17 Multivariable Calculus

We know (from Chapter 13) how to maximize a company's profit when both revenue and cost are written as functions of a single quantity, namely the number of units produced. But of course the production level is itself determined by other factors—and, in general, no single variable can represent them.

The amount of oil pumped from an oil field each week, for example, depends on both the number of pumps and the number of hours that the pumps are operated. The number of pumps in the field will depend on the amount of capital originally available to build the pumps as well as the size and shape of the field. The number of hours that the pumps can be operated depends on the labor available to run and maintain the pumps. In addition, the amount of oil that the owner will be willing to have pumped from the oil field will depend on the current demand for oil—which is related to the price of the oil.

Maximizing the weekly profit from an oil field will require a balance between the number of pumps and the amount of time each pump can be operated. The maximum profit will not be achieved by building more pumps than can be operated or by running a few pumps full time.

This is an example of the general problem of maximizing profit when production depends on several factors. The solution involves an analysis of the production function, which relates production output to resources allocated for production. Because, in general, several variables are needed to describe the resource allocation, the most profitable allocation cannot be found by differentiation with respect to a single variable, as in preceding chapters. The more advanced techniques necessary to do the job will be covered in this chapter.

Objective

To compute partial derivatives.

TO REVIEW functions of several variables, see Section 2.8.

17.1 Partial Derivatives

Throughout this book we have encountered many examples of functions of several variables. We recall, from Section 2.8, that the graph of a function of two variables is a surface. Figure 17.1 shows the surface $z = f(x, y)$ and a plane that is parallel to the x,z-plane and that passes through the point $(a, b, f(a, b))$ on the surface. The equation of this plane is $y = b$. Hence, any point on the curve that is the intersection of the surface $z = f(x, y)$ with the plane $y = b$ must have the form $(x, b, f(x, b))$. Thus, the curve can be described by the equation $z = f(x, b)$. Since b is constant, $z = f(x, b)$ can be considered a function of one variable, x. When the derivative of this function is evaluated at a, it gives the slope of the tangent line to this curve at the point $(a, b, f(a, b))$. (See Figure 17.1.) This slope is called the *partial derivative of f with respect to x* at (a, b) and is denoted $f_x(a, b)$. In terms of limits,

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} \tag{1}$$

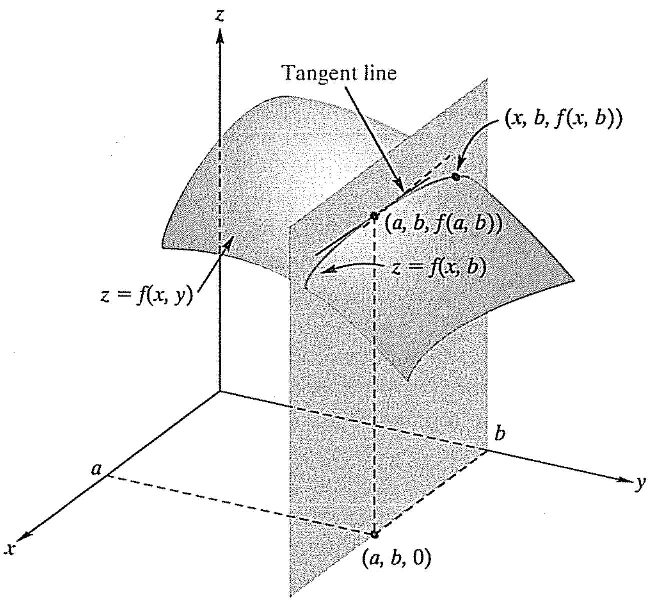

FIGURE 17.1 Geometric interpretation of $f_x(a, b)$.

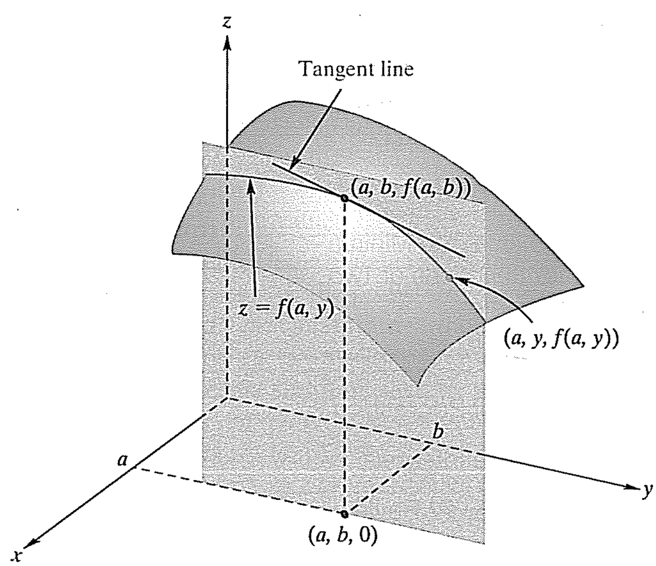

FIGURE 17.2 Geometric interpretation of $f_y(a, b)$.

On the other hand, in Figure 17.2, the plane $x = a$ is parallel to the y,z-plane and cuts the surface $z = f(x, y)$ in a curve given by $z = f(a, y)$, a function of y. When the derivative of this function is evaluated at b, it gives the slope of the tangent line to this curve at the point $(a, b, f(a, b))$. This slope is called the *partial derivative of f with respect to y* at (a, b) and is denoted $f_y(a, b)$. In terms of limits,

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h} \tag{2}$$

This gives us a geometric interpretation of a partial derivative.

We say that $f_x(a, b)$ is the slope of the tangent line to the graph of f at $(a, b, f(a, b))$ *in the x-direction;* similarly, $f_y(a, b)$ is the slope of the tangent line *in the y-direction.*

For generality, by replacing a and b in Equations (1) and (2) by x and y, respectively, we get the following definition.

Definition

If $z = f(x, y)$, the ***partial derivative of f with respect to x***, denoted f_x, is the function, of two variables, given by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided that the limit exists.

The ***partial derivative of f with respect to y***, denoted f_y, is the function, of two variables, given by

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided that the limit exists.

By analyzing the foregoing definition, we can state the following procedure to find f_x and f_y:

This gives us a mechanical way to find partial derivatives.

Procedure to Find $f_x(x, y)$ and $f_y(x, y)$

To find f_x, treat y as a constant, and differentiate f with respect to x in the usual way.

To find f_y, treat x as a constant, and differentiate f with respect to y in the usual way.

EXAMPLE 1 Finding Partial Derivatives

If $f(x, y) = xy^2 + x^2y$, find $f_x(x, y)$ and $f_y(x, y)$. Also, find $f_x(3, 4)$ and $f_y(3, 4)$.

Solution: To find $f_x(x, y)$, we treat y as a constant and differentiate f with respect to x:

$$f_x(x, y) = (1)y^2 + (2x)y = y^2 + 2xy$$

To find $f_y(x, y)$, we treat x as a constant and differentiate with respect to y:

$$f_y(x, y) = x(2y) + x^2(1) = 2xy + x^2$$

Note that $f_x(x, y)$ and $f_y(x, y)$ are each functions of the two variables x and y. To find $f_x(3, 4)$, we evaluate $f_x(x, y)$ when $x = 3$ and $y = 4$:

$$f_x(3, 4) = 4^2 + 2(3)(4) = 40$$

Similarly,

$$f_y(3, 4) = 2(3)(4) + 3^2 = 33$$

Now Work Problem 1 ◁

Notations for partial derivatives of $z = f(x, y)$ are in Table 17.1. Table 17.2 gives notations for partial derivatives evaluated at (a, b). Note that the symbol ∂ (not d) is used to denote a partial derivative. The symbol $\partial z/\partial x$ is read "the partial derivative of z with respect to x."

EXAMPLE 2 Finding Partial Derivatives

a. If $z = 3x^3y^3 - 9x^2y + xy^2 + 4y$, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \left.\frac{\partial z}{\partial x}\right|_{(1,0)}$ and $\left.\frac{\partial z}{\partial y}\right|_{(1,0)}$.

Solution: To find $\partial z/\partial x$, we differentiate z with respect to x while treating y as a constant:

$$\begin{aligned}\frac{\partial z}{\partial x} &= 3(3x^2)y^3 - 9(2x)y + (1)y^2 + 0 \\ &= 9x^2y^3 - 18xy + y^2\end{aligned}$$

Table 17.1

Partial Derivative of f (or z) with Respect to x	Partial Derivative of f (or z) with Respect to y
$f_x(x, y)$	$f_y(x, y)$
$\frac{\partial}{\partial x}(f(x, y))$	$\frac{\partial}{\partial y}(f(x, y))$
$\frac{\partial z}{\partial x}$	$\frac{\partial z}{\partial y}$

Table 17.2

Partial Derivative of f (or z) with Respect to x Evaluated at (a, b)	Partial Derivative of f (or z) with Respect to y Evaluated at (a, b)
$f_x(a, b)$	$f_y(a, b)$
$\left.\frac{\partial f}{\partial x}\right\vert_{(a,b)}$	$\left.\frac{\partial f}{\partial y}\right\vert_{(a,b)}$
$\left.\frac{\partial z}{\partial x}\right\vert_{\substack{x=a \\ y=b}}$	$\left.\frac{\partial z}{\partial y}\right\vert_{\substack{x=a \\ y=b}}$

Evaluating the latter equation at (1, 0), we obtain

$$\left.\frac{\partial z}{\partial x}\right\vert_{(1,0)} = 9(1)^2(0)^3 - 18(1)(0) + 0^2 = 0$$

To find $\partial z/\partial y$, we differentiate z with respect to y while treating x as a constant:

$$\begin{aligned}\frac{\partial z}{\partial y} &= 3x^3(3y^2) - 9x^2(1) + x(2y) + 4(1) \\ &= 9x^3y^2 - 9x^2 + 2xy + 4\end{aligned}$$

Thus,

$$\left.\frac{\partial z}{\partial y}\right\vert_{(1,0)} = 9(1)^3(0)^2 - 9(1)^2 + 2(1)(0) + 4 = -5$$

b. If $w = x^2e^{2x+3y}$, find $\partial w/\partial x$ and $\partial w/\partial y$.

Solution: To find $\partial w/\partial x$, we treat y as a constant and differentiate with respect to x. Since x^2e^{2x+3y} is a product of two functions, each involving x, we use the product rule:

$$\begin{aligned}\frac{\partial w}{\partial x} &= x^2\frac{\partial}{\partial x}(e^{2x+3y}) + e^{2x+3y}\frac{\partial}{\partial x}(x^2) \\ &= x^2(2e^{2x+3y}) + e^{2x+3y}(2x) \\ &= 2x(x+1)e^{2x+3y}\end{aligned}$$

To find $\partial w/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial w}{\partial y} = x^2\frac{\partial}{\partial y}(e^{2x+3y}) = 3x^2e^{2x+3y}$$

Now Work Problem 27 ◁

We have seen that, for a function of two variables, two partial derivatives can be considered. Actually, the concept of partial derivatives can be extended to functions of more than two variables. For example, with $w = f(x, y, z)$ we have three partial derivatives:

the partial with respect to x, denoted $f_x(x, y, z)$, $\partial w/\partial x$, and so on;
the partial with respect to y, denoted $f_y(x, y, z)$, $\partial w/\partial y$, and so on;

and

the partial with respect to z, denoted $f_z(x, y, z)$, $\partial w/\partial z$, and so on

To determine $\partial w/\partial x$, treat y and z as constants, and differentiate w with respect to x. For $\partial w/\partial y$, treat x and z as constants, and differentiate with respect to y. For $\partial w/\partial z$, treat x and y as constants, and differentiate with respect to z. For a function of n variables, we have n partial derivatives, which are determined in an analogous way.

EXAMPLE 3 **Partial Derivatives of a Function of Three Variables**

If $f(x, y, z) = x^2 + y^2z + z^3$, find $f_x(x, y, z)$, $f_y(x, y, z)$, and $f_z(x, y, z)$.

Solution: To find $f_x(x, y, z)$, we treat y and z as constants and differentiate f with respect to x:

$$f_x(x, y, z) = 2x$$

Treating x and z as constants and differentiating with respect to y, we have

$$f_y(x, y, z) = 2yz$$

Treating x and y as constants and differentiating with respect to z, we have

$$f_z(x, y, z) = y^2 + 3z^2$$

Now Work Problem 23 ◁

EXAMPLE 4 **Partial Derivatives of a Function of Four Variables**

If $p = g(r, s, t, u) = \dfrac{rsu}{rt^2 + s^2t}$, find $\dfrac{\partial p}{\partial s}$, $\dfrac{\partial p}{\partial t}$, and $\left.\dfrac{\partial p}{\partial t}\right|_{(0,1,1,1)}$.

