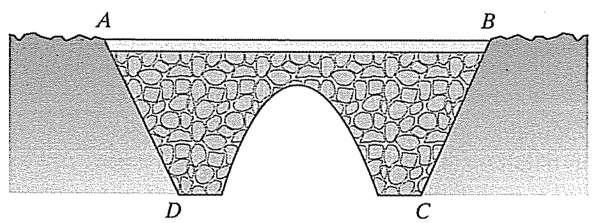
14 Integration

Anyone who runs a business knows the need for accurate cost estimates. When jobs are individually contracted, determining how much a job will cost is generally the first step in deciding how much to bid.

For example, a painter must determine how much paint a job will take. Since a gallon of paint will cover a certain number of square feet, the key is to determine the area of the surfaces to be painted. Normally, even this requires only simple arithmetic—walls and ceilings are rectangular, and so total area is a sum of products of base and height.

But not all area calculations are as simple. Suppose, for instance, that the bridge shown below must be sandblasted to remove accumulated soot. How would the contractor who charges for sandblasting by the square foot calculate the area of the vertical face on each side of the bridge?



The area could be estimated as perhaps three-quarters of the area of the trapezoid formed by points A, B, C, and D. But a more accurate calculation—which might be desirable if the bid were for dozens of bridges of the same dimensions (as along a stretch of railroad)—would require a more refined approach.

If the shape of the bridge's arch can be described mathematically by a function, the contractor could use the method introduced in this chapter: integration. Integration has many applications, the simplest of which is finding areas of regions bounded by curves. Other applications include calculating the total deflection of a beam due to bending stress, calculating the distance traveled underwater by a submarine, and calculating the electricity bill for a company that consumes power at differing rates over the course of a month. Chapters 11–13 dealt with differential calculus. We differentiated a function and obtained another function, its derivative. *Integral calculus* is concerned with the reverse process: We are given the derivative of a function and must find the original function. The need for doing this arises in a natural way. For example, we might have a marginal-revenue function and want to find the revenue function from it. Integral calculus also involves a concept that allows us to take the limit of a special kind of sum as the number of terms in the sum becomes infinite. This is the real power of integral calculus! With such a notion, we can find the area of a region that cannot be found by any other convenient method.

Objective

To define the differential, interpret it geometrically, and use it in approximations. Also, to restate the reciprocal relationship between dx/dy and dy/dx.

14.1 Differentials

We will soon give a reason for using the symbol dy/dx to denote the derivative of y with respect to x. To do this, we introduce the notion of the *differential* of a function.

> **Definition**
>
> Let $y = f(x)$ be a differentiable function of x, and let Δx denote a change in x, where Δx can be any real number. Then the ***differential*** of y, denoted dy or $d(f(x))$, is given by
>
> $$dy = f'(x)\,\Delta x$$

TO REVIEW functions of several variables, see Section 2.8.

Note that dy depends on two variables, namely, x and Δx. In fact, dy is a function of two variables.

EXAMPLE 1 Computing a Differential

Find the differential of $y = x^3-2x^2+3x-4$, and evaluate it when $x = 1$ and $\Delta x = 0.04$.

Solution: The differential is

$$\begin{aligned} dy &= \frac{d}{dx}(x^3 - 2x^2 + 3x - 4)\,\Delta x \\ &= (3x^2 - 4x + 3)\,\Delta x \end{aligned}$$

When $x = 1$ and $\Delta x = 0.04$,

$$dy = [3(1)^2 - 4(1) + 3](0.04) = 0.08$$

Now Work Problem 1 ◁

If $y = x$, then $dy = d(x) = 1\,\Delta x = \Delta x$. Hence, the differential of x is Δx. We abbreviate $d(x)$ by dx. Thus, $dx = \Delta x$. From now on, it will be our practice to write dx for Δx when finding a differential. For example,

$$d(x^2 + 5) = \frac{d}{dx}(x^2 + 5)\,dx = 2x\,dx$$

Summarizing, we say that if $y = f(x)$ defines a differentiable function of x, then

$$dy = f'(x)\,dx$$

where dx is any real number. Provided that $dx \neq 0$, we can divide both sides by dx:

$$\frac{dy}{dx} = f'(x)$$

That is, dy/dx can be viewed either as the quotient of two differentials, namely, dy divided by dx, or as one symbol for the derivative of f at x. It is for this reason that we introduced the symbol dy/dx to denote the derivative.

EXAMPLE 2 Finding a Differential in Terms of dx

a. If $f(x) = \sqrt{x}$, then

$$d(\sqrt{x}) = \frac{d}{dx}(\sqrt{x})\,dx = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$$

b. If $u = (x^2 + 3)^5$, then $du = 5(x^2 + 3)^4(2x)\,dx = 10x(x^2 + 3)^4\,dx$.

Now Work Problem 3 ◁

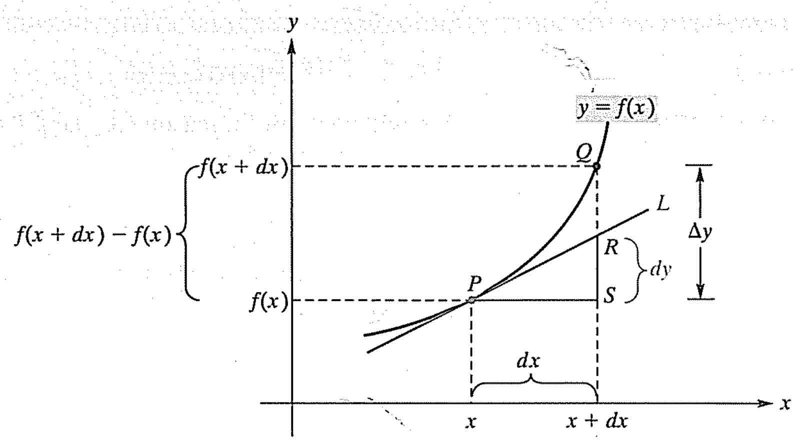


FIGURE 14.1 Geometric interpretation of dy and Δx.

The differential can be interpreted geometrically. In Figure 14.1, the point $P(x,f(x))$ is on the curve $y = f(x)$. Suppose x changes by dx, a real number, to the new value $x + dx$. Then the new function value is $f(x + dx)$, and the corresponding point on the curve is $Q(x + dx, f(x + dx))$. Passing through P and Q are horizontal and vertical lines, respectively, that intersect at S. A line L tangent to the curve at P intersects segment QS at R, forming the right triangle PRS. Observe that the graph of f near P is approximated by the tangent line at P. The slope of L is $f'(x)$ but it is also given by $\overline{SR}/\overline{PS}$ so that

$$f'(x) = \frac{\overline{SR}}{\overline{PS}}$$

Since $dy = f'(x)\,dx$ and $dx = \overline{PS}$,

$$dy = f'(x)\,dx = \frac{\overline{SR}}{\overline{PS}} \cdot \overline{PS} = \overline{SR}$$

Thus, if dx is a change in x at P, then dy is the corresponding vertical change along the **tangent line** at P. Note that for the same dx, the vertical change along the **curve** is $\Delta y = \overline{SQ} = f(x+dx) - f(x)$. Do not confuse Δy with dy. However, from Figure 14.1, the following is apparent:

When dx is close to 0, dy is an approximation to Δy. Therefore,

$$\Delta y \approx dy$$

This fact is useful in estimating Δy, a change in y, as Example 3 shows.

EXAMPLE 3 Using the Differential to Estimate a Change in a Quantity

A governmental health agency examined the records of a group of individuals who were hospitalized with a particular illness. It was found that the total proportion P that are discharged at the end of t days of hospitalization is given by

$$P = P(t) = 1 - \left(\frac{300}{300 + t}\right)^3$$

Use differentials to approximate the change in the proportion discharged if t changes from 300 to 305.

Solution: The change in t from 300 to 305 is $\Delta t = dt = 305 - 300 = 5$. The change in P is $\Delta P = P(305) - P(300)$. We approximate ΔP by dP:

$$\Delta P \approx dP = P'(t)\,dt = -3\left(\frac{300}{300+t}\right)^2\left(-\frac{300}{(300+t)^2}\right)dt = 3\frac{300^3}{(300+t)^4}\,dt$$

When $t = 300$ and $dt = 5$,

$$dP = 3\frac{300^3}{600^4}5 = \frac{15}{2^3600} = \frac{1}{2^340} = \frac{1}{320} \approx 0.0031$$

For a comparison, the true value of ΔP is

$$P(305) - P(300) = 0.87807 - 0.87500 = 0.00307$$

(to five decimal places).

Now Work Problem 11 ◁

We said that if $y = f(x)$, then $\Delta y \approx dy$ if dx is close to zero. Thus,

$$\Delta y = f(x + dx) - f(x) \approx dy$$

Formula (1) is used to approximate a function value, whereas the formula $\Delta y \approx dy$ is used to approximate a change in function values.

so that

$$f(x + dx) \approx f(x) + dy \qquad (1)$$

This formula gives us a way of estimating a function value $f(x + dx)$. For example, suppose we estimate $\ln(1.06)$. Letting $y = f(x) = \ln x$, we need to estimate $f(1.06)$. Since $d(\ln x) = (1/x)\,dx$, we have, from Formula (1),

$$f(x + dx) \approx f(x) + dy$$

$$\ln(x + dx) \approx \ln x + \frac{1}{x}\,dx$$

We know the exact value of ln 1, so we will let $x = 1$ and $dx = 0.06$. Then $x + dx = 1.06$, and dx is close to zero. Therefore,

$$\ln(1 + 0.06) \approx \ln(1) + \frac{1}{1}(0.06)$$

$$\ln(1.06) \approx 0 + 0.06 = 0.06$$

The true value of $\ln(1.06)$ to five decimal places is 0.05827.

EXAMPLE 4 Using the Differential to Estimate a Function Value

The demand function for a product is given by

$$p = f(q) = 20 - \sqrt{q}$$

where p is the price per unit in dollars for q units. By using differentials, approximate the price when 99 units are demanded.

