inada conditions and assume the savings function* (24) *to be derived from CES utility functions. If* $0 \leq \rho < 1$, *then there exists a unique perfect predictor.*

For nonmonotonic or monotonically decreasing savings functions, the existence of perfect predictors depends in various ways on the preferences and on the chosen technology. Let

$$g(k) = \frac{1}{\alpha} k^{\alpha}, \qquad 0 < \alpha < 1, \tag{25}$$

be the class of isoelastic production functions.

PROPOSITION 2. *Consider the standard growth model as introduced above with isoelastic production functions of the form* (25). *If consumption in both periods is a normal good, then there exists a unique perfect predictor that is a strictly decreasing function of the capital intensity.*

Proposition 2 implies in particular that perfect predictors exist for all isoelastic production functions and all CES savings functions. The situation of Proposition 2 is illustrated in Figure 2, which shows the contour lines of the error function for a particular parameter set (Case 1).

The intuition behind the two preceding propositions is as follows. The definition of the error function (23) implies that $e_F(k, R^e) = 0$ if and only if

$$g'^{-1}(R^e - (1-d)) = \frac{1}{1+n} s(k, R^e), \qquad k, R^e \in \mathbf{R}_+. \tag{26}$$

For an arbitrary but fixed capital intensity k, (26) is illustrated qualitatively in Figure 3.

On the one hand, a positive expectations effect of the savings function always works in the right direction no matter what the curvature of the production function is. On the other hand, the strong uniform curvature of isoelastic production functions seems to level off almost any negative or nonmonotonic expectations effect, as long as no inferiority in consumption is present. The following example indicates that, with weaker assumptions on either side, the existence of unique perfect predictors can no longer be expected. Consider exponential production functions of the form

$$g(k) = \frac{a}{b}[1 + c - e^{-bk}], \qquad a, b, c > 0 \tag{27}$$

and CES savings functions (24). In this case the contour lines of the error function in Figure 4 show that the constraint variety may fold back (Case 2). This is a clear indication that, in this case, there exists no unique perfect predictor.

More generally, it is evident that there are many economic situations in which the interplay of the savings behavior and of the income-generating features of competitive-factor rewards may prevent the existence of perfect predictors. Thus,

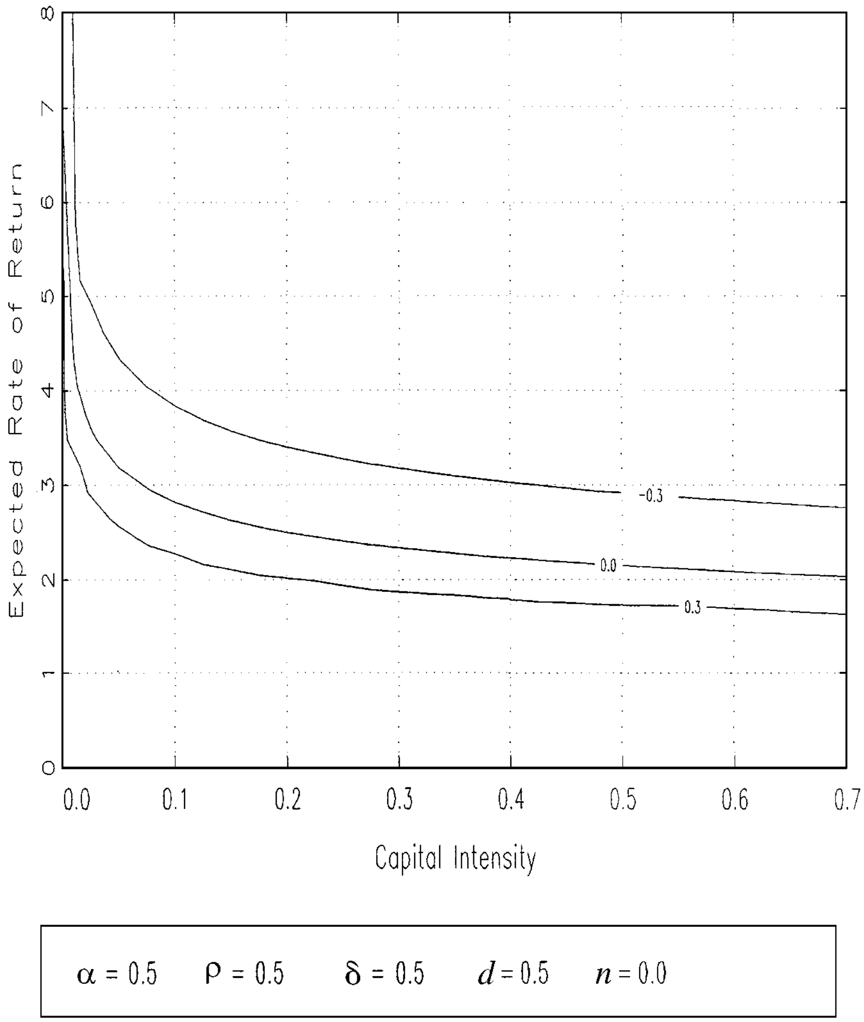

FIGURE 2. Percentage prediction-error contours for isoelastic production function and CES savings function.

the standard growth model, as one of the basic models in dynamic macroeconomics, does not seem to guarantee a priori the possibility for perfect predictions.