Solution: To find $\partial p/\partial s$, first note that p is a quotient of two functions, each involving the variable s. Thus, we use the quotient rule and treat r, t, and u as constants:

$$\frac{\partial p}{\partial s} = \frac{(rt^2 + s^2t)\dfrac{\partial}{\partial s}(rsu) - rsu\dfrac{\partial}{\partial s}(rt^2 + s^2t)}{(rt^2 + s^2t)^2}$$

$$= \frac{(rt^2 + s^2t)(ru) - (rsu)(2st)}{(rt^2 + s^2t)^2}$$

Simplification gives

$$\frac{\partial p}{\partial s} = \frac{ru(rt - s^2)}{t(rt + s^2)^2} \qquad \text{a factor of } t \text{ cancels}$$

To find $\partial p/\partial t$, we can first write p as

$$p = rsu(rt^2 + s^2t)^{-1}$$

Next, we use the power rule and treat r, s, and u as constants:

$$\frac{\partial p}{\partial t} = rsu(-1)(rt^2 + s^2t)^{-2}\frac{\partial}{\partial t}(rt^2 + s^2t)$$

$$= -rsu(rt^2 + s^2t)^{-2}(2rt + s^2)$$

so that

$$\frac{\partial p}{\partial s} = -\frac{rsu(2rt + s^2)}{(rt^2 + s^2t)^2}$$

Letting $r = 0, s = 1, t = 1$, and $u = 1$ gives

$$\left.\frac{\partial p}{\partial t}\right|_{(0,1,1,1)} = -\frac{0(1)(1)(2(0)(1) + (1)^2)}{(0(1)^2 + (1)^2(1))^2} = 0$$

Now Work Problem 31 ◁

PROBLEMS 17.1

In Problems 1–26, a function of two or more variables is given. Find the partial derivative of the function with respect to each of the variables.

1. $f(x, y) = 2x^2 + 3xy + 4y^2 + 5x + 6y - 7$

2. $f(x, y) = 2x^2 + 3xy$

3. $f(x, y) = 2y + 1$

4. $f(x, y) = \ln 2$

5. $g(x, y) = 3x^4y + 2xy^2 - 5xy + 8x - 9y$

6. $g(x, y) = (x^2 + 1)^2 + (y^3 - 3)^3 + 5xy^3 - 2x^2y^2$

7. $g(p,q) = \sqrt{pq}$

8. $g(w,z) = \sqrt[3]{w^2+z^2}$

9. $h(s,t) = \dfrac{s^2+4}{t-3}$

10. $h(u,v) = \dfrac{8uv^2}{u^2+v^2}$

11. $u(q_1,q_2) = \ln\sqrt{q_1+2} + \ln\sqrt[3]{q_2+5}$

12. $Q(l,k) = 2l^{0.38}k^{1.79} - 3l^{1.03} + 2k^{0.13}$

13. $h(x,y) = \dfrac{x^2+3xy+y^2}{\sqrt{x^2+y^2}}$

14. $h(x,y) = \dfrac{\sqrt{x+9}}{x^2y+y^2x}$

15. $z = e^{5xy}$

16. $z = (x^3+y^3)e^{xy+3x+3y}$

17. $z = 5x\ln(x^2+y)$

18. $z = \ln(5x^3y^2+2y^4)^4$

19. $f(r,s) = \sqrt{r+2s}(r^3-2rs+s^2)$

20. $f(r,s) = \sqrt{rs}\,e^{2+r}$

21. $f(r,s) = e^{3-r}\ln(7-s)$

22. $f(r,s) = (5r^2+3s^3)(2r-5s)$

23. $g(x,y,z) = 2x^3y^2+2xy^3z+4z^2$

24. $g(x,y,z) = 2xy^2z^6-4x^2y^3z^2+3xyz$

25. $g(r,s,t) = e^{s+t}(r^2+7s^3)$

26. $g(r,s,t,u) = rs\ln(t)e^u$

In Problems 27–34, evaluate the given partial derivatives.

27. $f(x,y) = x^3y+7x^2y^2;\quad f_x(1,-2)$

28. $z = \sqrt{2x^3+5xy+2y^2};\quad \left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1}}$

29. $g(x,y,z) = e^x\sqrt{y+2z};\quad g_z(0,6,4)$

30. $g(x,y,z) = \dfrac{3x^2y^2+2xy+x-y}{xy-yz+xz},\quad g_y(1,1,5)$

31. $h(r,s,t,u) = (rst^2u)\ln(1+rstu);\quad h_t(1,1,0,1)$

32. $h(r,s,t,u) = \dfrac{7r+3s^2u^2}{s};\quad h_t(4,3,2,1)$

33. $f(r,s,t) = rst(r^2+s^3+t^4);\quad f_s(1,-1,2)$

34. $z = \dfrac{x^2+y^2}{e^{x^2+y^2}};\quad \left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0\\y=0}},\quad \left.\dfrac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1}}$

35. If $z = xe^{x-y}+ye^{y-x}$, show that

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y} = e^{x-y}+e^{y-x}$$

36. Stock Prices of a Dividend Cycle In a discussion of stock prices of a dividend cycle, Palmon and Yaari[1] consider the function f given by

$$u = f(t,r,z) = \frac{(1+r)^{1-z}\ln(1+r)}{(1+r)^{1-z}-t}$$

where u is the instantaneous rate of ask-price appreciation, r is an annual opportunity rate of return, z is the fraction of a dividend cycle over which a share of stock is held by a midcycle seller, and t is the effective rate of capital gains tax. They claim that

$$\frac{\partial u}{\partial z} = \frac{t(1+r)^{1-z}\ln^2(1+r)}{[(1+r)^{1-z}-t]^2}$$

Verify this.

37. Money Demand In a discussion of inventory theory of money demand, Swanson[2] considers the function

$$F(b,C,T,i) = \frac{bT}{C}+\frac{iC}{2}$$

and determines that $\dfrac{\partial F}{\partial C} = -\dfrac{bT}{C^2}+\dfrac{i}{2}$. Verify this partial derivative.

38. Interest Rate Deregulation In an article on interest rate deregulation, Christofi and Agapos[3] arrive at the equation

$$r_L = r + D\frac{\partial r}{\partial D}+\frac{dC}{dD} \qquad (3)$$

where r is the deposit rate paid by commercial banks, r_L is the rate earned by commercial banks, C is the administrative cost of transforming deposits into return-earning assets, and D is the savings deposit level. Christofi and Agapos state that

$$r_L = r\left[\frac{1+\eta}{\eta}\right]+\frac{dC}{dD} \qquad (4)$$

where $\eta = \dfrac{r/D}{\partial r/\partial D}$ is the deposit elasticity with respect to the deposit rate. Express Equation (3) in terms of η to verify Equation (4).

39. Advertising and Profitability In an analysis of advertising and profitability, Swales[4] considers a function f given by

$$R = f(r,a,n) = \frac{r}{1+a\left(\dfrac{n-1}{2}\right)}$$

where R is the adjusted rate of profit, r is the accounting rate of profit, a is a measure of advertising expenditures, and n is the number of years that advertising fully depreciates. In the analysis, Swales determines $\partial R/\partial n$. Find this partial derivative.

Objective

To develop the notions of partial marginal cost, marginal productivity, and competitive and complementary products.

17.2 Applications of Partial Derivatives

From Section 17.1, we know that if $z = f(x,y)$, then $\partial z/\partial x$ and $\partial z/\partial y$ can be geometrically interpreted as giving the slopes of the tangent lines to the surface $z = f(x,y)$ in the x- and y-directions, respectively. There are other interpretations: Because $\partial z/\partial x$ is

[1] D. Palmon and U. Yaari, "Taxation of Capital Gains and the Behavior of Stock Prices over the Dividend Cycle," *The American Economist,* XXVII, no. 1 (1983), 13–22.

[2] P. E. Swanson, "Integer Constraints on the Inventory Theory of Money Demand," *Quarterly Journal of Business and Economics,* 23, no. 1 (1984), 32–37.

[3] A. Christofi and A. Agapos, "Interest Rate Deregulation: An Empirical Justification," *Review of Business and Economic Research,* XX (1984), 39–49.

[4] J. K. Swales, "Advertising as an Intangible Asset: Profitability and Entry Barriers: A Comment on Reekie and Bhoyrub," *Applied Economics,* 17, no. 4 (1985), 603–17.

the derivative of z with respect to x when y is held fixed, and because a derivative is a rate of change, we have

Here we have "rate of change" interpretations of partial derivatives.

$\dfrac{\partial z}{\partial x}$ is the rate of change of z with respect to x when y is held fixed.

Similarly,

$\dfrac{\partial z}{\partial y}$ is the rate of change of z with respect to y when x is held fixed.

We will now look at some applications in which the "rate of change" notion of a partial derivative is very useful.

Suppose a manufacturer produces x units of product X and y units of product Y. Then the total cost c of these units is a function of x and y and is called a **joint-cost function.** If such a function is $c = f(x, y)$, then $\partial c/\partial x$ is called the **(partial) marginal cost with respect to** x and is the rate of change of c with respect to x when y is held fixed. Similarly, $\partial c/\partial y$ is the **(partial) marginal cost with respect to** y and is the rate of change of c with respect to y when x is held fixed. It also follows that $\partial c/\partial x(x, y)$ is approximately the cost of producing one more unit of X when x units of X and y units of Y are produced. Similarly, $\partial c/\partial y(x, y)$ is approximately the cost of producing one more unit of Y when x units of X and y units of Y are produced.

For example, if c is expressed in dollars and $\partial c/\partial y = 2$, then the cost of producing an extra unit of Y when the level of production of X is fixed is approximately two dollars.

If a manufacturer produces n products, the joint-cost function is a function of n variables, and there are n (partial) marginal-cost functions.

EXAMPLE 1 Marginal Costs

A company manufactures two types of skis, the Lightning and the Alpine models. Suppose the joint-cost function for producing x pairs of the Lightning model and y pairs of the Alpine model per week is

$$c = f(x, y) = 0.07x^2 + 75x + 85y + 6000$$

where c is expressed in dollars. Determine the marginal costs $\partial c/\partial x$ and $\partial c/\partial y$ when $x = 100$ and $y = 50$, and interpret the results.

Solution: The marginal costs are

$$\frac{\partial c}{\partial x} = 0.14x + 75 \quad \text{and} \quad \frac{\partial c}{\partial y} = 85$$

Thus,

$$\left.\frac{\partial c}{\partial x}\right|_{(100,50)} = 0.14(100) + 75 = 89 \tag{1}$$

and

$$\left.\frac{\partial c}{\partial y}\right|_{(100,50)} = 85 \tag{2}$$

Equation (1) means that increasing the output of the Lightning model from 100 to 101 while maintaining production of the Alpine model at 50 increases costs by approximately \$89. Equation (2) means that increasing the output of the Alpine model from 50 to 51 and holding production of the Lightning model at 100 will increase costs by approximately \$85. In fact, since $\partial c/\partial y$ is a constant function, the marginal cost with respect to y is \$85 at all levels of production.

Now Work Problem 1 ◁

EXAMPLE 2 Loss of Body Heat

On a cold day, a person may feel colder when the wind is blowing than when the wind is calm because the rate of heat loss is a function of both temperature and wind speed. The equation

$$H = (10.45 + 10\sqrt{w} - w)(33 - t)$$

indicates the rate of heat loss H (in kilocalories per square meter per hour) when the air temperature is t (in degrees Celsius) and the wind speed is w (in meters per second). For $H = 2000$, exposed flesh will freeze in one minute.[5]

a. Evaluate H when $t = 0$ and $w = 4$.

Solution: When $t = 0$ and $w = 4$,

$$H = (10.45 + 10\sqrt{4} - 4)(33 - 0) = 872.85$$

b. Evaluate $\partial H/\partial w$ and $\partial H/\partial t$ when $t = 0$ and $w = 4$, and interpret the results.