Solution: We want to approximate $f(99)$. By Formula (1),

$$f(q + dq) \approx f(q) + dp$$

where

$$dp = -\frac{1}{2\sqrt{q}}\,dq \qquad \frac{dp}{dq} = -\frac{1}{2}q^{-1/2}$$

We choose $q = 100$ and $dq = -1$ because $q + dq = 99$, dq is small, and it is easy to compute $f(100) = 20 - \sqrt{100} = 10$. We thus have

$$f(99) = f[100 + (-1)] \approx f(100) - \frac{1}{2\sqrt{100}}(-1)$$

$$f(99) \approx 10 + 0.05 = 10.05$$

Hence, the price per unit when 99 units are demanded is approximately \$10.05.

Now Work Problem 17 ◁

The equation $y = x^3 + 4x + 5$ defines y as a function of x. We could write $f(x) = x^3 + 4x + 5$. However, the equation also defines x implicitly as a function of y. In fact, if we restrict the domain of f to some set of real numbers x so that $y = f(x)$ is a one-to-one function, then in principle we could solve for x in terms of y and get $x = f^{-1}(y)$. [Actually, no restriction of the domain is necessary here. Since $f'(x) = 3x^2 + 4 > 0$, for all x, we see that f is strictly increasing on $(-\infty, \infty)$ and is thus one-to-one on $(-\infty, \infty)$.] As we did in Section 12.2, we can look at the derivative of x with respect to y, dx/dy and we have seen that it is given by

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \qquad \text{provided that } dy/dx \neq 0$$

Since dx/dy can be considered a quotient of differentials, we now see that it is the reciprocal of the quotient of differentials dy/dx. Thus

$$\frac{dx}{dy} = \frac{1}{3x^2 + 4}$$

It is important to understand that it is not necessary to be able to solve $y = x^3 + 4x + 5$ for x in terms of y, and the equation $\frac{dx}{dy} = \frac{1}{3x^2 + 4}$ holds for all x.

EXAMPLE 5 **Finding dp/dq from dq/dp**

Find $\frac{dp}{dq}$ if $q = \sqrt{2500 - p^2}$.

Solution:

Strategy There are a number of ways to find dp/dq. One approach is to solve the given equation for p explicitly in terms of q and then differentiate directly. Another approach to find dp/dq is to use implicit differentiation. However, since q is given explicitly as a function of p, we can easily find dq/dp and then use the preceding reciprocal relation to find dp/dq. We will take this approach.

We have

$$\frac{dq}{dp} = \frac{1}{2}(2500 - p^2)^{-1/2}(-2p) = -\frac{p}{\sqrt{2500 - p^2}}$$

Hence,

$$\frac{dp}{dq} = \frac{1}{\frac{dq}{dp}} = -\frac{\sqrt{2500 - p^2}}{p}$$

Now Work Problem 27 ◁

PROBLEMS 14.1

In Problems 1–10, find the differential of the function in terms of x and dx.

1. $y = ax + b$

2. $y = 2$

3. $f(x) = \sqrt{x^4 - 9}$

4. $f(x) = (4x^2 - 5x + 2)^3$

5. $u = \frac{1}{x^2}$

6. $u = \sqrt{x}$

7. $p = \ln(x^2 + 7)$

8. $p = e^{x^3 + 2x - 5}$

9. $y = (9x + 3)e^{2x^2 + 3}$

10. $y = \ln\sqrt{x^2 + 12}$

In Problems 11–16, find Δy and dy for the given values of x and dx.

11. $y = ax + b$; for any x and any dx

12. $y = 5x^2$; $x = -1$, $dx = -0.02$

13. $y = 2x^2 + 5x - 7$; $x = -2$, $dx = 0.1$

14. $y = (3x + 2)^2$; $x = -1$, $dx = -0.03$

15. $y = \sqrt{32 - x^2}$; $x = 4$, $dx = -0.05$ Round your answer to three decimal places.

16. $y = \ln x$; $x = 1$, $dx = 0.01$

17. Let $f(x) = \frac{x + 5}{x + 1}$.

(a) Evaluate $f'(1)$.

(b) Use differentials to estimate the value of $f(1.1)$.

18. Let $f(x) = x^{3x}$.

(a) Evaluate $f'(1)$.

(b) Use differentials to estimate the value of $f(0.98)$.

In Problems 19–26, approximate each expression by using differentials.

19. $\sqrt{288}$ (*Hint:* $17^2 = 289$.)

20. $\sqrt{122}$

21. $\sqrt[3]{9}$

22. $\sqrt[4]{16.3}$

23. $\ln 0.97$

24. $\ln 1.01$

25. $e^{0.001}$

26. $e^{-0.002}$

In Problems 27–32, find dx/dy or dp/dq.

27. $y = 2x - 1$

28. $y = 5x^2 + 3x + 2$

29. $q = (p^2 + 5)^3$

30. $q = \sqrt{p + 5}$

31. $q = \dfrac{1}{p^2}$

32. $q = e^{4-2p}$

33. If $y = 7x^2 - 6x + 3$, find the value of dx/dy when $x = 3$.

34. If $y = \ln x^2$, find the value of dx/dy when $x = 3$.

In Problems 35 and 36, find the rate of change of q with respect to p for the indicated value of q.

35. $p = \dfrac{500}{q + 2}; q = 18$

36. $p = 60 - \sqrt{2q}; q = 50$

37. Profit Suppose that the profit (in dollars) of producing q units of a product is

$$P = 397q - 2.3q^2 - 400$$

Using differentials, find the approximate change in profit if the level of production changes from $q = 90$ to $q = 91$. Find the true change.

38. Revenue Given the revenue function

$$r = 250q + 45q^2 - q^3$$

use differentials to find the approximate change in revenue if the number of units increases from $q = 40$ to $q = 41$. Find the true change.

39. Demand The demand equation for a product is

$$p = \frac{10}{\sqrt{q}}$$

Using differentials, approximate the price when 24 units are demanded.

40. Demand Given the demand function

$$p = \frac{200}{\sqrt{q + 8}}$$

use differentials to estimate the price per unit when 40 units are demanded.

41. If $y = f(x)$, then the *proportional change in y* is defined to be $\Delta y/y$, which can be approximated with differentials by dy/y. Use this last form to approximate the proportional change in the cost function

$$c = f(q) = \frac{q^2}{2} + 5q + 300$$

when $q = 10$ and $dq = 2$. Round your answer to one decimal place.

42. Status/Income Suppose that S is a numerical value of status based on a person's annual income I (in thousands of dollars). For a certain population, suppose $S = 20\sqrt{I}$. Use differentials to approximate the change in S if annual income decreases from \$45,000 to \$44,500.

43. Biology The volume of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Estimate the change in volume when the radius changes from 6.5×10^{-4} cm to 6.6×10^{-4} cm.

44. Muscle Contraction The equation

$$(P + a)(v + b) = k$$

is called the "fundamental equation of muscle contraction."[1] Here P is the load imposed on the muscle, v is the velocity of the shortening of the muscle fibers, and a, b, and k are positive constants. Find P in terms of v, and then use the differential to approximate the change in P due to a small change in v.

45. Demand The demand, q, for a monopolist's product is related to the price per unit, p, according to the equation

$$2 + \frac{q^2}{200} = \frac{4000}{p^2}$$

(a) Verify that 40 units will be demanded when the price per unit is \$20.
(b) Show that $\dfrac{dq}{dp} = -2.5$ when the price per unit is \$20.
(c) Use differentials and the results of parts (a) and (b) to approximate the number of units that will be demanded if the price per unit is reduced to \$19.20.

46. Profit The demand equation for a monopolist's product is

$$p = \frac{1}{2}q^2 - 66q + 7000$$

and the average-cost function is

$$\bar{c} = 500 - q + \frac{80{,}000}{2q}$$

(a) Find the profit when 100 units are demanded.
(b) Use differentials and the result of part (a) to estimate the profit when 101 units are demanded.

Objective

To define the antiderivative and the indefinite integral and to apply basic integration formulas.

14.2 The Indefinite Integral

Given a function f, if F is a function such that

$$F'(x) = f(x) \tag{1}$$

then F is called an *antiderivative* of f. Thus,

> An antiderivative of f is simply a function whose derivative is f.

[1] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

Multiplying both sides of Equation (1) by the differential dx gives $F'(x)\,dx = f(x)\,dx$. However, because $F'(x)\,dx$ is the differential of F, we have $dF = f(x)\,dx$. Hence, we can think of an antiderivative of f as a function whose differential is $f(x)\,dx$.

Definition

An ***antiderivative*** of a function f is a function F such that

$$F'(x) = f(x)$$

Equivalently, in differential notation,

$$dF = f(x)\,dx$$

For example, because the derivative of x^2 is $2x$, x^2 is an antiderivative of $2x$. However, it is not the only antiderivative of $2x$: Since

$$\frac{d}{dx}(x^2+1) = 2x \qquad \text{and} \qquad \frac{d}{dx}(x^2-5) = 2x$$

both x^2+1 and x^2-5 are also antiderivatives of $2x$. In fact, it is obvious that because the derivative of a constant is zero, x^2+C is also an antiderivative of $2x$ for *any* constant C. Thus, $2x$ has infinitely many antiderivatives. More importantly, *all* antiderivatives of $2x$ must be functions of the form x^2+C, because of the following fact:

Any two antiderivatives of a function differ only by a constant.

Since x^2+C describes all antiderivatives of $2x$, we can refer to it as being the *most general antiderivative* of $2x$, denoted by $\int 2x\,dx$, which is read "the *indefinite integral* of $2x$ with respect to x." Thus, we write

$$\int 2x\,dx = x^2 + C$$

The symbol $\int$ is called the **integral sign,** $2x$ is the **integrand,** and C is the **constant of integration.** The dx is part of the integral notation and indicates the variable involved. Here x is the **variable of integration.**

More generally, the **indefinite integral** of any function f with respect to x is written $\int f(x)\,dx$ and denotes the most general antiderivative of f. Since all antiderivatives of f differ only by a constant, if F is any antiderivative of f, then

$$\int f(x)\,dx = F(x) + C, \quad \text{where } C \text{ is a constant}$$

To *integrate* f means to find $\int f(x)\,dx$. In summary,

$$\int f(x)\,dx = F(x) + C \qquad \text{if and only if} \qquad F'(x) = f(x)$$

Thus we have

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x) \quad \text{and} \quad \int \frac{d}{dx}(F(x))\,dx = F(x) + C$$

which shows the extent to which differentiation and indefinite integration are inverse procedures.