6. FINAL REMARKS AND CONCLUSIONS

The careful distinction between an economic law and a forecasting rule seems to be the necessary step to obtain an economic dynamical system that is defined explicitly and globally on the whole state space. This distinction allows

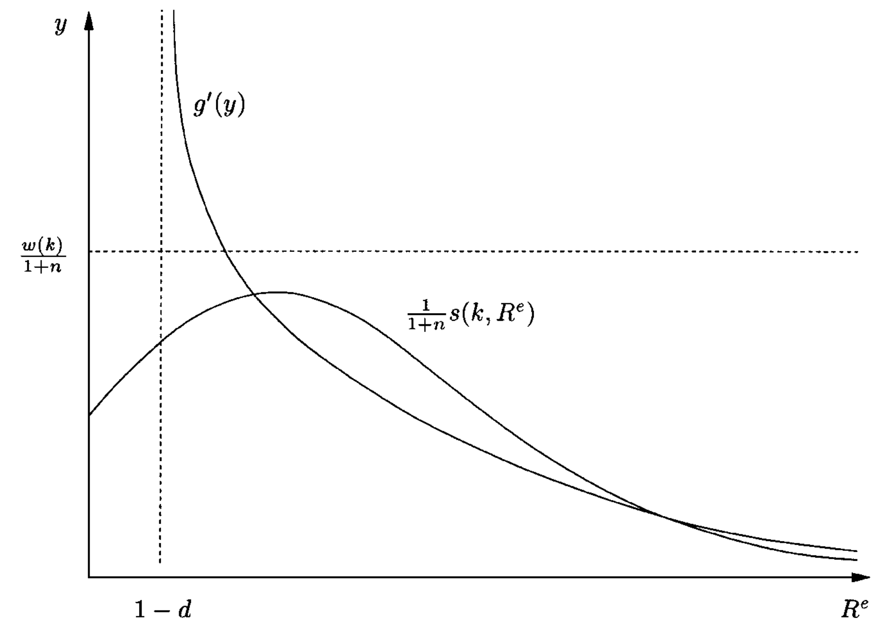

FIGURE 3. Expectations effects and production.

for a systematic analysis of perfect foresight as a property of forecasting rules. The results of this paper show in a striking fashion that the property of perfect foresight along orbits in dynamical economic systems imposes strong structural requirements on the economic fundamentals encoded in the economic law. The discussion of the examples indicates that these requirements are not a universal feature of economic systems, so that the perfect-foresight property of orbits in dynamical economic systems may well be the exception rather than the rule. The analysis also indicates that the amount of information necessary to construct a perfect predictor requires detailed knowledge of the whole economic system, which in turn is tantamount to the ability to compute the global inverse of the error function associated with the system. As a first step for an operational approach to learning in economic dynamical systems, it therefore seems to be reasonable to try to find predictors that, relative to a given class of predictors, are best approximations of a (locally) perfect predictor. For a dynamic economic analysis, this may be a successful line of research to understand the role of expectations.

NOTES

1. Here, by a metric on Y is meant a real-valued function ρ defined on $Y \times Y$ that satisfies (i) $\rho(y_1, y_2) \geq 0$ and $\rho(y_1, y_2) = 0$ iff $y_1 = y_2$, $(ii) \rho(y_1, y_2) = \rho(y_2, y_1)$, and $(iii) \rho(y_1, y_3) \leq \rho(y_1, y_2) + \rho(y_2, y_3)$ for all $y_1, y_2, y_3 \in Y$. Throughout this paper, ρ is assumed to be translation-invariant, that is, $\rho(y_1 + y_3, y_2 + y_3) = \rho(y_1, y_2)$ for all $y_1, y_2, y_3 \in Y$.