Solution:

$$\frac{\partial H}{\partial w} = \left(\frac{5}{\sqrt{w}} - 1\right)(33 - t), \quad \left.\frac{\partial H}{\partial w}\right|_{\substack{t=0\\w=4}} = 49.5$$

$$\frac{\partial H}{\partial t} = (10.45 + 10\sqrt{w} - w)(-1), \quad \left.\frac{\partial H}{\partial t}\right|_{\substack{t=0\\w=4}} = -26.45$$

These equations mean that when $t = 0$ and $w = 4$, increasing w by a small amount while keeping t fixed will make H increase approximately 49.5 times as much as w increases. Increasing t by a small amount while keeping w fixed will make H *decrease* approximately 26.45 times as much as t increases.

c. When $t = 0$ and $w = 4$, which has a greater effect on H: a change in wind speed of 1 m/s or a change in temperature of 1°C?

Solution: Since the partial derivative of H with respect to w is greater in magnitude than the partial with respect to t when $t = 0$ and $w = 4$, a change in wind speed of 1 m/s has a greater effect on H.

Now Work Problem 13 ◁

The output of a product depends on many factors of production. Among these may be labor, capital, land, machinery, and so on. For simplicity, let us suppose that output depends only on labor and capital. If the function $P = f(l, k)$ gives the output P when the producer uses l units of labor and k units of capital, then this function is called a **production function.** We define the **marginal productivity with respect to** l to be $\partial P/\partial l$. This is the rate of change of P with respect to l when k is held fixed. Likewise, the **marginal productivity with respect to** k is $\partial P/\partial k$ and is the rate of change of P with respect to k when l is held fixed.

EXAMPLE 3 Marginal Productivity

A manufacturer of a popular toy has determined that the production function is $P = \sqrt{lk}$, where l is the number of labor-hours per week and k is the capital (expressed in hundreds of dollars per week) required for a weekly production of P gross of the toy. (One gross is 144 units.) Determine the marginal productivity functions, and evaluate them when $l = 400$ and $k = 16$. Interpret the results.

[5] G. E. Folk, Jr., *Textbook of Environmental Physiology*, 2nd ed. (Philadelphia: Lea & Febiger, 1974).

Solution: Since $P = (lk)^{1/2}$,

$$\frac{\partial P}{\partial l} = \frac{1}{2}(lk)^{-1/2}k = \frac{k}{2\sqrt{lk}}$$

and

$$\frac{\partial P}{\partial k} = \frac{1}{2}(lk)^{-1/2}l = \frac{l}{2\sqrt{lk}}$$

Evaluating these equations when $l = 400$ and $k = 16$, we obtain

$$\left.\frac{\partial P}{\partial l}\right|_{\substack{l=400\\k=16}} = \frac{16}{2\sqrt{400(16)}} = \frac{1}{10}$$

and

$$\left.\frac{\partial P}{\partial k}\right|_{\substack{l=400\\k=16}} = \frac{400}{2\sqrt{400(16)}} = \frac{5}{2}$$

Thus, if $l = 400$ and $k = 16$, increasing l to 401 and holding k at 16 will increase output by approximately $\frac{1}{10}$ gross. But if k is increased to 17 while l is held at 400, the output increases by approximately $\frac{5}{2}$ gross.

Now Work Problem 5 ◁

Competitive and Complementary Products

Sometimes two products may be related such that changes in the price of one of them affect the demand for the other. A typical example is that of butter and margarine. If such a relationship exists between products A and B, then the demand for each product is dependent on the prices of both. Suppose q_A and q_B are the quantities demanded for A and B, respectively, and p_A and p_B are their respective prices. Then both q_A and q_B are functions of p_A and p_B:

$$q_A = f(p_A, p_B) \qquad \text{demand function for A}$$

$$q_B = g(p_A, p_B) \qquad \text{demand function for B}$$

We can find four partial derivatives:

$\dfrac{\partial q_A}{\partial p_A}$ *the marginal demand for* A *with respect to* p_A

$\dfrac{\partial q_A}{\partial p_B}$ *the marginal demand for* A *with respect to* p_B

$\dfrac{\partial q_B}{\partial p_A}$ *the marginal demand for* B *with respect to* p_A

$\dfrac{\partial q_B}{\partial p_B}$ *the marginal demand for* B *with respect to* p_B

Under typical conditions, if the price of B is fixed and the price of A increases, then the quantity of A demanded will decrease. Thus, $\partial q_A/\partial p_A < 0$. Similarly, $\partial q_B/\partial p_B < 0$. However, $\partial q_A/\partial p_B$ and $\partial q_B/\partial p_A$ may be either positive or negative. If

$$\frac{\partial q_A}{\partial p_B} > 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} > 0$$

then A and B are said to be **competitive products** or **substitutes.** In this situation, an increase in the price of B causes an increase in the demand for A, if it is assumed that the price of A does not change. Similarly, an increase in the price of A causes an increase in the demand for B when the price of B is held fixed. Butter and margarine are examples of substitutes.

Proceeding to a different situation, we say that if

$$\frac{\partial q_A}{\partial p_B} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} < 0$$

then A and B are **complementary products.** In this case, an increase in the price of B causes a decrease in the demand for A if the price of A does not change. Similarly, an increase in the price of A causes a decrease in the demand for B when the price of B is held fixed. For example, cars and gasoline are complementary products. An increase in the price of gasoline will make driving more expensive. Hence, the demand for cars will decrease. And an increase in the price of cars will reduce the demand for gasoline.

EXAMPLE 4 Determining Whether Products Are Competitive or Complementary

The demand functions for products A and B are each a function of the prices of A and B and are given by

$$q_A = \frac{50\sqrt[3]{p_B}}{\sqrt{p_A}} \quad \text{and} \quad q_B = \frac{75p_A}{\sqrt[3]{p_B^2}}$$

respectively. Find the four marginal-demand functions, and determine whether A and B are competitive products, complementary products, or neither.

Solution: Writing $q_A = 50p_A^{-1/2}p_B^{1/3}$ and $q_B = 75p_Ap_B^{-2/3}$, we have

$$\frac{\partial q_A}{\partial p_A} = 50\left(-\frac{1}{2}\right)p_A^{-3/2}p_B^{1/3} = -25p_A^{-3/2}p_B^{1/3}$$

$$\frac{\partial q_A}{\partial p_B} = 50p_A^{-1/2}\left(\frac{1}{3}\right)p_B^{-2/3} = \frac{50}{3}p_A^{-1/2}p_B^{-2/3}$$

$$\frac{\partial q_B}{\partial p_A} = 75(1)p_B^{-2/3} = 75p_B^{-2/3}$$

$$\frac{\partial q_B}{\partial p_B} = 75p_A\left(-\frac{2}{3}\right)p_B^{-5/3} = -50p_Ap_B^{-5/3}$$

Since p_A and p_B represent prices, they are both positive. Hence, $\partial q_A/\partial p_B > 0$ and $\partial q_B/\partial p_A > 0$. We conclude that A and B are competitive products.

Now Work Problem 19 ◁

PROBLEMS 17.2

For the joint-cost functions in Problems 1–3, find the indicated marginal cost at the given production level.

1. $c = 7x + 0.3y^2 + 2y + 900; \quad \frac{\partial c}{\partial y}, x = 20, y = 30$

2. $c = 2x\sqrt{x+y} + 6000; \quad \frac{\partial c}{\partial x}, x = 70, y = 74$

3. $c = 0.03(x+y)^3 - 0.6(x+y)^2 + 9.5(x+y) + 7700;$
$\frac{\partial c}{\partial x}, x = 50, y = 80$

For the production functions in Problems 4 and 5, find the marginal productivity functions $\partial P/\partial k$ and $\partial P/\partial l$.

4. $P = 15lk - 3l^2 + 5k^2 + 500$

5. $P = 2.314l^{0.357}k^{0.643}$

6. Cobb–Douglas Production Function In economics, a Cobb–Douglas production function is a production function of the form $P = Al^{\alpha}k^{\beta}$, where A, α, and β are constants and $\alpha + \beta = 1$. For such a function, show that

(a) $\partial P/\partial l = \alpha P/l$ **(b)** $\partial P/\partial k = \beta P/k$

(c) $l\frac{\partial P}{\partial l} + k\frac{\partial P}{\partial k} = P$. This means that summing the products of the marginal productivity of each factor and the amount of that factor results in the total product P.

In Problems 7–9, q_A and q_B are demand functions for products A and B, respectively. In each case, find $\partial q_A/\partial p_A$, $\partial q_A/\partial p_B$, $\partial q_B/\partial p_A$, $\partial q_B/\partial p_B$ and determine whether A and B are competitive, complementary, or neither.

7. $q_A = 1500 - 40p_A + 3p_B; \quad q_B = 900 + 5p_A - 20p_B$

8. $q_A = 20 - p_A - 2p_B; \quad q_B = 50 - 2p_A - 3p_B$

9. $q_A = \frac{100}{p_A\sqrt{p_B}}; \quad q_B = \frac{500}{p_B\sqrt[3]{p_A}}$

10. Canadian Manufacturing The production function for the Canadian manufacturing industries for 1927 is estimated by[6] $P = 33.0l^{0.46}k^{0.52}$, where P is product, l is labor, and k is capital. Find the marginal productivities for labor and capital, and evaluate when $l = 1$ and $k = 1$.

11. Dairy Farming An estimate of the production function for dairy farming in Iowa (1939) is given by[7]

$$P = A^{0.27}B^{0.01}C^{0.01}D^{0.23}E^{0.09}F^{0.27}$$

where P is product, A is land, B is labor, C is improvements, D is liquid assets, E is working assets, and F is cash operating expenses. Find the marginal productivities for labor and improvements.

12. Production Function Suppose a production function is given by $P = \dfrac{kl}{3k + 5l}$.
(a) Determine the marginal productivity functions.
(b) Show that when $k = l$, the marginal productivities sum to $\dfrac{1}{8}$.

13. M.B.A. Compensation In a study of success among graduates with master of business administration (M.B.A.) degrees, it was estimated that for staff managers (which include accountants, analysts, etc.), current annual compensation (in dollars) was given by

$$z = 43{,}960 + 4480x + 3492y$$

where x and y are the number of years of work experience before and after receiving the M.B.A. degree, respectively.[8] Find $\partial z/\partial x$ and interpret your result.

14. Status A person's general status S_g is believed to be a function of status attributable to education, S_e, and status attributable to income, S_i, where S_g, S_e, and S_i are represented numerically. If

$$S_g = 7\sqrt[3]{S_e}\sqrt{S_i}$$

determine $\partial S_g/\partial S_e$ and $\partial S_g/\partial S_i$ when $S_e = 125$ and $S_i = 100$, and interpret your results.[9]

15. Reading Ease Sometimes we want to evaluate the degree of readability of a piece of writing. Rudolf Flesch[10] developed a function of two variables that will do this, namely,

$$R = f(w, s) = 206.835 - (1.015w + 0.846s)$$

where R is called the *reading ease score*, w is the average number of words per sentence in 100-word samples, and s is the average number of syllables in such samples. Flesch says that an article for which $R = 0$ is "practically unreadable," but one with $R = 100$ is "easy for any literate person." (a) Find $\partial R/\partial w$ and $\partial R/\partial s$. (b) Which is "easier" to read: an article for which $w = w_0$ and $s = s_0$, or one for which $w = w_0 + 1$ and $s = s_0$?

16. Model for Voice The study of frequency of vibrations of a taut wire is useful in considering such things as an individual's voice. Suppose

$$\omega = \frac{1}{bL}\sqrt{\frac{\tau}{\pi\rho}}$$

where ω (a Greek letter read "omega") is frequency, b is diameter, L is length, ρ (a Greek letter read "rho") is density, and τ (a Greek letter read "tau") is tension.[11] Find $\partial\omega/\partial b$, $\partial\omega/\partial L$, $\partial\omega/\partial\rho$, and $\partial\omega/\partial\tau$.

17. Traffic Flow Consider the following traffic-flow situation. On a highway where two lanes of traffic flow in the same direction, there is a maintenance vehicle blocking the left lane. (See Figure 17.3.) Two vehicles (*lead* and *following*) are in the right lane with a gap between them. The *subject* vehicle can choose either to fill or not to fill the gap. That decision may be based not only on the distance x shown in the diagram, but also on other factors (such as the velocity of the *following* vehicle). A *gap index* g has been used in analyzing such a decision.[12,13] The greater the g-value, the greater is the propensity for the *subject* vehicle to fill the gap. Suppose

$$g = \frac{x}{V_F} - \left(0.75 + \frac{V_F - V_S}{19.2}\right)$$

where x (in feet) is as before, V_F is the velocity of the *following* vehicle (in feet per second), and V_S is the velocity of the *subject* vehicle (in feet per second). From the diagram, it seems reasonable that if both V_F and V_S are fixed and x increases, then g should increase. Show that this is true by applying calculus to the function g. Assume that x, V_F, and V_S are positive.