APPLY IT ▷

1. If the marginal cost for a company is $f(q) = 28.3$, find $\int 28.3\,dq$, which gives the form of the cost function.

EXAMPLE 1 **Finding an Indefinite Integral**

Find $\int 5\,dx$.

Solution:

Strategy First we must find (perhaps better words are *guess at*) a function whose derivative is 5. Then we add the constant of integration.

CAUTION!

A common mistake is to omit C, the constant of integration.

Since we know that the derivative of $5x$ is 5, $5x$ is an antiderivative of 5. Therefore,

$$\int 5\,dx = 5x + C$$

Now Work Problem 1 ◁

Table 14.1 Elementary Integration Formulas

1. $\int k\,dx = kx + C$	k is a constant
2. $\int x^a\,dx = \dfrac{x^{a+1}}{a+1} + C$	$a \neq -1$
3. $\int x^{-1}\,dx = \int \dfrac{1}{x}\,dx = \int \dfrac{dx}{x} = \ln x + C$	for $x > 0$
4. $\int e^x\,dx = e^x + C$	
5. $\int kf(x)\,dx = k\int f(x)\,dx$	k is a constant
6. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	

Using differentiation formulas from Chapters 11 and 12, we have compiled a list of elementary integration formulas in Table 14.1. These formulas are easily verified. For example, Formula (2) is true because the derivative of $x^{a+1}/(a+1)$ is x^a for $a \neq -1$. (We must have $a \neq -1$ because the denominator is 0 when $a = -1$.) Formula (2) states that the indefinite integral of a power of x, other than x^{-1}, is obtained by increasing the exponent of x by 1, dividing by the new exponent, and adding a constant of integration. The indefinite integral of x^{-1} will be discussed in Section 14.4.

To verify Formula (5), we must show that the derivative of $k\int f(x)\,dx$ is $kf(x)$. Since the derivative of $k\int f(x)\,dx$ is simply k times the derivative of $\int f(x)\,dx$, and the derivative of $\int f(x)\,dx$ is $f(x)$, Formula (5) is verified. The reader should verify the other formulas. Formula (6) can be extended to any number of terms.

EXAMPLE 2 **Indefinite Integrals of a Constant and of a Power of x**

a. Find $\int 1\,dx$.

Solution: By Formula (1) with $k = 1$

$$\int 1\,dx = 1x + C = x + C$$

Usually, we write $\int 1\,dx$ as $\int dx$. Thus, $\int dx = x + C$.

b. Find $\int x^5\,dx$.

Solution: By Formula (2) with $n = 5$,

$$\int x^5\,dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$$

Now Work Problem 3 ◁

APPLY IT ▷

2. If the rate of change of a company's revenues can be modeled by $\frac{dR}{dt} = 0.12t^2$, then find $\int 0.12t^2\,dt$, which gives the form of the company's revenue function.

EXAMPLE 3 Indefinite Integral of a Constant Times a Function

Find $\int 7x\,dx$.

Solution: By Formula (5) with $k = 7$ and $f(x) = x$,

$$\int 7x\,dx = 7\int x\,dx$$

Since x is x^1, by Formula (2) we have

$$\int x^1\,dx = \frac{x^{1+1}}{1+1} + C_1 = \frac{x^2}{2} + C_1$$

where C_1 is the constant of integration. Therefore,

$$\int 7x\,dx = 7\int x\,dx = 7\left(\frac{x^2}{2} + C_1\right) = \frac{7}{2}x^2 + 7C_1$$

CAUTION!

Only a *constant* factor of the integrand can pass through an integral sign.

Since $7C_1$ is just an arbitrary constant, we will replace it by C for simplicity. Thus,

$$\int 7x\,dx = \frac{7}{2}x^2 + C$$

It is not necessary to write all intermediate steps when integrating. More simply, we write

$$\int 7x\,dx = (7)\frac{x^2}{2} + C = \frac{7}{2}x^2 + C$$

Now Work Problem 5 ◁

EXAMPLE 4 Indefinite Integral of a Constant Times a Function

Find $\int -\frac{3}{5}e^x\,dx$.

Solution:

$$\begin{aligned}\int -\frac{3}{5}e^x\,dx &= -\frac{3}{5}\int e^x\,dx && \text{Formula (5)}\\ &= -\frac{3}{5}e^x + C && \text{Formula (4)}\end{aligned}$$

Now Work Problem 21 ◁

APPLY IT ▷

3. Due to new competition, the number of subscriptions to a certain magazine is declining at a rate of $\frac{dS}{dt} = -\frac{480}{t^3}$ subscriptions per month, where t is the number of months since the competition entered the market. Find the form of the equation for the number of subscribers to the magazine.

EXAMPLE 5 Finding Indefinite Integrals

a. Find $\int \frac{1}{\sqrt{t}}\,dt$.

Solution: Here t is the variable of integration. We rewrite the integrand so that a basic formula can be used. Since $1/\sqrt{t} = t^{-1/2}$, applying Formula (2) gives

$$\int \frac{1}{\sqrt{t}}\,dt = \int t^{-1/2}\,dt = \frac{t^{(-1/2)+1}}{-\frac{1}{2}+1} + C = \frac{t^{1/2}}{\frac{1}{2}} + C = 2\sqrt{t} + C$$

b. Find $\int \frac{1}{6x^3}\,dx$.

Solution:

$$\int \frac{1}{6x^3}\,dx = \frac{1}{6}\int x^{-3}\,dx = \left(\frac{1}{6}\right)\frac{x^{-3+1}}{-3+1} + C$$

$$= -\frac{x^{-2}}{12} + C = -\frac{1}{12x^2} + C$$

Now Work Problem 9 ◁

APPLY IT ▷

4. The rate of growth of the population of a new city is estimated by $\frac{dN}{dt} = 500 + 300\sqrt{t}$, where t is in years. Find

$$\int (500 + 300\sqrt{t})\,dt$$

EXAMPLE 6 Indefinite Integral of a Sum

Find $\int (x^2 + 2x)\,dx$.

Solution: By Formula (6),

$$\int (x^2 + 2x)\,dx = \int x^2\,dx + \int 2x\,dx$$

Now,

$$\int x^2\,dx = \frac{x^{2+1}}{2+1} + C_1 = \frac{x^3}{3} + C_1$$

and

$$\int 2x\,dx = 2\int x\,dx = (2)\frac{x^{1+1}}{1+1} + C_2 = x^2 + C_2$$

Thus,

$$\int (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 + C_1 + C_2$$

When integrating an expression involving more than one term, only one constant of integration is needed.

For convenience, we will replace the constant $C_1 + C_2$ by C. We then have

$$\int (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 + C$$

Omitting intermediate steps, we simply integrate term by term and write

$$\int (x^2 + 2x)\,dx = \frac{x^3}{3} + (2)\frac{x^2}{2} + C = \frac{x^3}{3} + x^2 + C$$

Now Work Problem 11 ◁

APPLY IT ▷

5. Suppose the rate of savings in the United States is given by $\frac{dS}{dt} = 2.1t^2 - 65.4t + 491.6$, where t is the time in years and S is the amount of money saved in billions of dollars. Find the form of the equation for the amount of money saved.

EXAMPLE 7 Indefinite Integral of a Sum and Difference

Find $\int (2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1)\,dx$.

Solution:

$$\int (2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1)\,dx$$

$$= 2\int x^{4/5}\,dx - 7\int x^3\,dx + 10\int e^x\,dx - \int 1\,dx \qquad \text{Formulas (5) and (6)}$$

$$= (2)\frac{x^{9/5}}{\frac{9}{5}} - (7)\frac{x^4}{4} + 10e^x - x + C \qquad \text{Formulas (1), (2), and (4)}$$

$$= \frac{10}{9}x^{9/5} - \frac{7}{4}x^4 + 10e^x - x + C$$

Now Work Problem 15 ◁

Sometimes, in order to apply the basic integration formulas, it is necessary first to perform algebraic manipulations on the integrand, as Example 8 shows.

EXAMPLE 8 **Using Algebraic Manipulation to Find an Indefinite Integral**

Find $\int y^2\left(y+\frac{2}{3}\right)dy$.

Solution: The integrand does not fit a familiar integration form. However, by multiplying the integrand we get

$$\int y^2\left(y+\frac{2}{3}\right)dy = \int\left(y^3+\frac{2}{3}y^2\right)dy$$
$$= \frac{y^4}{4}+\left(\frac{2}{3}\right)\frac{y^3}{3}+C = \frac{y^4}{4}+\frac{2y^3}{9}+C$$

Now Work Problem 41 ◁

CAUTION!

In Example 8, we first multiplied the factors in the integrand. The answer could not have been found simply in terms of $\int y^2\,dy$ and $\int (y+\frac{2}{3})\,dy$. There is not a formula for the integral of a general product of functions.

EXAMPLE 9 **Using Algebraic Manipulation to Find an Indefinite Integral**

a. Find $\int \frac{(2x-1)(x+3)}{6}\,dx$.

Solution: By factoring out the constant $\frac{1}{6}$ and multiplying the binomials, we get

$$\int \frac{(2x-1)(x+3)}{6}\,dx = \frac{1}{6}\int (2x^2+5x-3)\,dx$$
$$= \frac{1}{6}\left((2)\frac{x^3}{3}+(5)\frac{x^2}{2}-3x\right)+C$$
$$= \frac{x^3}{9}+\frac{5x^2}{12}-\frac{x}{2}+C$$

b. Find $\int \frac{x^3-1}{x^2}\,dx$.