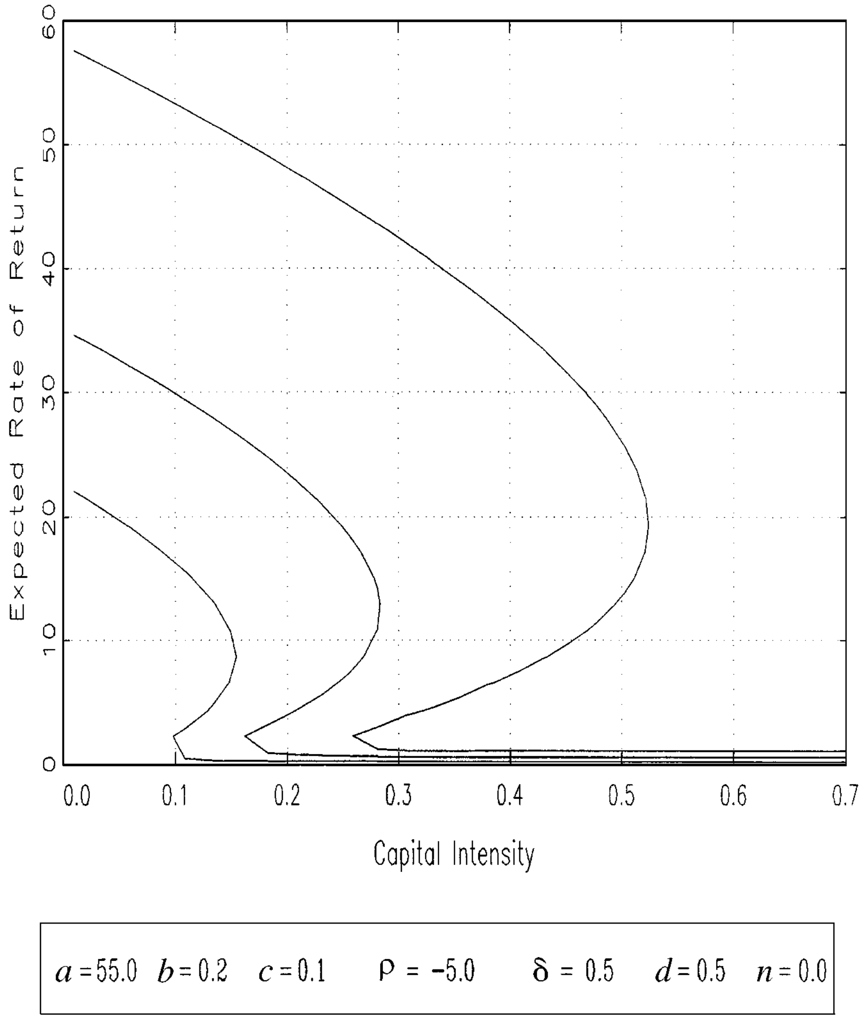

FIGURE 4. Percentage prediction-error contours for exponential production function and CES savings function.

2. In general, such a variety may well have singularities.

3. In particular, if $\|DF(x_\star, y_\star)\| < 1$ for a steady state $x_\star = (\bar{x}_\star, y_\star)$ with the perfect-foresight property, then there exists a locally perfect predictor on an open neighborhood U of $x_\star$ that is invariant under F_ψ. In this case, $x_\star$ is an asymptotically stable fixed point such that orbits starting in U will be perfect-foresight orbits converging to $x_\star$.

4. The condition for existence of local solutions of (12) near the steady state $\bar{x}$ is that $D_2T(\bar{x}, \bar{x}, \bar{x}) + D_3T(\bar{x}, \bar{x}, \bar{x})$ is invertible. This condition is weaker than the condition for the existence of F_{loc} and the locally perfect predictor $\psi_\star$.

5. In the case in which $Id - B$ is not invertible, there may exist a continuum of perfect predictors; cf. Böhm and Wenzelburger (1997a).

REFERENCES

Azariadis, C. (1995) *Intertemporal Macroeconomics.* Oxford: Basil Blackwell.

Blanchard, O.J. & S. Fischer (1989) *Lectures on Macroeconomics.* London: MIT Press.

Böhm, V., M. Lohmann & H.-W. Lorenz (1994) Dynamic Complexity in a Keynesian Macroeconomic Model. Discussion paper 288, University of Bielefeld.

Böhm, V. & J. Wenzelburger (1997a) Expectations, Forecasting, and Perfect Foresight—A Dynamical Systems Approach, rev. Sept. 1997. Discussion paper 307, University of Bielefeld.

——— (1997b) Expectational Leads in Economic Dynamical Systems. Discussion paper 373, University of Bielefeld.

Brock, W. & C. Hommes (1997) A rational route to randomness. *Econometrica* 65, 1059–1095.

Chiappori, P. & R. Guesnerie (1991) Sunspot equilibria in sequential market models. In W. Hildenbrand & H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, vol 4. Amsterdam: North-Holland.