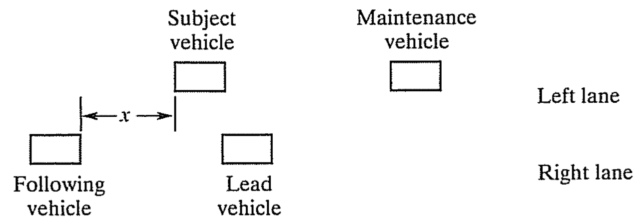

FIGURE 17.3

18. Demand Suppose the demand equations for related products A and B are

$$q_A = e^{-(p_A+p_B)} \quad \text{and} \quad q_B = \frac{16}{p_A^2 p_B^2}$$

[6] P. Daly and P. Douglas, "The Production Function for Canadian Manufactures," *Journal of the American Statistical Association*, 38 (1943), 178–86.

[7] G. Tintner and O. H. Brownlee, "Production Functions Derived from Farm Records," *American Journal of Agricultural Economics*, 26 (1944), 566–71.

[8] Adapted from A. G. Weinstein and V. Srinivasen, "Predicting Managerial Success of Master of Business Administration (M.B.A.) Graduates," *Journal of Applied Psychology*, 59, no. 2 (1974), 207–12.

[9] Adapted from R. K. Leik and B. F. Meeker, *Mathematical Sociology* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975).

[10] R. Flesch, *The Art of Readable Writing* (New York: Harper & Row Publishers, Inc., 1949).

[11] R. M. Thrall, J. A. Mortimer, K. R. Rebman, and R. F. Baum, eds., *Some Mathematical Models in Biology*, rev. ed., Report No. 40241-R-7. Prepared at University of Michigan, 1967.

[12] P. M. Hurst, K. Perchonok, and E. L. Seguin, "Vehicle Kinematics and Gap Acceptance," *Journal of Applied Psychology*, 52, no. 4 (1968), 321–24.

[13] K. Perchonok and P. M. Hurst, "Effect of Lane-Closure Signals upon Driver Decision Making and Traffic Flow," *Journal of Applied Psychology*, 52, no. 5 (1968), 410–13.

where q_A and q_B are the number of units of A and B demanded when the unit prices (in thousands of dollars) are p_A and p_B, respectively.

(a) Classify A and B as competitive, complementary, or neither.
(b) If the unit prices of A and B are \$1000 and \$2000, respectively, estimate the change in the demand for A when the price of B is decreased by \$20 and the price of A is held constant.

19. Demand The demand equations for related products A and B are given by

$$q_A = 10\sqrt{\frac{p_B}{p_A}} \quad \text{and} \quad q_B = 3\sqrt[3]{\frac{p_A}{p_B}}$$

where q_A and q_B are the quantities of A and B demanded and p_A and p_B are the corresponding prices (in dollars) per unit.

(a) Find the values of the two marginal demands for product A when $p_A = 9$ and $p_B = 16$.
(b) If p_B were reduced to 14 from 16, with p_A fixed at 9, use part (a) to estimate the corresponding change in demand for product A.

20. Joint-Cost Function A manufacturer's joint-cost function for producing q_A units of product A and q_B units of product B is given by

$$c = \frac{q_A^2(q_B^3 + q_A)^{1/2}}{17} + q_A q_B^{1/3} + 600$$

where c is in dollars.

(a) Find the marginal-cost functions with respect to q_A and q_B.
(b) Evaluate the marginal-cost function with respect to q_A when $q_A = 17$ and $q_B = 8$. Round your answer to two decimal places.
(c) Use your answer to part (a) to estimate the change in cost if production of product A is decreased from 17 to 16 units, while production of product B is held constant at 8 units.

21. Elections For the congressional elections of 1974, the Republican percentage, R, of the Republican–Democratic vote in a district is given (approximately) by[14]

$$\begin{aligned} R &= f(E_r, E_d, I_r, I_d, N) \\ &= 15.4725 + 2.5945E_r - 0.0804E_r^2 - 2.3648E_d \\ &\quad + 0.0687E_d^2 + 2.1914I_r - 0.0912I_r^2 \\ &\quad - 0.8096I_d + 0.0081I_d^2 - 0.0277E_rI_r \\ &\quad + 0.0493E_dI_d + 0.8579N - 0.0061N^2 \end{aligned}$$

Here E_r and E_d are the campaign expenditures (in units of \$10,000) by Republicans and Democrats, respectively; I_r and I_d are the number of terms served in Congress, *plus one*, for the Republican and Democratic candidates, respectively; and N is the percentage of the two-party presidential vote that Richard Nixon received in the district for 1968. The variable N gives a measure of Republican strength in the district.

(a) In the Federal Election Campaign Act of 1974, Congress set a limit of \$188,000 on campaign expenditures. By analyzing $\partial R/\partial E_r$, would you have advised a Republican candidate who served nine terms in Congress to spend \$188,000 on his or her campaign?
(b) Find the percentage above which the Nixon vote had a negative effect on R; that is, find N when $\partial R/\partial N < 0$. Give your answer to the nearest percent.

22. Sales After a new product has been launched onto the market, its sales volume (in thousands of units) is given by

$$S = \frac{AT + 450}{\sqrt{A + T^2}}$$

where T is the time (in months) since the product was first introduced and A is the amount (in hundreds of dollars) spent each month on advertising.

(a) Verify that the partial derivative of sales volume with respect to time is given by

$$\frac{\partial S}{\partial T} = \frac{A^2 - 450T}{(A + T^2)^{3/2}}$$

(b) Use the result in part (a) to predict the number of months that will elapse before the sales volume begins to decrease if the amount allocated to advertising is held fixed at \$9000 per month.

Let f be a demand function for product A and $q_A = f(p_A, p_B)$, where q_A is the quantity of A demanded when the price per unit of A is p_A and the price per unit of product B is p_B. The partial elasticity of demand for A with respect to p_A, denoted η_{p_A}, is defined as $\eta_{p_A} = (p_A/q_A)(\partial q_A/\partial p_A)$. The partial elasticity of demand for A with respect to p_B, denoted η_{p_B}, is defined as $\eta_{p_B} = (p_B/q_A)(\partial q_A/\partial p_B)$. Loosely speaking, η_{p_A} is the ratio of a percentage change in the quantity of A demanded to a percentage change in the price of A when the price of B is fixed. Similarly, η_{p_B} can be loosely interpreted as the ratio of a percentage change in the quantity of A demanded to a percentage change in the price of B when the price of A is fixed. In Problems 23–25, find η_{p_A} and η_{p_B} for the given values of p_A and p_B.

23. $q_A = 1000 - 50p_A + 2p_B$; $p_A = 2, p_B = 10$
24. $q_A = 60 - 3p_A - 2p_B$; $p_A = 5, p_B = 3$
25. $q_A = 100/(p_A\sqrt{p_B})$; $p_A = 1, p_B = 4$

Objective

To find partial derivatives of a function defined implicitly.

17.3 Implicit Partial Differentiation[15]

An equation in x, y, and z does not necessarily define z as a function of x and y. For example, in the equation

$$z^2 - x^2 - y^2 = 0 \qquad (1)$$

[14] J. Silberman and G. Yochum, "The Role of Money in Determining Election Outcomes," *Social Science Quarterly*, 58, no. 4 (1978), 671–82.

[15] This section can be omitted without loss of continuity.

if $x = 1$ and $y = 1$, then $z^2 - 1 - 1 = 0$, so $z = \pm\sqrt{2}$. Thus, Equation (1) does not define z as a function of x and y. However, solving Equation (1) for z gives

$$z = \sqrt{x^2 + y^2} \quad \text{or} \quad z = -\sqrt{x^2 + y^2}$$

each of which defines z as a function of x and y. Although Equation (1) does not explicitly express z as a function of x and y, it can be thought of as expressing z *implicitly* as one of two different functions of x and y. Note that the equation $z^2 - x^2 - y^2 = 0$ has the form $F(x, y, z) = 0$, where F is a function of three variables. Any equation of the form $F(x, y, z) = 0$ can be thought of as expressing z implicitly as one of a set of possible functions of x and y. Moreover, we can find $\partial z/\partial x$ and $\partial z/\partial y$ directly from the form $F(x, y, z) = 0$.

To find $\partial z/\partial x$ for

$$z^2 - x^2 - y^2 = 0 \tag{2}$$

we first differentiate both sides of Equation (2) with respect to x while treating z as a function of x and y and treating y as a constant:

$$\frac{\partial}{\partial x}(z^2 - x^2 - y^2) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(y^2) = 0$$

$$2z\frac{\partial z}{\partial x} - 2x - 0 = 0$$

Because y is treated as a constant, $\dfrac{\partial y}{\partial x} = 0$.

Solving for $\partial z/\partial x$, we obtain

$$2z\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial x} = \frac{x}{z}$$

To find $\partial z/\partial y$, we differentiate both sides of Equation (2) with respect to y while treating z as a function of x and y and treating x as a constant:

$$\frac{\partial}{\partial y}(z^2 - x^2 - y^2) = \frac{\partial}{\partial y}(0)$$

$$2z\frac{\partial z}{\partial y} - 0 - 2y = 0 \qquad \frac{\partial x}{\partial y} = 0$$

$$2z\frac{\partial z}{\partial y} = 2y$$

Hence,

$$\frac{\partial z}{\partial y} = \frac{y}{z}$$

The method we used to find $\partial z/\partial x$ and $\partial z/\partial y$ is called *implicit partial differentiation.*

EXAMPLE 1 Implicit Partial Differentiation

If $\dfrac{xz^2}{x+y} + y^2 = 0$, evaluate $\dfrac{\partial z}{\partial x}$ when $x = -1, y = 2$, and $z = 2$.

Solution: We treat z as a function of x and y and differentiate both sides of the equation with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{xz^2}{x+y}\right) + \frac{\partial}{\partial x}(y^2) = \frac{\partial}{\partial x}(0)$$

Using the quotient rule for the first term on the left, we have

$$\frac{(x+y)\dfrac{\partial}{\partial x}(xz^2) - xz^2\dfrac{\partial}{\partial x}(x+y)}{(x+y)^2} + 0 = 0$$

Using the product rule for $\dfrac{\partial}{\partial x}(xz^2)$ gives

$$\frac{(x+y)\left[x\left(2z\dfrac{\partial z}{\partial x}\right) + z^2(1)\right] - xz^2(1)}{(x+y)^2} = 0$$

Solving for $\partial z/\partial x$, we obtain

$$2xz(x+y)\frac{\partial z}{\partial x} + z^2(x+y) - xz^2 = 0$$

$$\frac{\partial z}{\partial x} = \frac{xz^2 - z^2(x+y)}{2xz(x+y)} = -\frac{yz}{2x(x+y)} \qquad z \neq 0$$

Thus,

$$\left.\frac{\partial z}{\partial x}\right|_{(-1,2,2)} = 2$$

Now Work Problem 13 ◁

EXAMPLE 2 **Implicit Partial Differentiation**

If $se^{r^2+u^2} = u\ln(t^2+1)$, determine $\partial t/\partial u$.

Solution: We consider t as a function of r, s, and u. By differentiating both sides with respect to u while treating r and s as constants, we get

$$\frac{\partial}{\partial u}(se^{r^2+u^2}) = \frac{\partial}{\partial u}(u\ln(t^2+1))$$

$$2sue^{r^2+u^2} = u\frac{\partial}{\partial u}(\ln(t^2+1)) + \ln(t^2+1)\frac{\partial}{\partial u}(u) \qquad \text{product rule}$$

$$2sue^{r^2+u^2} = u\frac{2t}{t^2+1}\frac{\partial t}{\partial u} + \ln(t^2+1)$$

Therefore,

$$\frac{\partial t}{\partial u} = \frac{(t^2+1)(2sue^{r^2+u^2} - \ln(t^2+1))}{2ut}$$

Now Work Problem 1 ◁

PROBLEMS 17.3

In Problems 1–11, find the indicated partial derivatives by the method of implicit partial differentiation.