Solution: We can break up the integrand into fractions by dividing each term in the numerator by the denominator:

$$\int \frac{x^3-1}{x^2}\,dx = \int\left(\frac{x^3}{x^2}-\frac{1}{x^2}\right)dx = \int (x-x^{-2})\,dx$$
$$= \frac{x^2}{2}-\frac{x^{-1}}{-1}+C = \frac{x^2}{2}+\frac{1}{x}+C$$

Now Work Problem 49 ◁

Another algebraic approach to part (b) is

$$\int \frac{x^3-1}{x^2}\,dx = \int (x^3-1)x^{-2}\,dx$$
$$= \int (x-x^{-2})\,dx$$

and so on.

PROBLEMS 14.2

In Problems 1–52, find the indefinite integrals.

1. $\int 7\,dx$
2. $\int \frac{1}{x}\,dx$
3. $\int x^8\,dx$
4. $\int 5x^{24}\,dx$
5. $\int 5x^{-7}\,dx$
6. $\int \frac{z^{-3}}{3}\,dz$
7. $\int \frac{5}{x^7}\,dx$
8. $\int \frac{7}{x^4\,dx}$
9. $\int \frac{1}{t^{7/4}}\,dt$
10. $\int \frac{7}{2x^{9/4}}\,dx$
11. $\int (4+t)\,dt$
12. $\int (7r^5+4r^2+1)\,dr$
13. $\int (y^5-5y)\,dy$
14. $\int (5-2w-6w^2)\,dw$
15. $\int (3t^2-4t+5)\,dt$
16. $\int (1+t^2+t^4+t^6)\,dt$

17. $\int (\sqrt{2} + e)\, dx$

18. $\int (5 - 2^{-1})\, dx$

19. $\int \left(\frac{x}{7} - \frac{3}{4}x^4\right) dx$

20. $\int \left(\frac{2x^2}{7} - \frac{8}{3}x^4\right) dx$

21. $\int \pi e^x\, dx$

22. $\int (e^x + 3x^2 + 2x)\, dx$

23. $\int (x^{8.3} - 9x^6 + 3x^{-4} + x^{-3})\, dx$

24. $\int (0.7y^3 + 10 + 2y^{-3})\, dy$

25. $\int \frac{-2\sqrt{x}}{3}\, dx$

26. $\int dz$

27. $\int \frac{5}{3\sqrt[3]{x^2}}\, dx$

28. $\int \frac{-4}{(3x)^3}\, dx$

29. $\int \left(\frac{x^3}{3} - \frac{3}{x^3}\right) dx$

30. $\int \left(\frac{1}{2x^3} - \frac{1}{x^4}\right) dx$

31. $\int \left(\frac{3w^2}{2} - \frac{2}{3w^2}\right) dw$

32. $\int 7e^{-s}\, ds$

33. $\int \frac{3u - 4}{5}\, du$

34. $\int \frac{1}{12}\left(\frac{1}{3}e^x\right) dx$

35. $\int (u^e + e^u)\, du$

36. $\int \left(3y^3 - 2y^2 + \frac{e^y}{6}\right) dy$

37. $\int \left(\frac{3}{\sqrt{x}} - 12\sqrt{x}\right) dx$

38. $\int 0\, dt$

39. $\int \left(-\frac{\sqrt[3]{x^2}}{5} - \frac{7}{2\sqrt{x}} + 6x\right) dx$

40. $\int \left(\sqrt[3]{u} + \frac{1}{\sqrt{u}}\right) du$

41. $\int (x^2 + 5)(x - 3)\, dx$

42. $\int x^3(x^2 + 5x + 2)\, dx$

43. $\int \sqrt{x}(x + 3)\, dx$

44. $\int (z + 2)^2\, dz$

45. $\int (3u + 2)^3\, du$

46. $\int \left(\frac{2}{\sqrt[5]{x}} - 1\right)^2 dx$

47. $\int x^{-2}(3x^4 + 4x^2 - 5)\, dx$

48. $\int (6e^u - u^3(\sqrt{u} + 1))\, du$

49. $\int \frac{z^4 + 10z^3}{2z^2}\, dz$

50. $\int \frac{x^4 - 5x^2 + 2x}{5x^2}\, dx$

51. $\int \frac{e^x + e^{2x}}{e^x}\, dx$

52. $\int \frac{(x^2 + 1)^3}{x}\, dx$

53. If $F(x)$ and $G(x)$ are such that $F'(x) = G'(x)$, is it true that $F(x) - G(x)$ must be zero?

54. **(a)** Find a function F such that $\int F(x)\, dx = xe^x + C$.
(b) Is there only one function F satisfying the equation given in part (a), or are there many such functions?

55. Find $\int \frac{d}{dx}\left(\frac{1}{\sqrt{x^2 + 1}}\right) dx$.

Objective

To find a particular antiderivative of a function that satisfies certain conditions. This involves evaluating constants of integration.

14.3 Integration with Initial Conditions

If we know the rate of change, f', of the function f, then the function f itself is an antiderivative of f' (since the derivative of f is f'). Of course, there are many antiderivatives of f', and the most general one is denoted by the indefinite integral. For example, if

$$f'(x) = 2x$$

then

$$f(x) = \int f'(x)\, dx = \int 2x\, dx = x^2 + C. \tag{1}$$

That is, *any* function of the form $f(x) = x^2 + C$ has its derivative equal to $2x$. Because of the constant of integration, notice that we do not know $f(x)$ specifically. However, if f must assume a certain function value for a particular value of x, then we can determine the value of C and thus determine $f(x)$ specifically. For instance, if $f(1) = 4$, then, from Equation (1),

$$f(1) = 1^2 + C$$
$$4 = 1 + C$$
$$C = 3$$

Thus,

$$f(x) = x^2 + 3$$

That is, we now know the particular function $f(x)$ for which $f'(x) = 2x$ and $f(1) = 4$. The condition $f(1) = 4$, which gives a function value of f for a specific value of x, is called an *initial condition*.

APPLY IT ▷

6. The rate of growth of a species of bacteria is estimated by $\frac{dN}{dt} = 800 + 200e^t$, where N is the number of bacteria (in thousands) after t hours. If $N(5) = 40{,}000$, find $N(t)$.

EXAMPLE 1 Initial-Condition Problem

If y is a function of x such that $y' = 8x - 4$ and $y(2) = 5$, find y. [Note: $y(2) = 5$ means that $y = 5$ when $x = 2$.] Also, find $y(4)$.

Solution: Here $y(2) = 5$ is the initial condition. Since $y' = 8x - 4$, y is an antiderivative of $8x - 4$:

$$y = \int (8x - 4)\,dx = 8 \cdot \frac{x^2}{2} - 4x + C = 4x^2 - 4x + C \tag{2}$$

We can determine the value of C by using the initial condition. Because $y = 5$ when $x = 2$, from Equation (2), we have

$$5 = 4(2)^2 - 4(2) + C$$
$$5 = 16 - 8 + C$$
$$C = -3$$

Replacing C by -3 in Equation (2) gives the function that we seek:

$$y = 4x^2 - 4x - 3 \tag{3}$$

To find $y(4)$, we let $x = 4$ in Equation (3):

$$y(4) = 4(4)^2 - 4(4) - 3 = 64 - 16 - 3 = 45$$

Now Work Problem 1 ◁

APPLY IT ▷

7. The acceleration of an object after t seconds is given by $y'' = 84t + 24$, the velocity at 8 seconds is given by $y'(8) = 2891$ ft/s, and the position at 2 seconds is given by $y(2) = 185$ ft. Find $y(t)$.

EXAMPLE 2 Initial-Condition Problem Involving y''

Given that $y'' = x^2 - 6$, $y'(0) = 2$, and $y(1) = -1$, find y.

Solution:

Strategy To go from y'' to y, two integrations are needed: the first to take us from y'' to y' and the other to take us from y' to y. Hence, there will be two constants of integration, which we will denote by C_1 and C_2.

Since $y'' = \frac{d}{dx}(y') = x^2 - 6$, y' is an antiderivative of $x^2 - 6$. Thus,

$$y' = \int (x^2 - 6)\,dx = \frac{x^3}{3} - 6x + C_1 \tag{4}$$

Now, $y'(0) = 2$ means that $y' = 2$ when $x = 0$; therefore, from Equation (4), we have

$$2 = \frac{0^3}{3} - 6(0) + C_1$$

Hence, $C_1 = 2$, so

$$y' = \frac{x^3}{3} - 6x + 2$$

By integration, we can find y:

$$y = \int \left(\frac{x^3}{3} - 6x + 2\right) dx$$
$$= \left(\frac{1}{3}\right)\frac{x^4}{4} - (6)\frac{x^2}{2} + 2x + C_2$$

so

$$y = \frac{x^4}{12} - 3x^2 + 2x + C_2 \tag{5}$$

Now, since $y = -1$ when $x = 1$, we have, from Equation (5),

$$-1 = \frac{1^4}{12} - 3(1)^2 + 2(1) + C_2$$

Thus, $C_2 = -\frac{1}{12}$, so

$$y = \frac{x^4}{12} - 3x^2 + 2x - \frac{1}{12}$$

Now Work Problem 5 ◁

Integration with initial conditions is applicable to many applied situations, as the next three examples illustrate.

EXAMPLE 3 Income and Education

For a particular urban group, sociologists studied the current average yearly income y (in dollars) that a person can expect to receive with x years of education before seeking regular employment. They estimated that the rate at which income changes with respect to education is given by

$$\frac{dy}{dx} = 100x^{3/2} \quad 4 \leq x \leq 16$$

where $y = 28{,}720$ when $x = 9$. Find y.

Solution: Here y is an antiderivative of $100x^{3/2}$. Thus,

$$y = \int 100x^{3/2}\,dx = 100\int x^{3/2}\,dx$$

$$= (100)\frac{x^{5/2}}{\frac{5}{2}} + C$$

$$y = 40x^{5/2} + C \tag{6}$$

The initial condition is that $y = 28{,}720$ when $x = 9$. By putting these values into Equation (6), we can determine the value of C:

$$28{,}720 = 40(9)^{5/2} + C$$

$$= 40(243) + C$$

$$28{,}720 = 9720 + C$$

Therefore, $C = 19{,}000$, and

$$y = 40x^{5/2} + 19{,}000$$

Now Work Problem 17 ◁

EXAMPLE 4 Finding the Demand Function from Marginal Revenue

If the marginal-revenue function for a manufacturer's product is

$$\frac{dr}{dq} = 2000 - 20q - 3q^2$$

find the demand function.