Chiarella, C. (1992) The cobweb model: Its instability and the onset of chaos. *Economic Modelling* 5, 377–384.

Deimling, K. (1980) *Nonlinear Functional Analysis.* Berlin: Springer-Verlag.

Evans, G.W. & S. Honkapohja (1986) A complete characterization of ARMA solutions to linear rational expectations models. *Review of Economic Studies* 53, 227–239.

——— (1997) Learning Dynamics. Discussion paper 412, University of Helsinki.

Grandmont, J.-M. (1985) On endogenous competitive business cycles. *Econometrica* 53, 995–1046.

——— (1988) Introduction. In J.-M. Grandmont (ed.), *Temporary Equilibrium, Selected Readings*, pp. xiii–xxiv. London: Academic Press.

Grandmont, J.-M. & G. Laroque (1986) Stability of cycles and expectations. *Journal of Economic Theory* 40, 138–151.

Hasselblatt, B. & A. Katok (1995) *Introduction to the Modern Theory of Dynamical Systems.* Cambridge: Cambridge University Press.

Lang, S. (1968) *Analysis I.* Reading, MA: Addison-Wesley.

Marcet, A. & T.J. Sargent (1989) Convergence of least squares learning mechanisms in self-referential stochastic models. *Journal of Economic Theory* 48, 337–368.

Sargent, T.J. (1993) *Bounded Rationality in Macroeconomics.* Oxford: Clarendon Press.

Zenner, M. (1996) *Learning to Become Rational.* Lecture Notes in Economics and Mathematical Systems, vol. 439, Berlin: Springer-Verlag.

APPENDIX: PROOFS OF RESULTS

Proof of Proposition 1. The case that s is constant in R^e is trivial because then there is no expectations feedback, implying that perfect foresight exists for all k. Consider now the nonconstant case. The partial derivative of e_F with respect to R^e reads

$$\partial_2 e_F(k, R^e) = g''\left(\frac{1}{1+n} s(k, R^e)\right) \frac{1}{1+n} \partial_2 s(k, R^e) - 1.$$

Because $g'' < 0$ and $\partial_2 s(k, R^e) \geq 0$, this implies that

$$\partial_2 e_F(k, R^e) \leq -1 < 0 \quad \text{for all} \quad k, R^e. \qquad \textbf{(A.1)}$$

Let $k \in \mathbf{R}_+$ be arbitrary but fixed. Because $s(k, \cdot)$ is bounded from above,

$$\lim_{R^e \to \infty} e_F(k, R^e) = -\infty.$$

On the other hand, for each k,

$$\lim_{R^e \to 0} e_F(k, R^e) > 0.$$

Using (A.1), the latter two equations imply that, for each $k \in \mathbf{R}_+$, there exists a unique R^e_k such that $e_F(k, R^e_k) = 0$. Then, the function $\psi_\star : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ defined by $\psi_\star(k) := R^e_k$ is the desired unique perfect predictor.

Now, implicit differentiation of $\psi_\star$ gives

$$\psi'_\star(k) = \frac{g''\left(\frac{1}{1+n} s(k, \psi_\star(k))\right) \partial_1 s(k, \psi_\star(k))}{1 + n - g''\left(\frac{1}{1+n} s(k, \psi_\star(k))\right) \partial_2 s(k, \psi_\star(k))}, \qquad k \in \mathbf{R}_+. \tag{A.2}$$

Observe that g is concave and s is nondecreasing in R^e and increasing in k because both goods are normal. Therefore, $\psi_\star$ must be decreasing. ■

Proof of Proposition 2. The normality of both consumption goods implies that the product $R^e s(k, R^e)$ is increasing in R^e for any $k > 0$. Then, the error function e_F has the form

$$e_F(k, R^e) = R^e \left[\left(\frac{1+n}{R^e s(k, R^e)} \right)^{1-\alpha} \frac{1}{(R^e)^\alpha} - \left(1 - \frac{1-d}{R^e} \right) \right].$$

For arbitrary but fixed $k \in \mathbf{R}_+$, one has

$$\lim_{R^e \to \infty} e_F(k, R^e) = -1 \quad \text{and} \quad \lim_{R^e \to 0} e_F(k, R^e) > 1 - d \geq 0.$$

Because the term in brackets is strictly decreasing, for each $k \in \mathbf{R}_+$ there exists a unique $R^e_k = \psi_\star(k)$ such that $e_F(k, \psi_\star(k)) = 0$. By an argument analogous to that of the proof of Proposition 1, the normality of the consumption goods implies that $\psi_\star$ is strictly decreasing. ■