1. $2x^2 + 3y^2 + 5z^2 = 900$; $\partial z/\partial x$
2. $z^2 - 5x^2 + y^2 = 0$; $\partial z/\partial x$
3. $3z^2 - 5x^2 - 7y^2 = 0$; $\partial z/\partial y$
4. $3x^2 + y^2 + 2z^3 = 9$; $\partial z/\partial y$
5. $x^2 - 2y - z^2 + x^2yz^2 = 20$; $\partial z/\partial x$
6. $z^3 + 2x^2z^2 - xy = 0$; $\partial z/\partial x$
7. $e^x + e^y + e^z = 10$; $\partial z/\partial y$
8. $xyz + xy^2z^3 - \ln z^4 = 0$; $\partial z/\partial y$
9. $\ln(z) + 9z - xy = 1$; $\partial z/\partial x$
10. $\ln x + \ln y - \ln z = e^y$; $\partial z/\partial x$
11. $(z^2 + 6xy)\sqrt{x^3+5} = 2$; $\partial z/\partial y$

In Problems 12–20, evaluate the indicated partial derivatives for the given values of the variables.

12. $xz + xyz - 5 = 0$; $\partial z/\partial x, x = 1, y = 4, z = 1$
13. $xz^2 + yz^2 - x^2y = 1$; $\partial z/\partial x, x = 1, y = 0, z = 1$
14. $e^{zx} = xyz$; $\partial z/\partial y, x = 1, y = -e^{-1}, z = -1$
15. $e^{yz} = -xyz$; $\partial z/\partial x, x = -e^2/2, y = 1, z = 2$
16. $\sqrt{xz + y^2} - xy = 0$; $\partial z/\partial y, x = 2, y = 2, z = 6$
17. $\ln z = 4x + y$; $\partial z/\partial x, x = 5, y = -20, z = 1$
18. $\dfrac{r^2s^2}{s^2+t^2} = \dfrac{t^2}{2}$; $\partial r/\partial t, r = 1, s = 1, t = 1$
19. $\dfrac{s^2+t^2}{rs} = 10$; $\partial t/\partial r, r = 1, s = 2, t = 4$
20. $\ln(x + y + z) + xyz = ze^{x+y+z}$; $\partial z/\partial x, x = 0, y = 1, z = 0$

21. Joint-Cost Function A joint-cost function is defined implicitly by the equation

$$c + \sqrt{c} = 12 + q_A\sqrt{9 + q_B^2}$$

where c denotes the total cost (in dollars) for producing q_A units of product A and q_B units of product B.

(a) If $q_A = 6$ and $q_B = 4$, find the corresponding value of c.

(b) Determine the marginal costs with respect to q_A and q_B when $q_A = 6$ and $q_B = 4$.

Objective

To compute higher-order partial derivatives.

17.4 Higher-Order Partial Derivatives

If $z = f(x, y)$, then not only is z a function of x and y, but also f_x and f_y are each functions of x and y, which may themselves have partial derivatives. If we can differentiate f_x and f_y, we obtain **second-order partial derivatives** of f. Symbolically,

$$f_{xx} \text{ means } (f_x)_x \quad f_{xy} \text{ means } (f_x)_y$$
$$f_{yx} \text{ means } (f_y)_x \quad f_{yy} \text{ means } (f_y)_y$$

In terms of ∂-notation,

$$\frac{\partial^2 z}{\partial x^2} \text{ means } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y\,\partial x} \text{ means } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} \text{ means } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \qquad \frac{\partial^2 z}{\partial y^2} \text{ means } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

CAUTION!

For $z = f(x, y), f_{xy} = \partial^2 z/\partial y\,\partial x$.

Note that to find f_{xy}, we first differentiate f with respect to x. For $\partial^2 z/\partial x\,\partial y$, we first differentiate with respect to y.

We can extend our notation beyond second-order partial derivatives. For example, f_{xxy} $(= \partial^3 z/\partial y\,\partial x^2)$ is a third-order partial derivative of f, namely, the partial derivative of f_{xx} $(= \partial^2 z/\partial x^2)$ with respect to y. The generalization of higher-order partial derivatives to functions of more than two variables should be obvious.

EXAMPLE 1 Second-Order Partial Derivatives

Find the four second-order partial derivatives of $f(x, y) = x^2y + x^2y^2$.

Solution: Since

$$f_x(x, y) = 2xy + 2xy^2$$

we have

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(2xy + 2xy^2) = 2y + 2y^2$$

and

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(2xy + 2xy^2) = 2x + 4xy$$

Also, since

$$f_y(x, y) = x^2 + 2x^2y$$

we have

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(x^2 + 2x^2y) = 2x^2$$

and

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(x^2 + 2x^2y) = 2x + 4xy$$

Now Work Problem 1 ◁

The derivatives f_{xy} and f_{yx} are called **mixed partial derivatives.** Observe in Example 1 that $f_{xy}(x, y) = f_{yx}(x, y)$. Under suitable conditions, mixed partial derivatives of a function are equal; that is, the order of differentiation is of no concern. You may assume that this is the case for all the functions that we consider.

EXAMPLE 2 **Mixed Partial Derivative**

Find the value of $\left.\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x}\right|_{(1,2,3)}$ if $w = (2x + 3y + 4z)^3$.

Solution:

$$\frac{\partial w}{\partial x} = 3(2x+3y+4z)^2\frac{\partial}{\partial x}(2x+3y+4z) = 6(2x+3y+4z)^2$$

$$\frac{\partial^2 w}{\partial y\,\partial x} = 6\cdot 2(2x+3y+4z)\frac{\partial}{\partial y}(2x+3y+4z) = 36(2x+3y+4z)$$

$$\frac{\partial^3 w}{\partial z\,\partial y\,\partial x} = 36\cdot 4 = 144$$

Thus,

$$\left.\frac{\partial^3 w}{\partial z\,\partial y\,\partial x}\right|_{(1,2,3)} = 144$$

Now Work Problem 3 ◁

EXAMPLE 3 **Second-Order Partial Derivative of an Implicit Function**[16]

Determine $\dfrac{\partial^2 z}{\partial x^2}$ if $z^2 = xy$.

Solution: By implicit differentiation, we first determine $\partial z/\partial x$:

$$\frac{\partial}{\partial x}(z^2) = \frac{\partial}{\partial x}(xy)$$

$$2z\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial x} = \frac{y}{2z} \qquad z \neq 0$$

Differentiating both sides with respect to x, we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}yz^{-1}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{2}yz^{-2}\frac{\partial z}{\partial x}$$

Substituting $y/(2z)$ for $\partial z/\partial x$, we have

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{2}yz^{-2}\left(\frac{y}{2z}\right) = -\frac{y^2}{4z^3} \qquad z \neq 0$$

Now Work Problem 23 ◁

PROBLEMS 17.4

In Problems 1–10, find the indicated partial derivatives.

1. $f(x, y) = 6xy^2$; $f_x(x, y)$, $f_{xy}(x, y)$, $f_{yx}(x, y)$
2. $f(x, y) = 2x^3y^2 + 6x^2y^3 - 3xy$; $f_x(x, y)$, $f_{xx}(x, y)$
3. $f(x, y) = 7x^2 + 3y$; $f_y(x, y)$, $f_{yy}(x, y)$, $f_{yyx}(x, y)$
4. $f(x, y) = (x^2 + xy + y^2)(xy + x + y)$; $f_x(x, y)$, $f_{xy}(x, y)$
5. $f(x, y) = 9e^{2xy}$; $f_y(x, y)$, $f_{yx}(x, y)$, $f_{yxy}(x, y)$
6. $f(x, y) = \ln(x^2 + y^2) + 2$; $f_x(x, y)$, $f_{xx}(x, y)$, $f_{xy}(x, y)$
7. $f(x, y) = (x + y)^2(xy)$; $f_x(x, y)$, $f_y(x, y)$, $f_{xx}(x, y)$, $f_{yy}(x, y)$
8. $f(x, y, z) = x^2y^3z^4$; $f_x(x, y, z)$, $f_{xz}(x, y, z)$, $f_{zx}(x, y, z)$
9. $z = \ln\sqrt{x^2 + y^2}$; $\dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial y^2}$
10. $z = \dfrac{\ln(x^2 + 5)}{y}$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial y\,\partial x}$

In Problems 11–16, find the indicated value.

11. If $f(x, y, z) = 7$, find $f_{yxx}(4, 3, -2)$.
12. If $f(x, y, z) = z^2(3x^2 - 4xy^3)$, find $f_{xyz}(1, 2, 3)$.

[16]Omit if Section 17.3 was not covered.

13. If $f(l,k) = 3l^3k^6 - 2l^2k^7$, find $f_{klk}(2,1)$.

14. If $f(x,y) = x^3y^2 + x^2y - x^2y^2$, find $f_{xxy}(2,3)$ and $f_{xyx}(2,3)$.

15. If $f(x,y) = y^2e^x + \ln(xy)$, find $f_{xyy}(1,1)$.

16. If $f(x,y) = x^3 - 6xy^2 + x^2 - y^3$, find $f_{xy}(1,-1)$.

17. Cost Function Suppose the cost c of producing q_A units of product A and q_B units of product B is given by

$$c = (3q_A^2 + q_B^3 + 4)^{1/3}$$

and the coupled demand functions for the products are given by

$$q_A = 10 - p_A + p_B^2$$

and

$$q_B = 20 + p_A - 11p_B$$

Find the value of

$$\frac{\partial^2 c}{\partial q_A\,\partial q_B}$$

when $p_A = 25$ and $p_B = 4$.

18. For $f(x,y) = x^4y^4 + 3x^3y^2 - 7x + 4$, show that

$$f_{xyx}(x,y) = f_{xxy}(x,y)$$

19. For $f(x,y) = e^{x^2+xy+y^2}$, show that

$$f_{xy}(x,y) = f_{yx}(x,y)$$

20. For $f(x,y) = e^{xy}$, show that

$$f_{xx}(x,y) + f_{xy}(x,y) + f_{yx}(x,y) + f_{yy}(x,y) = f(x,y)((x+y)^2 + 2)$$

21. For $z = \ln(x^2 + y^2)$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

[17]**22.** If $z^3 - x^3 - x^2y - xy^2 - y^3 = 0$, find $\dfrac{\partial^2 z}{\partial x^2}$.

[17]**23.** If $z^2 - 3x^2 + y^2 = 0$, find $\dfrac{\partial^2 z}{\partial y^2}$.

[17]**24.** If $2z^2 = x^2 + 2xy + xz$, find $\dfrac{\partial^2 z}{\partial x\,\partial y}$.

Objective

To show how to find partial derivatives of composite functions by using the chain rule.

17.5 Chain Rule[18]

Suppose a manufacturer of two related products A and B has a joint-cost function given by

$$c = f(q_A, q_B)$$

where c is the total cost of producing quantities q_A and q_B of A and B, respectively. Furthermore, suppose the demand functions for the products are

$$q_A = g(p_A, p_B) \quad \text{and} \quad q_B = h(p_A, p_B)$$

where p_A and p_B are the prices per unit of A and B, respectively. Since c is a function of q_A and q_B, and since both q_A and q_B are themselves functions of p_A and p_B, c can be viewed as a function of p_A and p_B. (Appropriately, the variables q_A and q_B are called *intermediate variables* of c.) Consequently, we should be able to determine $\partial c/\partial p_A$, the rate of change of total cost with respect to the price of A. One way to do this is to substitute the expressions $g(p_A, p_B)$ and $h(p_A, p_B)$ for q_A and q_B, respectively, into $c = f(q_A, q_B)$. Then c is a function of p_A and p_B, and we can differentiate c with respect to p_A directly. This approach has some drawbacks—especially when f, g, or h is given by a complicated expression. Another way to approach the problem would be to use the chain rule (actually *a* chain rule), which we now state without proof.

Chain Rule

Let $z = f(x,y)$, where both x and y are functions of r and s given by $x = x(r,s)$ and $y = y(r,s)$. If f, x, and y have continuous partial derivatives, then z is a function of r and s, and

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

[17]Omit if Section 17.3 was not covered.

[18]This section can be omitted without loss of continuity.

Note that in the chain rule, the number of intermediate variables of z (two) is the same as the number of terms that compose each of $\partial z/\partial r$ and $\partial z/\partial s$.

Returning to the original situation concerning the manufacturer, we see that if f, q_A, and q_B have continuous partial derivatives, then, by the chain rule,

$$\frac{\partial c}{\partial p_A} = \frac{\partial c}{\partial q_A}\frac{\partial q_A}{\partial p_A} + \frac{\partial c}{\partial q_B}\frac{\partial q_B}{\partial p_A}$$

EXAMPLE 1 Rate of Change of Cost

For a manufacturer of cameras and film, the total cost c of producing q_C cameras and q_F units of film is given by

$$c = 30q_C + 0.015q_Cq_F + q_F + 900$$

The demand functions for the cameras and film are given by

$$q_C = \frac{9000}{p_C\sqrt{p_F}} \quad \text{and} \quad q_F = 2000 - p_C - 400p_F$$

where p_C is the price per camera and p_F is the price per unit of film. Find the rate of change of total cost with respect to the price of the camera when $p_C = 50$ and $p_F = 2$.