Solution:

Strategy By integrating dr/dq and using an initial condition, we can find the revenue function r. But revenue is also given by the general relationship $r = pq$, where p is the price per unit. Thus, $p = r/q$. Replacing r in this equation by the revenue function yields the demand function.

Since dr/dq is the derivative of total revenue r,

$$r = \int (2000 - 20q - 3q^2)\,dq$$

$$= 2000q - (20)\frac{q^2}{2} - (3)\frac{q^3}{3} + C$$

so that

$$r = 2000q - 10q^2 - q^3 + C \tag{7}$$

Revenue is 0 when q is 0.

We assume that **when no units are sold, there is no revenue;** that is, $r = 0$ when $q = 0$. This is our initial condition. Putting these values into Equation (7) gives

$$0 = 2000(0) - 10(0)^2 - 0^3 + C$$

Although $q = 0$ gives $C = 0$, this is not true in general. It occurs in this section because the revenue functions are polynomials. In later sections, evaluating at $q = 0$ may produce a nonzero value for C.

Hence, $C = 0$, and

$$r = 2000q - 10q^2 - q^3$$

To find the demand function, we use the fact that $p = r/q$ and substitute for r:

$$p = \frac{r}{q} = \frac{2000q - 10q^2 - q^3}{q}$$

$$p = 2000 - 10q - q^2$$

Now Work Problem 11 ◁

EXAMPLE 5 Finding Cost from Marginal Cost

In the manufacture of a product, fixed costs per week are \$4000. (Fixed costs are costs, such as rent and insurance, that remain constant at all levels of production during a given time period.) If the marginal-cost function is

$$\frac{dc}{dq} = 0.000001(0.002q^2 - 25q) + 0.2$$

where c is the total cost (in dollars) of producing q pounds of product per week, find the cost of producing 10,000 lb in 1 week.

Solution: Since dc/dq is the derivative of the total cost c,

$$c(q) = \int [0.000001(0.002q^2 - 25q) + 0.2]\,dq$$

$$= 0.000001 \int (0.002q^2 - 25q)\,dq + \int 0.2\,dq$$

$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + C$$

When q is 0, total cost is equal to fixed cost.

Fixed costs are constant regardless of output. Therefore, when $q = 0, c = 4000$, which is our initial condition. Putting $c(0) = 4000$ in the last equation, we find that $C = 4000$, so

$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + 4000 \tag{8}$$

Although $q = 0$ gives C a value equal to fixed costs, this is not true in general. It occurs in this section because the cost functions are polynomials. In later sections, evaluating at $q = 0$ may produce a value for C that is different from fixed cost.

From Equation (8), we have $c(10{,}000) = 5416\frac{2}{3}$. Thus, the total cost for producing 10,000 pounds of product in 1 week is \$5416.67.

Now Work Problem 15 ◁

PROBLEMS 14.3

In Problems 1 and 2, find y subject to the given conditions.

1. $dy/dx = 3x - 4$; $y(-1) = \frac{13}{2}$

2. $dy/dx = x^2 - x$; $y(3) = \frac{19}{2}$

In Problems 3 and 4, if y satisfies the given conditions, find y(x) for the given value of x.

3. $y' = \dfrac{9}{8\sqrt{x}}$, $y(16) = 10$; $x = 9$

4. $y' = -x^2 + 2x$, $y(2) = 1$; $x = 1$

In Problems 5–8, find y subject to the given conditions.

5. $y'' = -3x^2 + 4x$; $y'(1) = 2$, $y(1) = 3$

6. $y'' = x + 1$; $y'(0) = 0$, $y(0) = 5$

7. $y''' = 2x$; $y''(-1) = 3$, $y'(3) = 10$, $y(0) = 13$

8. $y''' = 2e^{-x} + 3$; $y''(0) = 7$, $y'(0) = 5$, $y(0) = 1$

In Problems 9–12, dr/dq is a marginal-revenue function. Find the demand function.

9. $dr/dq = 0.7$

10. $dr/dq = 10 - \dfrac{1}{16}q$

11. $dr/dq = 275 - q - 0.3q^2$

12. $dr/dq = 5{,}000 - 3(2q + 2q^3)$

In Problems 13–16, dc/dq is a marginal-cost function and fixed costs are indicated in braces. For Problems 13 and 14, find the total-cost function. For Problems 15 and 16, find the total cost for the indicated value of q.

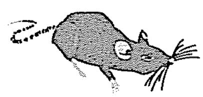
13. $dc/dq = 2.47$; $\{159\}$

14. $dc/dq = 2q + 75$; $\{2000\}$

15. $dc/dq = 0.08q^2 - 1.6q + 6.5$; $\{8000\}$; $q = 25$

16. $dc/dq = 0.000204q^2 - 0.046q + 6$; $\{15{,}000\}$; $q = 200$

17. Diet for Rats A group of biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.[2] The protein consisted of yeast and corn flour.



Over a period of time, the group found that the (approximate) rate of change of the average weight gain G (in grams) of a rat with respect to the percentage P of yeast in the protein mix was

$$\frac{dG}{dP} = -\frac{P}{25} + 2 \qquad 0 \le P \le 100$$

If $G = 38$ when $P = 10$, find G.

18. Winter Moth A study of the winter moth was made in Nova Scotia.[3] The prepupae of the moth fall onto the ground from host trees. It was found that the (approximate) rate at which prepupal density y (the number of prepupae per square foot of soil) changes with respect to distance x (in feet) from the base of a host tree is

$$\frac{dy}{dx} = -1.5 - x \qquad 1 \le x \le 9$$

If $y = 59.6$ when $x = 1$, find y.

19. Fluid Flow In the study of the flow of fluid in a tube of constant radius R, such as blood flow in portions of the body, one can think of the tube as consisting of concentric tubes of radius r, where $0 \le r \le R$. The velocity v of the fluid is a function of r and is given by[4]

$$v = \int -\frac{(P_1 - P_2)r}{2l\eta}\, dr$$

where P_1 and P_2 are pressures at the ends of the tube, η (a Greek letter read "eta") is fluid viscosity, and l is the length of the tube. If $v = 0$ when $r = R$, show that

$$v = \frac{(P_1 - P_2)(R^2 - r^2)}{4l\eta}$$

20. Elasticity of Demand The sole producer of a product has determined that the marginal-revenue function is

$$\frac{dr}{dq} = 100 - 3q^2$$

Determine the point elasticity of demand for the product when $q = 5$. (*Hint:* First find the demand function.)

21. Average Cost A manufacturer has determined that the marginal-cost function is

$$\frac{dc}{dq} = 0.003q^2 - 0.4q + 40$$

where q is the number of units produced. If marginal cost is \$27.50 when $q = 50$ and fixed costs are \$5000, what is the *average* cost of producing 100 units?

22. If $f''(x) = 30x^4 + 12x$ and $f'(1) = 10$, evaluate

$$f(965.335245) - f(-965.335245)$$

Objective

To learn and apply the formulas for $\int u^a\, du$, $\int e^u\, du$, and $\int \frac{1}{u}\, du$.

14.4 More Integration Formulas

Power Rule for Integration

The formula

$$\int x^a\, dx = \frac{x^{a+1}}{n+1} + C \qquad \text{if } a \ne -1$$

[2]Adapted from R. Bressani, "The Use of Yeast in Human Foods," in *Single-Cell Protein*, eds. R. I. Mateles and S. R. Tannenbaum (Cambridge, MA: MIT Press, 1968).

[3]Adapted from D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada*, no. 46 (1965).

[4]R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

which applies to a power of x, can be generalized to handle a power of a *function* of x. Let u be a differentiable function of x. By the power rule for differentiation, if $a \neq -1$, then

$$\frac{d}{dx}\left(\frac{(u(x))^{a+1}}{a+1}\right) = \frac{(a+1)(u(x))^a \cdot u'(x)}{a+1} = (u(x))^a \cdot u'(x)$$

Thus,

$$\int (u(x))^a \cdot u'(x)\, dx = \frac{(u(x))^{a+1}}{a+1} + C \quad a \neq -1$$

We call this the *power rule for integration*. Note that $u'(x)dx$ is the differential of u, namely du. In mathematical shorthand, we can replace $u(x)$ by u and $u'(x)\,dx$ by du:

Power Rule for Integration

If u is differentiable, then

$$\int u^a\, du = \frac{u^{a+1}}{a+1} + C \qquad \text{if } a \neq -1 \tag{1}$$

It is important to appreciate the difference between the power rule for integration and the formula for $\int x^a\, dx$. In the power rule, u represents a function, whereas in $\int x^a\, dx$, x is a variable.

EXAMPLE 1 **Applying the Power Rule for Integration**

a. Find $\int (x+1)^{20}\, dx$.

Solution: Since the integrand is a power of the function $x+1$, we will set $u = x+1$. Then $du = dx$, and $\int (x+1)^{20}\, dx$ has the form $\int u^{20}\, du$. By the power rule for integration,

$$\int (x+1)^{20}\, dx = \int u^{20}\, du = \frac{u^{21}}{21} + C = \frac{(x+1)^{21}}{21} + C$$

Note that we give our answer not in terms of u, but explicitly in terms of x.

b. Find $\int 3x^2(x^3+7)^3\, dx$.

Solution: We observe that the integrand contains a power of the function x^3+7. Let $u = x^3+7$. Then $du = 3x^2\, dx$. Fortunately, $3x^2$ appears as a factor in the integrand and we have

$$\int 3x^2(x^3+7)^3\, dx = \int (x^3+7)^3[3x^2\, dx] = \int u^3\, du$$

$$= \frac{u^4}{4} + C = \frac{(x^3+7)^4}{4} + C$$

After integrating, you may wonder what happened to $3x^2$. We note again that $du = 3x^2\, dx$.

Now Work Problem 3 ◁

In order to apply the power rule for integration, sometimes an adjustment must be made to obtain du in the integrand, as Example 2 illustrates.

EXAMPLE 2 **Adjusting for *du***

Find $\int x\sqrt{x^2+5}\, dx$.

Solution: We can write this as $\int x(x^2+5)^{1/2}\, dx$. Notice that the integrand contains a power of the function x^2+5. If $u = x^2+5$, then $du = 2x\, dx$. Since the *constant* factor 2 in du does *not* appear in the integrand, this integral does not have the form $\int u^n\,du$. However, from $du = 2x\,dx$ we can write $x\,dx = \frac{du}{2}$ so that the integral becomes

$$\int x(x^2+5)^{1/2}\,dx = \int (x^2+5)^{1/2}[x\,dx] = \int u^{1/2}\frac{du}{2}$$

Moving the *constant* factor $\frac{1}{2}$ in front of the integral sign, we have

$$\int x(x^2+5)^{1/2}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C = \frac{1}{3}u^{3/2} + C$$

which in terms of x (as is required) gives

$$\int x\sqrt{x^2+5}\,dx = \frac{(x^2+5)^{3/2}}{3} + C$$

CAUTION!

The answer to an integration problem must be expressed in terms of the original variable.

Now Work Problem 15 ◁

In Example 2, the integrand $x\sqrt{x^2+5}$ missed being of the form $(u(x))^{1/2}u'(x)$ by the *constant factor* of 2. In general, if we have $\int (u(x))^a \frac{u'(x)}{k}dx$, for k a nonzero constant, then we can write

$$\int (u(x))^a \frac{u'(x)}{k}\,dx = \int u^a\frac{du}{k} = \frac{1}{k}\int u^a\,du$$

to simplify the integral, but such *adjustments* of the integrand are *not possible for variable factors.*

CAUTION!

We can adjust for constant factors, but not variable factors.

When using the form $\int u^a\,du$, do not neglect du. For example,

$$\int (4x+1)^2\,dx \neq \frac{(4x+1)^3}{3} + C$$

The correct way to do this problem is as follows. Let $u = 4x+1$, from which it follows that $du = 4\,dx$. Thus $dx = \frac{du}{4}$ and

$$\int (4x+1)^2\,dx = \int u^2\left[\frac{du}{4}\right] = \frac{1}{4}\int u^2\,du = \frac{1}{4}\cdot\frac{u^3}{3} + C = \frac{(4x+1)^3}{12} + C$$

EXAMPLE 3 Applying the Power Rule for Integration

a. Find $\int \sqrt[3]{6y}\,dy$.

Solution: The integrand is $(6y)^{1/3}$, a power of a function. However, in this case the obvious substitution $u = 6y$ can be avoided. More simply, we have

$$\int \sqrt[3]{6y}\,dy = \int 6^{1/3}y^{1/3}\,dy = \sqrt[3]{6}\int y^{1/3}\,dy = \sqrt[3]{6}\frac{y^{4/3}}{\frac{4}{3}} + C = \frac{3\sqrt[3]{6}}{4}y^{4/3} + C$$

b. Find $\int \frac{2x^3+3x}{(x^4+3x^2+7)^4}\,dx$.

Solution: We can write this as $\int (x^4+3x^2+7)^{-4}(2x^3+3x)\,dx$. Let us try to use the power rule for integration. If $u = x^4+3x^2+7$, then $du = (4x^3+6x)\,dx$, which is two times the quantity $(2x^3+3x)\,dx$ in the integral. Thus $(2x^3+3x)\,dx = \frac{du}{2}$ and we again illustrate the *adjustment* technique:

$$\int (x^4+3x^2+7)^{-4}[(2x^3+3x)\,dx] = \int u^{-4}\left[\frac{du}{2}\right] = \frac{1}{2}\int u^{-4}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{-3}}{-3} + C = -\frac{1}{6u^3} + C = -\frac{1}{6(x^4+3x^2+7)^3} + C$$

Now Work Problem 5 ◁

In using the power rule for integration, take care when making a choice for u. In Example 3(b), letting $u = 2x^3 + 3x$ does not lead very far. At times it may be necessary to try many different choices. Sometimes a wrong choice will provide a hint as to what does work. **Skill at integration comes only after many hours of practice and conscientious study.**

EXAMPLE 4 An Integral to Which the Power Rule Does Not Apply

Find $\int 4x^2(x^4+1)^2\,dx$.

Solution: If we set $u = x^4 + 1$, then $du = 4x^3\,dx$. To get du in the integral, we need an additional factor of the *variable* x. However, we can adjust only for **constant** factors. Thus, we cannot use the power rule. Instead, to find the integral, we will first expand $(x^4+1)^2$:

$$\begin{aligned}\int 4x^2(x^4+1)^2\,dx &= 4\int x^2(x^8+2x^4+1)\,dx\\ &= 4\int (x^{10}+2x^6+x^2)\,dx\\ &= 4\left(\frac{x^{11}}{11}+\frac{2x^7}{7}+\frac{x^3}{3}\right)+C\end{aligned}$$

Now Work Problem 67 ◁

Integrating Natural Exponential Functions

We now turn our attention to integrating exponential functions. If u is a differentiable function of x, then

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

CAUTION!

Do not apply the power-rule formula for $\int u^a\,du$ to $\int e^u\,du$.

Corresponding to this differentiation formula is the integration formula

$$\int e^u\frac{du}{dx}\,dx = e^u + C$$

But $\frac{du}{dx}\,dx$ is the differential of u, namely, du. Thus,

$$\int e^u\,du = e^u + C \tag{2}$$

APPLY IT ▷

8. When an object is moved from one environment to another, its temperature T changes at a rate given by $\frac{dT}{dt} = kCe^{kt}$, where t is the time (in hours) after changing environments, C is the temperature difference (original minus new) between the environments, and k is a constant. If the original environment is 70°, the new environment is 60°, and $k = -0.5$, find the general form of $T(t)$.

EXAMPLE 5 Integrals Involving Exponential Functions

a. Find $\int 2xe^{x^2}\,dx$.

Solution: Let $u = x^2$. Then $du = 2x\,dx$, and, by Equation (2),

$$\begin{aligned}\int 2xe^{x^2}\,dx &= \int e^{x^2}[2x\,dx] = \int e^u\,du\\ &= e^u + C = e^{x^2} + C\end{aligned}$$

b. Find $\int (x^2+1)e^{x^3+3x}\,dx$.

Solution: If $u = x^3 + 3x$, then $du = (3x^2+3)\,dx = 3(x^2+1)\,dx$. If the integrand contained a factor of 3, the integral would have the form $\int e^u\,du$. Thus, we write

$$\begin{aligned}\int (x^2+1)e^{x^3+3x}\,dx &= \int e^{x^3+3x}[(x^2+1)\,dx]\\ &= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C\\ &= \frac{1}{3}e^{x^3+3x} + C\end{aligned}$$

where in the second step we replaced $(x^2+1)\,dx$ by $\frac{1}{3}\,du$ but wrote $\frac{1}{3}$ outside the integral.

Now Work Problem 41 ◁

Integrals Involving Logarithmic Functions

As we know, the power-rule formula $\int u^a\,du = u^{a+1}/(a+1) + C$ does not apply when $a = -1$. To handle that situation, namely, $\int u^{-1}\,du = \int \frac{1}{u}\,du$, we first recall from Section 12.1 that

$$\frac{d}{dx}(\ln|u|) = \frac{1}{u}\frac{du}{dx} \quad \text{for } u \neq 0$$

which gives us the integration formula

$$\int \frac{1}{u}\,du = \ln|u| + C \quad \text{for } u \neq 0 \tag{3}$$

In particular, if $u = x$, then $du = dx$, and

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{for } x \neq 0 \tag{4}$$

APPLY IT ▷

9. If the rate of vocabulary memorization of the average student in a foreign language is given by $\frac{dv}{dt} = \frac{35}{t+1}$, where v is the number of vocabulary words memorized in t hours of study, find the general form of $v(t)$.

EXAMPLE 6 Integrals Involving $\frac{1}{u}\,du$

a. Find $\int \frac{7}{x}\,dx$.

Solution: From Equation (4),

$$\int \frac{7}{x}\,dx = 7\int \frac{1}{x}\,dx = 7\ln|x| + C$$

Using properties of logarithms, we can write this answer another way:

$$\int \frac{7}{x}\,dx = \ln|x^7| + C$$

b. Find $\int \frac{2x}{x^2+5}\,dx$.

Solution: Let $u = x^2 + 5$. Then $du = 2x\,dx$. From Equation (3),

$$\begin{aligned}\int \frac{2x}{x^2+5}\,dx &= \int \frac{1}{x^2+5}[2x\,dx] = \int \frac{1}{u}\,du\\ &= \ln|u| + C = \ln|x^2+5| + C\end{aligned}$$

Since $x^2 + 5$ is always positive, we can omit the absolute-value bars:

$$\int \frac{2x}{x^2+5}\,dx = \ln(x^2+5) + C$$

Now Work Problem 31 ◁

EXAMPLE 7 **An Integral Involving $\frac{1}{u}\,du$**

Find $\int \frac{(2x^3+3x)\,dx}{x^4+3x^2+7}$.

Solution: If $u = x^4+3x^2+7$, then $du = (4x^3+6x)\,dx$, which is two times the numerator giving $(2x^3+3x)\,dx = \frac{du}{2}$. To apply Equation (3), we write

$$\int \frac{2x^3+3x}{x^4+3x^2+7}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^4+3x^2+7| + C \qquad \text{Rewrite } u \text{ in terms of } x.$$

$$= \frac{1}{2}\ln(x^4+3x^2+7) + C \qquad x^4+3x^2+7>0 \quad \text{for all } x$$

Now Work Problem 51 ◁

EXAMPLE 8 **An Integral Involving Two Forms**

Find $\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw$.

Solution:

$$\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw = \int (1-w)^{-2}dw + \int \frac{1}{w-1}dw$$

$$= -1\int (1-w)^{-2}[-dw] + \int \frac{1}{w-1}dw$$

The first integral has the form $\int u^{-2}\,du$, and the second has the form $\int \frac{1}{v}\,dv$. Thus,

$$\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right) dw = -\frac{(1-w)^{-1}}{-1} + \ln|w-1| + C$$

$$= \frac{1}{1-w} + \ln|w-1| + C$$

◁

PROBLEMS 14.4

In Problems 1–80, find the indefinite integrals.