Solution: We must first determine $\partial c/\partial p_C$. By the chain rule,

$$\frac{\partial c}{\partial p_C} = \frac{\partial c}{\partial q_C}\frac{\partial q_C}{\partial p_C} + \frac{\partial c}{\partial q_F}\frac{\partial q_F}{\partial p_C}$$

$$= (30 + 0.015q_F)\left[\frac{-9000}{p_C^2\sqrt{p_F}}\right] + (0.015q_C + 1)(-1)$$

When $p_C = 50$ and $p_F = 2$, then $q_C = 90\sqrt{2}$ and $q_F = 1150$. Substituting these values into $\partial c/\partial p_C$ and simplifying, we have

$$\left.\frac{\partial c}{\partial p_C}\right|_{\substack{p_C=50\\ p_F=2}} \approx -123.2$$

Now Work Problem 1 ◁

The chain rule can be extended. For example, suppose $z = f(v, w, x, y)$ and v, w, x, and y are all functions of r, s, and t. Then, if certain conditions of continuity are assumed, z is a function of r, s, and t, and we have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Observe that the number of intermediate variables of z (four) is the same as the number of terms that form each of $\partial z/\partial r$, $\partial z/\partial s$, and $\partial z/\partial t$.

Now consider the situation where $z = f(x, y)$ such that $x = x(t)$ and $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Use the partial derivative symbols and the ordinary derivative symbols appropriately.

Here we use the symbol dz/dt rather than $\partial z/\partial t$, since z can be considered a function of *one* variable t. Likewise, the symbols dx/dt and dy/dt are used rather than $\partial x/\partial t$ and $\partial y/\partial t$. As is typical, the number of terms that compose dz/dt equals the number of intermediate variables of z. Other situations would be treated in a similar way.

EXAMPLE 2 **Chain Rule**

a. If $w = f(x, y, z) = 3x^2y + xyz - 4y^2z^3$, where

$$x = 2r - 3s \quad y = 6r + s \quad z = r - s$$

determine $\partial w/\partial r$ *and* $\partial w/\partial s$.

Solution: Since x, y, and z are functions of r and s, then, by the chain rule,

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ &= (6xy + yz)(2) + (3x^2 + xz - 8yz^3)(6) + (xy - 12y^2z^2)(1) \\ &= x(18x + 13y + 6z) + 2yz(1 - 24z^2 - 6yz)\end{aligned}$$

Also,

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (6xy + yz)(-3) + (3x^2 + xz - 8yz^3)(1) + (xy - 12y^2z^2)(-1) \\ &= x(3x - 19y + z) - yz(3 + 8z^2 - 12yz)\end{aligned}$$

b. If $z = \dfrac{x + e^y}{y}$, where $x = rs + se^{rt}$ and $y = 9 + rt$, evaluate $\partial z/\partial s$ when $r = -2$, $s = 5$, and $t = 4$.

Solution: Since x and y are functions of r, s, and t (note that we can write $y = 9 + rt + 0 \cdot s$), by the chain rule,

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= \left(\frac{1}{y}\right)(r + e^{rt}) + \frac{\partial z}{\partial y} \cdot (0) = \frac{r + e^{rt}}{y}\end{aligned}$$

If $r = -2, s = 5$, and $t = 4$, then $y = 1$. Thus,

$$\left.\frac{\partial z}{\partial s}\right|_{\substack{r=-2\\ s=5\\ t=4}} = \frac{-2 + e^{-8}}{1} = -2 + e^{-8}$$

Now Work Problem 13 ◁

EXAMPLE 3 **Chain Rule**

a. Determine $\partial y/\partial r$ if $y = x^2 \ln (x^4 + 6)$ and $x = (r + 3s)^6$.

Solution: By the chain rule,

$$\begin{aligned}\frac{\partial y}{\partial r} &= \frac{dy}{dx}\frac{\partial x}{\partial r} \\ &= \left[x^2 \cdot \frac{4x^3}{x^4 + 6} + 2x \cdot \ln (x^4 + 6)\right][6(r + 3s)^5] \\ &= 12x(r + 3s)^5\left[\frac{2x^4}{x^4 + 6} + \ln (x^4 + 6)\right]\end{aligned}$$

b. Given that $z = e^{xy}, x = r - 4s$, and $y = r - s$, find $\partial z/\partial r$ in terms of r and s.

Solution:

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= (ye^{xy})(1) + (xe^{xy})(1) \\ &= (x+y)e^{xy}\end{aligned}$$

Since $x = r - 4s$ and $y = r - s$,

$$\begin{aligned}\frac{\partial z}{\partial r} &= [(r-4s)+(r-s)]e^{(r-4s)(r-s)} \\ &= (2r-5s)e^{r^2-5rs+4s^2}\end{aligned}$$

Now Work Problem 15 ◁

PROBLEMS 17.5

In Problems 1–12, find the indicated derivatives by using the chain rule.

1. $z = 5x + 3y, x = 2r + 3s, y = r - 2s;\quad \partial z/\partial r, \partial z/\partial s$

2. $z = 2x^2 + 3xy + 2y^2, x = r^2 - s^2, y = r^2 + s^2;\quad \partial z/\partial r, \partial z/\partial s$

3. $z = e^{x+y}, x = t^2 + 3, y = \sqrt{t^3};\quad dz/dt$

4. $z = \sqrt{8x+y}, x = t^2 + 3t + 4,\ y = t^3 + 4;\quad dz/dt$

5. $w = x^2yz + xy^2z + xyz^2, x = e^t, y = te^t, z = t^2e^t;\quad dw/dt$

6. $w = \ln(x^2+y^2+z^2),\ x = 2 - 3t, y = t^2 + 3, z = 4 - t; dw/dt$

7. $z = (x^2 + xy^2)^3, x = r + s + t,\ y = 2r - 3s + 8t;\quad \partial z/\partial t$

8. $z = \sqrt{x^2+y^2}, x = r^2 + s - t,\ y = r - s + t;\quad \partial z/\partial r$

9. $w = x^2 + xyz + z^2, x = r^2 - s^2,\ y = rs, z = r^2 + s^2;\quad \partial w/\partial s$

10. $w = \ln(xyz), x = r^2s, y = rs, z = rs^2;\quad \partial w/\partial r$

11. $y = x^2 - 7x + 5, x = 19rs + 2s^2t^2;\quad \partial y/\partial r$

12. $y = 4 - x^2, x = 2r + 3s - 4t;\quad \partial y/\partial t$

13. If $z = (4x+3y)^3$, where $x = r^2s$ and $y = r - 2s$, evaluate $\partial z/\partial r$ when $r = 0$ and $s = 1$.

14. If $z = \sqrt{2x+3y}$, where $x = 3t + 5$ and $y = t^2 + 2t + 1$, evaluate dz/dt when $t = 1$.

15. If $w = e^{x+y+z}(x^2+y^2+z^2)$, where $x = (r-s)^2, y = (r+s)^2$, and $z = (s-r)^2$, evaluate $\partial w/\partial s$ when $r = 1$ and $s = 1$.

16. If $y = x/(x-5)$, where $x = 2t^2 - 3rs - r^2t$, evaluate $\partial y/\partial t$ when $r = 0, s = 2$, and $t = -1$.

17. Cost Function Suppose the cost c of producing q_A units of product A and q_B units of product B is given by

$$c = (3q_A^2 + q_B^3 + 4)^{1/3}$$

and the coupled demand functions for the products are given by

$$q_A = 10 - p_A + p_B^2$$

and

$$q_B = 20 + p_A - 11p_B$$

Use a chain rule to evaluate $\dfrac{\partial c}{\partial p_A}$ and $\dfrac{\partial c}{\partial p_B}$ when $p_A = 25$ and $p_B = 4$.

18. Suppose $w = f(x,y)$, where $x = g(t)$ and $y = h(t)$.
(a) State a chain rule that gives dw/dt.
(b) Suppose $h(t) = t$, so that $w = f(x,t)$, where $x = g(t)$. Use part (a) to find dw/dt and simplify your answer.

19. (a) Suppose w is a function of x and y, where both x and y are functions of s and t. State a chain rule that expresses $\partial w/\partial t$ in terms of derivatives of these functions.
(b) Let $w = 2x^2 \ln|3x - 5y|$, where $x = s\sqrt{t^2+2}$ and $y = t - 3e^{2-s}$. Use part (a) to evaluate $\partial w/\partial t$ when $s = 1$ and $t = 0$.

20. Production Function In considering a production function $P = f(l,k)$, where l is labor input and k is capital input, Fon, Boulier, and Goldfarb[19] assume that $l = Lg(h)$, where L is the number of workers, h is the number of hours per day per worker, and $g(h)$ is a labor effectiveness function. In maximizing profit p given by

$$p = aP - whL$$

where a is the price per unit of output and w is the hourly wage per worker, Fon, Boulier, and Goldfarb determine $\partial p/\partial L$ and $\partial p/\partial h$. Assume that k is independent of L and h, and determine these partial derivatives.

Objective

To discuss relative maxima and relative minima, to find critical points, and to apply the second-derivative test for a function of two variables.

17.6 Maxima and Minima for Functions of Two Variables

We now extend the notion of relative maxima and minima (or relative extrema) to functions of two variables.

[19] V. Fon, B. L. Boulier, and R. S. Goldfarb, "The Firm's Demand for Daily Hours of Work: Some Implications," *Atlantic Economic Journal*, XIII, no. 1 (1985), 36–42.

Definition

A function $z = f(x, y)$ is said to have a ***relative maximum*** at the point (a, b) if, for all points (x, y) in the plane that are sufficiently close to (a, b), we have

$$f(a, b) \geq f(x, y) \tag{1}$$

For a ***relative minimum***, we replace $\geq$ by $\leq$ in Equation (1).

To say that $z = f(x, y)$ has a relative maximum at (a, b) means, geometrically, that the point $(a, b, f(a, b))$ on the graph of f is higher than (or is as high as) all other points on the surface that are "near" $(a, b, f(a, b))$. In Figure 17.4(a), f has a relative maximum at (a, b). Similarly, the function f in Figure 17.4(b) has a relative minimum when $x = y = 0$, which corresponds to a *low* point on the surface.

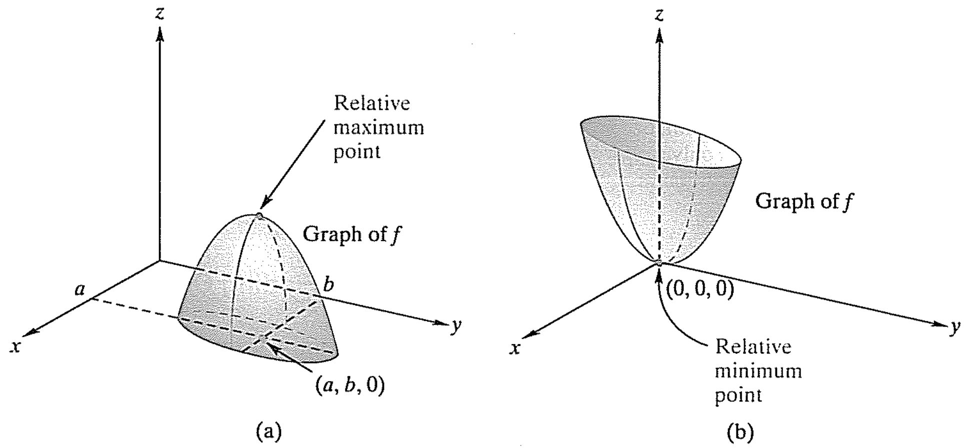

FIGURE 17.4 Relative extrema.

Recall that in locating extrema for a function $y = f(x)$ of one variable, we examine those values of x in the domain of f for which $f'(x) = 0$ or $f'(x)$ does not exist. For functions of two (or more) variables, a similar procedure is followed. However, for the functions that concern us, extrema will not occur where a derivative does not exist, and such situations will be excluded from our consideration.

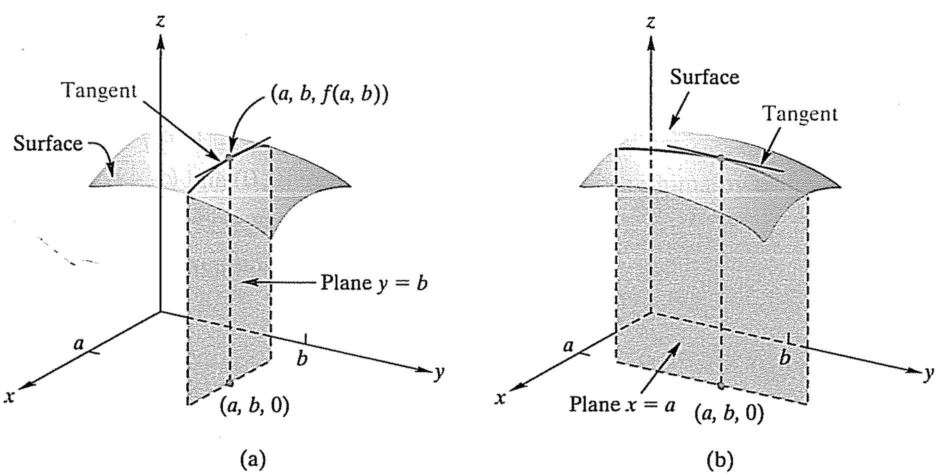

FIGURE 17.5 At relative extremum, $f_x(x, y) = 0$ and $f_y(x, y) = 0$.

Suppose $z = f(x, y)$ has a relative maximum at (a, b), as indicated in Figure 17.5(a). Then the curve where the plane $y = b$ intersects the surface must have a relative maximum when $x = a$. Hence, the slope of the tangent line to the surface in the x-direction must be 0 at (a, b). Equivalently, $f_x(x, y) = 0$ at (a, b). Similarly, on the curve where the plane $x = a$ intersects the surface [Figure 17.5(b)], there must be a relative maximum when $y = b$. Thus, in the y-direction, the slope of the tangent to the surface must be 0 at (a, b). Equivalently, $f_y(x, y) = 0$ at (a, b). Since a similar discussion applies to a relative minimum, we can combine these results as follows:

Rule 1

If $z = f(x, y)$ has a relative maximum or minimum at (a, b), and if both f_x and f_y are defined for all points close to (a, b), it is necessary that (a, b) be a solution of the system

$$\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases}$$

CAUTION!

Rule 1 does not imply that there must be an extremum at a critical point. Just as in the case of functions of one variable, a critical point can give rise to a relative maximum, a relative minimum, or neither. A critical point is only a *candidate* for a relative extremum.

A point (a, b) for which $f_x(a, b) = f_y(a, b) = 0$ is called a **critical point** of f. Thus, from Rule 1, we infer that, to locate relative extrema for a function, we should examine its critical points.

Two additional comments are in order: First, Rule 1, as well as the notion of a critical point, can be extended to functions of more than two variables. For example, to locate possible extrema for $w = f(x, y, z)$, we would examine those points for which $w_x = w_y = w_z = 0$. Second, for a function whose domain is restricted, a thorough examination for absolute extrema would include a consideration of boundary points.

EXAMPLE 1 Finding Critical Points

Find the critical points of the following functions.

a. $f(x, y) = 2x^2 + y^2 - 2xy + 5x - 3y + 1$.

Solution: Since $f_x(x, y) = 4x - 2y + 5$ and $f_y(x, y) = 2y - 2x - 3$, we solve the system

$$\begin{cases} 4x - 2y + 5 = 0 \\ -2x + 2y - 3 = 0 \end{cases}$$

This gives $x = -1$ and $y = \frac{1}{2}$. Thus, $\left(-1, \frac{1}{2}\right)$ is the only critical point.

b. $f(l, k) = l^3 + k^3 - lk$.

Solution:

$$\begin{cases} f_l(l, k) = 3l^2 - k = 0 & (2) \\ f_k(l, k) = 3k^2 - l = 0 & (3) \end{cases}$$

From Equation (2), $k = 3l^2$. Substituting for k in Equation (3) gives

$$0 = 27l^4 - l = l(27l^3 - 1)$$

Hence, either $l = 0$ or $l = \frac{1}{3}$. If $l = 0$, then $k = 0$; if $l = \frac{1}{3}$, then $k = \frac{1}{3}$. The critical points are therefore $(0, 0)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$.

c. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 100)$.

Solution: Solving the system

$$\begin{cases} f_x(x, y, z) = 4x + y - z = 0 \\ f_y(x, y, z) = x + 2y - z = 0 \\ f_z(x, y, z) = -x - y + 100 = 0 \end{cases}$$

gives the critical point $(25, 75, 175)$, as the reader should verify.

Now Work Problem 1 ◁

EXAMPLE 2 Finding Critical Points

Find the critical points of

$$f(x, y) = x^2 - 4x + 2y^2 + 4y + 7$$

Solution: We have $f_x(x, y) = 2x - 4$ and $f_y(x, y) = 4y + 4$. The system

$$\begin{cases} 2x - 4 = 0 \\ 4y + 4 = 0 \end{cases}$$

gives the critical point $(2, -1)$. Observe that we can write the given function as

$$\begin{aligned} f(x, y) &= x^2 - 4x + 4 + 2(y^2 + 2y + 1) + 1 \\ &= (x - 2)^2 + 2(y + 1)^2 + 1 \end{aligned}$$

and $f(2, -1) = 1$. Clearly, if $(x, y) \neq (2, -1)$, then $f(x, y) > 1$. Hence, a relative minimum occurs at $(2, -1)$. Moreover, there is an *absolute minimum* at $(2, -1)$, since $f(x, y) > f(2, -1)$ for *all* $(x, y) \neq (2, -1)$.

Now Work Problem 3 ◁

Although in Example 2 we were able to show that the critical point gave rise to a relative extremum, in many cases this is not so easy to do. There is, however, a second-derivative test that gives conditions under which a critical point will be a relative maximum or minimum. We state it now, omitting the proof.

Rule 2 Second-Derivative Test for Functions of Two Variables

Suppose $z = f(x, y)$ has continuous partial derivatives f_{xx}, f_{yy}, and f_{xy} at all points (x, y) near a critical point (a, b). Let D be the function defined by

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$$

Then

1. if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b);
2. if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b);
3. if $D(a, b) < 0$, then f has a *saddle point* at (a, b) (see Example 4);
4. if $D(a, b) = 0$, then no conclusion about an extremum at (a, b) can be drawn, and further analysis is required.

We remark that when $D(a, b) > 0$, the sign of $f_{xx}(a, b)$ is necessarily the same as the sign of $f_{yy}(a, b)$. Thus, when $D(a, b) > 0$ we can test either $f_{xx}(a, b)$ or $f_{yy}(a, b)$, whichever is easiest, to make the determination required in parts 1 and 2 of the second derivative test.

EXAMPLE 3 Applying the Second-Derivative Test

Examine $f(x, y) = x^3 + y^3 - xy$ for relative maxima or minima by using the second-derivative test.

Solution: First we find critical points:

$$f_x(x, y) = 3x^2 - y \quad f_y(x, y) = 3y^2 - x$$

In the same manner as in Example 1(b), solving $f_x(x, y) = f_y(x, y) = 0$ gives the critical points $(0, 0)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$. Now,

$$f_{xx}(x, y) = 6x \qquad f_{yy}(x, y) = 6y \qquad f_{xy}(x, y) = -1$$

Thus,

$$D(x, y) = (6x)(6y) - (-1)^2 = 36xy - 1$$

Since $D(0, 0) = 36(0)(0) - 1 = -1 < 0$, there is no relative extremum at $(0, 0)$. Also, since $D\left(\frac{1}{3}, \frac{1}{3}\right) = 36\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - 1 = 3 > 0$ and $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = 6\left(\frac{1}{3}\right) = 2 > 0$, there is a relative minimum at $\left(\frac{1}{3}, \frac{1}{3}\right)$. At this point, the value of the function is

$$f\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{27}$$

Now Work Problem 7 ◁

EXAMPLE 4 **A Saddle Point**

The surface in Figure 17.6 is called a hyperbolic paraboloid.

Examine $f(x, y) = y^2 - x^2$ for relative extrema.

Solution: Solving

$$f_x(x, y) = -2x = 0 \quad \text{and} \quad f_y(x, y) = 2y = 0$$

we get the critical point (0, 0). Now we apply the second-derivative test. At (0, 0), and indeed at any point,

$$f_{xx}(x, y) = -2 \quad f_{yy}(x, y) = 2 \quad f_{xy}(x, y) = 0$$

Because $D(0, 0) = (-2)(2) - (0)^2 = -4 < 0$, no relative extremum exists at (0, 0). A sketch of $z = f(x, y) = y^2 - x^2$ appears in Figure 17.6. Note that, for the surface curve cut by the plane $y = 0$, there is a *maximum* at (0, 0); but for the surface curve cut by the plane $x = 0$, there is a *minimum* at (0, 0). Thus, on the *surface,* no relative extremum can exist at the origin, although (0, 0) is a critical point. Around the origin the curve is saddle shaped, and (0, 0) is called a *saddle point* of f.

Now Work Problem 11 ◁

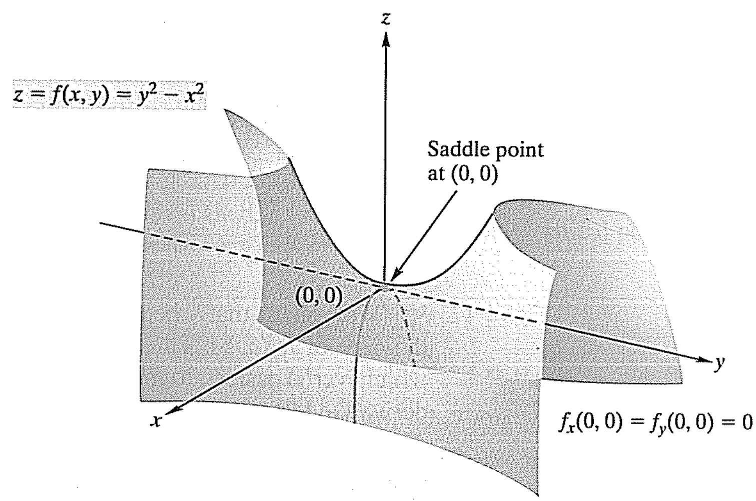

FIGURE 17.6 Saddle point.

EXAMPLE 5 **Finding Relative Extrema**

Examine $f(x, y) = x^4 + (x - y)^4$ for relative extrema.

Solution: If we set

$$f_x(x, y) = 4x^3 + 4(x - y)^3 = 0 \tag{4}$$

and

$$f_y(x, y) = -4(x - y)^3 = 0 \tag{5}$$

then, from Equation (5), we have $x - y = 0$, or $x = y$. Substituting into Equation (4) gives $4x^3 = 0$, or $x = 0$. Thus, $x = y = 0$, and (0, 0) is the only critical point. At (0, 0),

$$f_{xx}(x, y) = 12x^2 + 12(x - y)^2 = 0$$
$$f_{yy}(x, y) = 12(x - y)^2 = 0$$

and

$$f_{xy}(x, y) = -12(x - y)^2 = 0$$

Hence, $D(0, 0) = 0$, and the second-derivative test gives no information. However, for all $(x, y) \neq (0, 0)$, we have $f(x, y) > 0$, whereas $f(0, 0) = 0$. Therefore, at (0, 0) the graph of f has a low point, and we conclude that f has a relative (and absolute) minimum at (0, 0).

Now Work Problem 13 ◁

Applications

In many situations involving functions of two variables, and especially in their applications, the nature of the given problem is an indicator of whether a critical point is in fact a relative (or absolute) maximum or a relative (or absolute) minimum. In such cases, the second-derivative test is not needed. Often, in mathematical studies of applied problems, the appropriate second-order conditions are assumed to hold.

EXAMPLE 6 Maximizing Output

Let P be a production function given by

$$P = f(l, k) = 0.54l^2 - 0.02l^3 + 1.89k^2 - 0.09k^3$$

where l and k are the amounts of labor and capital, respectively, and P is the quantity of output produced. Find the values of l and k that maximize P.

Solution: To find the critical points, we solve the system $P_l = 0$ and $P_k = 0$:

$$\begin{aligned} P_l &= 1.08l - 0.06l^2 & P_k &= 3.78k - 0.27k^2 \\ &= 0.06l(18 - l) = 0 & &= 0.27k(14 - k) = 0 \\ l &= 0, l = 18 & k &= 0, k = 14 \end{aligned}$$

There are four critical points: (0, 0), (0, 14), (18, 0), and (18, 14).