1. $\int (x+5)^7\,dx$
2. $\int 15(x+2)^4\,dx$
3. $\int 2x(x^2+3)^5\,dx$
4. $\int (4x+3)(2x^2+3x+1)\,dx$
5. $\int (3y^2+6y)(y^3+3y^2+1)^{2/3}\,dy$
6. $\int (15t^2-6t+1)(5t^3-3t^2+t)^{17}\,dt$
7. $\int \frac{5}{(3x-1)^3}\,dx$
8. $\int \frac{4x}{(2x^2-7)^{10}}\,dx$
9. $\int \sqrt{7x+3}\,dx$
10. $\int \frac{1}{\sqrt{x-5}}\,dx$
11. $\int (7x-6)^4\,dx$
12. $\int x^2(3x^3+7)^3\,dx$
13. $\int u(5u^2-9)^{14}\,du$
14. $\int x\sqrt{3+5x^2}\,dx$
15. $\int 4x^4(27+x^5)^{1/3}\,dx$
16. $\int (4-5x)^9\,dx$
17. $\int 3e^{3x}\,dx$
18. $\int 5e^{3t+7}\,dt$
19. $\int (3t+1)e^{3t^2+2t+1}\,dt$
20. $\int -3w^2e^{-w^3}\,dw$
21. $\int xe^{7x^2}\,dx$
22. $\int x^3e^{4x^4}\,dx$
23. $\int 4e^{-3x}\,dx$
24. $\int 24x^5e^{-2x^6+7}\,dx$
25. $\int \frac{1}{x+5}\,dx$
26. $\int \frac{12x^2+4x+2}{x+x^2+2x^3}\,dx$
27. $\int \frac{3x^2+4x^3}{x^3+x^4}\,dx$
28. $\int \frac{6x^2-6x}{1-3x^2+2x^3}\,dx$
29. $\int \frac{8z}{(z^2-5)^7}\,dz$
30. $\int \frac{3}{(5v-1)^4}\,dv$

31. $\int \frac{4}{x}\,dx$

32. $\int \frac{3}{1+2y}\,dy$

33. $\int \frac{s^2}{s^3+5}\,ds$

34. $\int \frac{32x^3}{4x^4+9}\,dx$

35. $\int \frac{5}{4-2x}\,dx$

36. $\int \frac{7t}{5t^2-6}\,dt$

37. $\int \sqrt{5x}\,dx$

38. $\int \frac{1}{(3x)^6}\,dx$

39. $\int \frac{x}{\sqrt{ax^2+b}}\,dx$

40. $\int \frac{9}{1-3x}\,dx$

41. $\int 2y^3e^{y^4+1}\,dy$

42. $\int 2\sqrt{2x-1}\,dx$

43. $\int v^2e^{-2v^3+1}\,dv$

44. $\int \frac{x^2+x+1}{\sqrt[3]{x^3+\frac{3}{2}x^2+3x}}\,dx$

45. $\int (e^{-5x}+2e^x)\,dx$

46. $\int 4\sqrt[3]{y+1}\,dy$

47. $\int (8x+10)(7-2x^2-5x)^3\,dx$

48. $\int 2ye^{3y^2}\,dy$

49. $\int \frac{6x^2+8}{x^3+4x}\,dx$

50. $\int (e^x+2e^{-3x}-e^{5x})\,dx$

51. $\int \frac{16s-4}{3-2s+4s^2}\,ds$

52. $\int (6t^2+4t)(t^3+t^2+1)^6\,dt$

53. $\int x(2x^2+1)^{-1}\,dx$

54. $\int (45w^4+18w^2+12)(3w^5+2w^3+4)^{-4}\,dw$

55. $\int -(x^2-2x^5)(x^3-x^6)^{-10}\,dx$

56. $\int \frac{3}{5}(v-2)e^{2-4v+v^2}\,dv$

57. $\int (2x^3+x)(x^4+x^2)\,dx$

58. $\int (e^{3.1})^2\,dx$

59. $\int \frac{9+18x}{(5-x-x^2)^4}\,dx$

60. $\int (e^x-e^{-x})^2\,dx$

61. $\int x(2x+1)e^{4x^3+3x^2-4}\,dx$

62. $\int (u^3-ue^{6-3u^2})\,du$

63. $\int x\sqrt{(8-5x^2)^3}\,dx$

64. $\int e^{ax}\,dx$

65. $\int \left(\sqrt{2x}-\frac{1}{\sqrt{2x}}\right)dx$

66. $\int 3\frac{x^4}{e^{x^5}}\,dx$

67. $\int (x^2+1)^2\,dx$

68. $\int \left[x(x^2-16)^2-\frac{1}{2x+5}\right]dx$

69. $\int \left(\frac{x}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx$

70. $\int \left[\frac{3}{x-1}+\frac{1}{(x-1)^2}\right]dx$

71. $\int \left[\frac{2}{4x+1}-(4x^2-8x^5)(x^3-x^6)^{-8}\right]dx$

72. $\int (r^3+5)^2\,dr$

73. $\int \left[\sqrt{3x+1}-\frac{x}{x^2+3}\right]dx$

74. $\int \left(\frac{x}{7x^2+2}-\frac{x^2}{(x^3+2)^4}\right)dx$

75. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

76. $\int (e^5-3^e)\,dx$

77. $\int \frac{1+e^{2x}}{4e^x}\,dx$

78. $\int \frac{2}{t^2}\sqrt{\frac{1}{t}+9}\,dt$

79. $\int \frac{4x+3}{2x^2+3x}\ln(2x^2+3x)dx$

80. $\int \sqrt[3]{x}e^{\sqrt[3]{8x^4}}\,dx$

In Problems 81–84, find y subject to the given conditions.

81. $y'=(3-2x)^2;\quad y(0)=1$ **82.** $y'=\frac{x}{x^2+6};\quad y(1)=0$

83. $y''=\frac{1}{x^2};\quad y'(-2)=3, y(1)=2$

84. $y''=(x+1)^{1/2};\quad y'(8)=19, y(24)=\frac{2572}{3}$

85. Real Estate The rate of change of the value of a house that cost \$350,000 to build can be modeled by $\frac{dV}{dt}=8e^{0.05t}$, where t is the time in years since the house was built and V is the value (in thousands of dollars) of the house. Find $V(t)$.

86. Life Span If the rate of change of the expected life span l at birth of people born in the United States can be modeled by $\frac{dl}{dt}=\frac{12}{2t+50}$, where t is the number of years after 1940 and the expected life span was 63 years in 1940, find the expected life span for people born in 1998.

87. Oxygen in Capillary In a discussion of the diffusion of oxygen from capillaries,[5] concentric cylinders of radius r are used as a model for a capillary. The concentration C of oxygen in the capillary is given by

$$C=\int\left(\frac{Rr}{2K}+\frac{B_1}{r}\right)dr$$

where R is the constant rate at which oxygen diffuses from the capillary, and K and B_1 are constants. Find C. (Write the constant of integration as B_2.)

88. Find $f(2)$ if $f\left(\frac{1}{3}\right)=2$ and $f'(x)=e^{3x+2}-3x$.

Objective

To discuss techniques of handling more challenging integration problems, namely, by algebraic manipulation and by fitting the integrand to a familiar form. To integrate an exponential function with a base different from *e* and to find the consumption function, given the marginal propensity to consume.

14.5 Techniques of Integration

We turn now to some more difficult integration problems.

When integrating fractions, sometimes a preliminary division is needed to get familiar integration forms, as the next example shows.

[5] W. Simon, *Mathematical Techniques for Physiology and Medicine* (New York: Academic Press, Inc., 1972).

EXAMPLE 1 Preliminary Division before Integration

a. Find $\int \frac{x^3 + x}{x^2}\, dx$.

Solution: A familiar integration form is not apparent. However, we can break up the integrand into two fractions by dividing each term in the numerator by the denominator. We then have

Here we split up the integrand.

$$\int \frac{x^3 + x}{x^2}\, dx = \int \left(\frac{x^3}{x^2} + \frac{x}{x^2}\right) dx = \int \left(x + \frac{1}{x}\right) dx$$

$$= \frac{x^2}{2} + \ln|x| + C$$

b. Find $\int \frac{2x^3 + 3x^2 + x + 1}{2x + 1}\, dx$.

Solution: Here the integrand is a quotient of polynomials in which the degree of the numerator is greater than or equal to that of the denominator. In such a situation we first use long division. Recall that if f and g are polynomials, with the degree of f greater than or equal to the degree of g, then long division allows us to find (uniquely) polynomials q and r, where either r is the zero polynomial or the degree of r is strictly less than the degree of g, satisfying

$$\frac{f}{g} = q + \frac{r}{g}$$

Using an obvious, abbreviated notation, we see that

$$\int \frac{f}{g} = \int \left(q + \frac{r}{g}\right) = \int q + \int \frac{r}{g}$$

Since integrating a polynomial is easy, we see that integrating rational functions reduces to the task of integrating *proper rational functions*—those for which the degree of the numerator is strictly less than the degree of the denominator. In this case we obtain

Here we used long division to rewrite the integrand.

$$\int \frac{2x^3 + 3x^2 + x + 1}{2x + 1}\, dx = \int \left(x^2 + x + \frac{1}{2x + 1}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + \int \frac{1}{2x + 1}\, dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{2}\int \frac{1}{2x + 1}\, d(2x + 1)$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{2}\ln|2x + 1| + C$$

Now Work Problem 1 ◁

EXAMPLE 2 Indefinite Integrals

a. Find $\int \frac{1}{\sqrt{x}(\sqrt{x} - 2)^3}\, dx$.