Now we apply the second-derivative test to each critical point. We have

$$P_{ll} = 1.08 - 0.12l \quad P_{kk} = 3.78 - 0.54k \quad P_{lk} = 0$$

Thus,

$$\begin{aligned} D(l, k) &= P_{ll}P_{kk} - [P_{lk}]^2 \\ &= (1.08 - 0.12l)(3.78 - 0.54k) \end{aligned}$$

At (0, 0),

$$D(0, 0) = 1.08(3.78) > 0$$

Since $D(0, 0) > 0$ and $P_{ll} = 1.08 > 0$, there is a relative minimum at (0, 0). At (0, 14),

$$D(0, 14) = 1.08(-3.78) < 0$$

Because $D(0, 14) < 0$, there is no relative extremum at (0, 14). At (18, 0),

$$D(18, 0) = (-1.08)(3.78) < 0$$

Since $D(18, 0) < 0$, there is no relative extremum at (18, 0). At (18, 14),

$$D(18, 14) = (-1.08)(-3.78) > 0$$

Because $D(18, 14) > 0$ and $P_{ll} = -1.08 < 0$, there is a relative maximum at (18, 14). Hence, the maximum output is obtained when $l = 18$ and $k = 14$.

Now Work Problem 21 ◁

EXAMPLE 7 **Profit Maximization**

A candy company produces two types of candy, A and B, for which the average costs of production are constant at \$2 and \$3 per pound, respectively. The quantities q_A, q_B (in pounds) of A and B that can be sold each week are given by the joint-demand functions

$$q_A = 400(p_B - p_A)$$

and

$$q_B = 400(9 + p_A - 2p_B)$$

where p_A and p_B are the selling prices (in dollars per pound) of A and B, respectively. Determine the selling prices that will maximize the company's profit P.

Solution: The total profit is given by

$$P = \begin{pmatrix}\text{profit}\\ \text{per pound}\\ \text{of A}\end{pmatrix}\begin{pmatrix}\text{pounds}\\ \text{of A}\\ \text{sold}\end{pmatrix} + \begin{pmatrix}\text{profit}\\ \text{per pound}\\ \text{of B}\end{pmatrix}\begin{pmatrix}\text{pounds}\\ \text{of B}\\ \text{sold}\end{pmatrix}$$

For A and B, the profits per pound are $p_A - 2$ and $p_B - 3$, respectively. Thus,

$$\begin{aligned} P &= (p_A - 2)q_A + (p_B - 3)q_B \\ &= (p_A - 2)[400(p_B - p_A)] + (p_B - 3)[400(9 + p_A - 2p_B)] \end{aligned}$$

Notice that P is expressed as a function of two variables, p_A and p_B. To maximize P, we set its partial derivatives equal to 0:

$$\begin{aligned} \frac{\partial P}{\partial p_A} &= (p_A - 2)[400(-1)] + [400(p_B - p_A)](1) + (p_B - 3)[400(1)] \\ &= 0 \\ \frac{\partial P}{\partial p_B} &= (p_A - 2)[400(1)] + (p_B - 3)[400(-2)] + 400(9 + p_A - 2p_B)](1) \\ &= 0 \end{aligned}$$

Simplifying the preceding two equations gives

$$\begin{cases} -2p_A + 2p_B - 1 = 0 \\ 2p_A - 4p_B + 13 = 0 \end{cases}$$

whose solution is $p_A = 5.5$ and $p_B = 6$. Moreover, we find that

$$\frac{\partial^2 P}{\partial p_A^2} = -800 \qquad \frac{\partial^2 P}{\partial p_B^2} = -1600 \qquad \frac{\partial^2 P}{\partial p_B \partial p_A} = 800$$

Therefore,

$$D(5.5, 6) = (-800)(-1600) - (800)^2 > 0$$

Since $\partial^2 P/\partial p_A^2 < 0$, we indeed have a maximum, and the company should sell candy A at \$5.50 per pound and B at \$6.00 per pound.

Now Work Problem 23 ◁

EXAMPLE 8 **Profit Maximization for a Monopolist**[20]

Suppose a monopolist is practicing price discrimination by selling the same product in two separate markets at different prices. Let q_A be the number of units sold in market

[20]Omit if Section 17.5 was not covered.

A, where the demand function is $p_A = f(q_A)$, and let q_B be the number of units sold in market B, where the demand function is $p_B = g(q_B)$. Then the revenue functions for the two markets are

$$r_A = q_A f(q_A) \quad \text{and} \quad r_B = q_B g(q_B)$$

Assume that all units are produced at one plant, and let the cost function for producing $q = q_A + q_B$ units be $c = c(q)$. Keep in mind that r_A is a function of q_A and r_B is a function of q_B. The monopolist's profit P is

$$P = r_A + r_B - c$$

To maximize P with respect to outputs q_A and q_B, we set its partial derivatives equal to zero. To begin with,

$$\begin{aligned}\frac{\partial P}{\partial q_A} &= \frac{dr_A}{dq_A} + 0 - \frac{\partial c}{\partial q_A} \\ &= \frac{dr_A}{dq_A} - \frac{dc}{dq}\frac{\partial q}{\partial q_A} = 0 \qquad \text{chain rule}\end{aligned}$$

Because

$$\frac{\partial q}{\partial q_A} = \frac{\partial}{\partial q_A}(q_A + q_B) = 1$$

we have

$$\frac{\partial P}{\partial q_A} = \frac{dr_A}{dq_A} - \frac{dc}{dq} = 0 \tag{6}$$

Similarly,

$$\frac{\partial P}{\partial q_B} = \frac{dr_B}{dq_B} - \frac{dc}{dq} = 0 \tag{7}$$

From Equations (6) and (7), we get

$$\frac{dr_A}{dq_A} = \frac{dc}{dq} = \frac{dr_B}{dq_B}$$

But dr_A/dq_A and dr_B/dq_B are marginal revenues, and dc/dq is marginal cost. Hence, to maximize profit, it is necessary to charge prices (and distribute output) so that the marginal revenues in both markets will be the same and, loosely speaking, will also be equal to the cost of the last unit produced in the plant.

Now Work Problem 25 ◁

PROBLEMS 17.6

In Problems 1–6, find the critical points of the functions.

1. $f(x, y) = x^2 - 3y^2 - 8x + 9y + 3xy$

2. $f(x, y) = x^2 + 4y^2 - 6x + 16y$

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

4. $f(x, y) = xy - x + y$

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

In Problems 7–20, find the critical points of the functions. For each critical point, determine, by the second-derivative test, whether it corresponds to a relative maximum, to a relative minimum, or to neither, or whether the test gives no information.

7. $f(x, y) = x^2 + 3y^2 + 4x - 9y + 3$

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

9. $f(x, y) = y - y^2 - 3x - 6x^2$

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

11. $f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3$

12. $f(x, y) = \frac{x^3}{3} + y^2 - 2x + 2y - 2xy$

13. $f(x, y) = \frac{1}{3}(x^3 + 8y^3) - 2(x^2 + y^2) + 1$

14. $f(x, y) = x^2 + y^2 - xy + x^3$

15. $f(l, k) = \frac{l^2}{2} + 2lk + 3k^2 - 69l - 164k + 17$

16. $f(l, k) = l^2 + 4k^2 - 4lk$

17. $f(p, q) = pq - \frac{1}{p} - \frac{1}{q}$

18. $f(x, y) = (x - 3)(y - 3)(x + y - 3)$

19. $f(x, y) = (y^2 - 4)(e^x - 1)$

20. $f(x, y) = \ln(xy) + 2x^2 - xy - 6x$

21. Maximizing Output Suppose

$$P = f(l, k) = 2.18l^2 - 0.02l^3 + 1.97k^2 - 0.03k^3$$

is a production function for a firm. Find the quantities of inputs l and k that maximize output P.

22. Maximizing Output In a certain office, computers C and D are utilized for c and d hours, respectively. If daily output Q is a function of c and d, namely,

$$Q = 18c + 20d - 2c^2 - 4d^2 - cd$$

find the values of c and d that maximize Q.

In Problems 23–35, unless otherwise indicated, the variables p_A and p_B denote selling prices of products A and B, respectively. Similarly, q_A and q_B denote quantities of A and B that are produced and sold during some time period. In all cases, the variables employed will be assumed to be units of output, input, money, and so on.

23. Profit A candy company produces two varieties of candy, A and B, for which the constant average costs of production are 60 and 70 (cents per lb), respectively. The demand functions for A and B are given by

$$q_A = 5(p_B - p_A) \quad \text{and} \quad q_B = 500 + 5(p_A - 2p_B)$$

Find the selling prices p_A and p_B that maximize the company's profit.

24. Profit Repeat Problem 23 if the constant costs of production of A and B are a and b (cents per lb), respectively.

25. Price Discrimination Suppose a monopolist is practicing price discrimination in the sale of a product by charging different prices in two separate markets. In market A the demand function is

$$p_A = 100 - q_A$$

and in B it is

$$p_B = 84 - q_B$$

where q_A and q_B are the quantities sold per week in A and B, and p_A and p_B are the respective prices per unit. If the monopolist's cost function is

$$c = 600 + 4(q_A + q_B)$$

how much should be sold in each market to maximize profit? What selling prices give this maximum profit? Find the maximum profit.

26. Profit A monopolist sells two competitive products, A and B, for which the demand functions are

$$q_A = 16 - p_A + p_B \quad \text{and} \quad q_B = 24 + 2p_A - 4p_B$$

If the constant average cost of producing a unit of A is 2 and a unit of B is 4, how many units of A and B should be sold to maximize the monopolist's profit?

27. Profit For products A and B, the joint-cost function for a manufacturer is

$$c = \frac{3}{2}q_A^2 + 3q_B^2$$

and the demand functions are $p_A = 60 - q_A^2$ and $p_B = 72 - 2q_B^2$. Find the level of production that maximizes profit.

28. Profit For a monopolist's products A and B, the joint-cost function is $c = 2(q_A + q_B + q_A q_B)$, and the demand functions are $q_A = 20 - 2p_A$ and $q_B = 10 - p_B$. Find the values of p_A and p_B that maximize profit. What are the quantities of A and B that correspond to these prices? What is the total profit?

29. Cost An open-top rectangular box is to have a volume of 6 ft^3. The cost per square foot of materials is \$3 for the bottom, \$1 for the front and back, and \$0.50 for the other two sides. Find the dimensions of the box so that the cost of materials is minimized. (See Figure 17.7.)

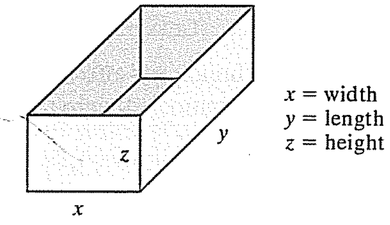

FIGURE 17.7

30. Collusion Suppose A and B are the only two firms in the market selling the same product. (We say that they are *duopolists*.) The industry demand function for the product is

$$p = 92 - q_A - q_B$$

where q_A and q_B denote the output produced and sold by A and B, respectively. For A, the cost function is $c_A = 10q_A$; for B, it is $c_B = 0.5q_B^2$. Suppose the firms decide to enter into an agreement on output and price control by jointly acting as a monopoly. In this case, we say they enter into *collusion*. Show that the profit function for the monopoly is given by

$$P = pq_A - c_A + pq_B - c_B$$

Express P as a function of q_A and q_B, and determine how output should be allocated so as to maximize the profit of the monopoly.

31. Suppose $f(x, y) = x^2 + 3y^2 + 9$, where x and y must satisfy the equation $x + y = 2$. Find the relative extrema of f, subject to the given condition on x and y, by first solving the second equation for y (or x). Substitute the result in the first equation. Thus, f is expressed as a function of one variable. Now find where relative extrema for f occur.

32. Repeat Problem 31 if $f(x, y) = x^2 + 4y^2 + 6$, subject to the condition that $2x - 8y = 20$.

33. Suppose the joint-cost function

$$c = q_A^2 + 3q_B^2 + 2q_A q_B + aq_A + bq_B + d$$

has a relative minimum value of 15 when $q_A = 3$ and $q_B = 1$. Determine the values of the constants a, b, and d.

34. Suppose that the function $f(x, y)$ has continuous partial derivatives f_{xx}, f_{yy}, and f_{xy} at all points (x, y) near a critical point (a, b). Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$ and suppose that $D(a, b) > 0$.

(a) Show that $f_{xx}(a, b) < 0$ if and only if $f_{yy}(a, b) < 0$.
(b) Show that $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

35. Profit from Competitive Products A monopolist sells two competitive products, A and B, for which the demand equations are

$$p_A = 35 - 2q_A^2 + q_B$$