Solution: We can write this integral as $\int \frac{(\sqrt{x} - 2)^{-3}}{\sqrt{x}}\, dx$. Let us try the power rule for integration with $u = \sqrt{x} - 2$. Then $du = \frac{1}{2\sqrt{x}}\, dx$, so that $\frac{dx}{\sqrt{x}} = 2\, du$, and

Here the integral is fit to the form to which the power rule for integration applies.

$$\int \frac{(\sqrt{x} - 2)^{-3}}{\sqrt{x}}\, dx = \int (\sqrt{x} - 2)^{-3} \left[\frac{dx}{\sqrt{x}}\right]$$

$$= 2\int u^{-3}\, du = 2\left(\frac{u^{-2}}{-2}\right) + C$$

$$= -\frac{1}{u^2} + C = -\frac{1}{(\sqrt{x} - 2)^2} + C$$

b. Find $\int \frac{1}{x \ln x}\, dx$.

Here the integral fits the familiar form $\int \frac{1}{u}\, du$.

Solution: If $u = \ln x$, then $du = \frac{1}{x}\, dx$, and

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{\ln x}\left(\frac{1}{x}\, dx\right) = \int \frac{1}{u}\, du$$
$$= \ln |u| + C = \ln |\ln x| + C$$

c. Find $\int \frac{5}{w(\ln w)^{3/2}}\, dw$.

Here the integral is fit to the form to which the power rule for integration applies.

Solution: If $u = \ln w$, then $du = \frac{1}{w} dw$. Applying the power rule for integration, we have

$$\int \frac{5}{w(\ln w)^{3/2}} dw = 5 \int (\ln w)^{-3/2} \left[\frac{1}{w} dw\right]$$
$$= 5 \int u^{-3/2}\, du = 5 \cdot \frac{u^{-1/2}}{-\frac{1}{2}} + C$$
$$= \frac{-10}{u^{1/2}} + C = -\frac{10}{(\ln w)^{1/2}} + C$$

Now Work Problem 23 ◁

Integrating b^u

In Section 14.4, we integrated an exponential function to the base e:

$$\int e^u\, du = e^u + C$$

Now let us consider the integral of an exponential function with an arbitrary base, b.

$$\int b^u\, du$$

To find this integral, we first convert to base e using

$$b^u = e^{(\ln b)u} \tag{1}$$

(as we did in many differentiation examples too). Example 3 will illustrate.

EXAMPLE 3 An Integral Involving b^u

Find $\int 2^{3-x}\, dx$.

Solution:

Strategy We want to integrate an exponential function to the base 2. To do this, we will first convert from base 2 to base e by using Equation (1).

$$\int 2^{3-x}\, dx = \int e^{(\ln 2)(3-x)}\, dx$$

The integrand of the second integral is of the form e^u, where $u = (\ln 2)(3 - x)$. Since $du = -\ln 2\, dx$, we can solve for dx and write

$$\int e^{(\ln 2)(3-x)}\, dx = -\frac{1}{\ln 2} \int e^u\, du$$
$$= -\frac{1}{\ln 2} e^u + C = -\frac{1}{\ln 2} e^{(\ln 2)(3-x)} + C = -\frac{1}{\ln 2} 2^{3-x} + C$$

Thus,

$$\int 2^{3-x}\,dx = -\frac{1}{\ln 2}2^{3-x} + C$$

Notice that we expressed our answer in terms of an exponential function to the base 2, the base of the original integrand.

Now Work Problem 27 ◁

Generalizing the procedure described in Example 3, we can obtain a formula for integrating b^u:

$$\begin{aligned}\int b^u du &= \int e^{(\ln b)u} du \\ &= \frac{1}{\ln b}\int e^{(\ln b)u}\, d((\ln b)u) && \ln b \text{ is a constant} \\ &= \frac{1}{\ln b}e^{(\ln b)u} + C \\ &= \frac{1}{\ln b}b^u + C\end{aligned}$$

Hence, we have

$$\int b^u du = \frac{1}{\ln b}b^u + C$$

Applying this formula to the integral in Example 3 gives

$$\begin{aligned}&\int 2^{3-x}\,dx && b = 2, u = 3 - x \\ &= -\int 2^{3-x}\,d(3-x) && -d(3-x) = dx \\ &= -\frac{1}{\ln 2}2^{3-x} + C\end{aligned}$$

which is the same result that we obtained before.

Application of Integration

We will now consider an application of integration that relates a consumption function to the marginal propensity to consume.

EXAMPLE 4 Finding a Consumption Function from Marginal Propensity to Consume

For a certain country, the marginal propensity to consume is given by

$$\frac{dC}{dI} = \frac{3}{4} - \frac{1}{2\sqrt{3I}}$$

where consumption C is a function of national income I. Here I is expressed in large denominations of money. Determine the consumption function for the country if it is known that consumption is 10 ($C = 10$) when $I = 12$.

Solution: Since the marginal propensity to consume is the derivative of C, we have

$$\begin{aligned}C = C(I) &= \int\left(\frac{3}{4} - \frac{1}{2\sqrt{3I}}\right)dI = \int \frac{3}{4}dI - \frac{1}{2}\int (3I)^{-1/2}\,dI \\ &= \frac{3}{4}I - \frac{1}{2}\int (3I)^{-1/2}\,dI\end{aligned}$$

If we let $u = 3I$, then $du = 3\,dI = d(3I)$, and

$$C = \frac{3}{4}I - \left(\frac{1}{2}\right)\frac{1}{3}\int (3I)^{-1/2}\,d(3I)$$

$$= \frac{3}{4}I - \frac{1}{6}\frac{(3I)^{1/2}}{\frac{1}{2}} + K$$

$$C = \frac{3}{4}I - \frac{\sqrt{3I}}{3} + K$$

This is an example of an initial-value problem.

When $I = 12$, $C = 10$, so

$$10 = \frac{3}{4}(12) - \frac{\sqrt{3(12)}}{3} + K$$

$$10 = 9 - 2 + K$$

Thus, $K = 3$, and the consumption function is

$$C = \frac{3}{4}I - \frac{\sqrt{3I}}{3} + 3$$

Now Work Problem 61 ◁

PROBLEMS 14.5

In Problems 1–56, determine the indefinite integrals.

1. $\int \frac{2x^6 + 8x^4 - 4x}{2x^2}\,dx$

2. $\int \frac{9x^2 + 5}{3x}\,dx$

3. $\int (3x^2 + 2)\sqrt{2x^3 + 4x + 1}\,dx$

4. $\int \frac{x}{\sqrt[4]{x^2 + 1}}\,dx$

5. $\int \frac{3}{\sqrt{4 - 5x}}\,dx$

6. $\int \frac{2xe^{x^2}\,dx}{e^{x^2} - 2}$

7. $\int 4^{7x}\,dx$

8. $\int 5^t\,dt$

9. $\int 2x(7 - e^{x^2/4})\,dx$

10. $\int \frac{e^x + 1}{e^x}\,dx$

11. $\int \frac{6x^2 - 11x + 5}{3x - 1}\,dx$

12. $\int \frac{(3x + 2)(x - 4)}{x - 3}\,dx$

13. $\int \frac{5e^{2x}}{7e^{2x} + 4}\,dx$

14. $\int 6(e^{4-3x})^2\,dx$

15. $\int \frac{5e^{13/x}}{x^2}\,dx$

16. $\int \frac{2x^4 - 6x^3 + x - 2}{x - 2}\,dx$

17. $\int \frac{5x^3}{x^2 + 9}\,dx$

18. $\int \frac{5 - 4x^2}{3 + 2x}\,dx$

19. $\int \frac{(\sqrt{x} + 2)^2}{3\sqrt{x}}\,dx$

20. $\int \frac{5e^s}{1 + 3e^s}\,ds$

21. $\int \frac{5(x^{1/3} + 2)^4}{\sqrt[3]{x^2}}\,dx$

22. $\int \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}}\,dx$

23. $\int \frac{\ln x}{x}\,dx$

24. $\int \sqrt{t}(3 - t\sqrt{t})^{0.6}\,dt$

25. $\int \frac{r\sqrt{\ln (r^2 + 1)}}{r^2 + 1}\,dr$

26. $\int \frac{9x^5 - 6x^4 - ex^3}{7x^2}\,dx$

27. $\int \frac{3^{\ln x}}{x}\,dx$

28. $\int \frac{4}{x \ln (2x^2)}\,dx$

29. $\int x^2\sqrt{e^{x^3+1}}\,dx$

30. $\int \frac{ax + b}{cx + d}\,dx \quad c \neq 0$

31. $\int \frac{8}{(x + 3)\ln (x + 3)}\,dx$

32. $\int (e^{e^2} + x^e - 2x)\,dx$

33. $\int \frac{x^3 + x^2 - x - 3}{x^2 - 3}\,dx$

34. $\int \frac{4x \ln \sqrt{1 + x^2}}{1 + x^2}\,dx$

35. $\int \frac{12x^3\sqrt{\ln (x^4 + 1)^3}}{x^4 + 1}\,dx$

36. $\int 3(x^2 + 2)^{-1/2} x e^{\sqrt{x^2+2}}\,dx$

37. $\int \left(\frac{x^3 - 1}{\sqrt{x^4 - 4x}} - \ln 7\right)\,dx$

38. $\int \frac{x - x^{-2}}{x^2 + 2x^{-1}}\,dx$

39. $\int \frac{2x^4 - 8x^3 - 6x^2 + 4}{x^3}\,dx$

40. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

41. $\int \frac{x}{x + 1}\,dx$

42. $\int \frac{2x}{(x^2 + 1)\ln (x^2 + 1)}\,dx$

43. $\int \frac{xe^{x^2}}{\sqrt{e^{x^2} + 2}}\,dx$

44. $\int \frac{5}{(3x + 1)[1 + \ln (3x + 1)]^2}\,dx$

45. $\int \frac{(e^{-x} + 5)^3}{e^x}\,dx$

46. $\int \left[\frac{1}{8x + 1} - \frac{1}{e^x(8 + e^{-x})^2}\right]\,dx$

47. $\int (x^3 + ex)\sqrt{x^2 + e}\,dx$

48. $\int 3^{x \ln x}(1 + \ln x)\,dx$ [*Hint:* $\frac{d}{dx}(x \ln x) = 1 + \ln x$]

49. $\int \sqrt{x}\sqrt{(8x)^{3/2} + 3}\,dx$

50. $\int \frac{7}{x(\ln x)^{\pi}}\,dx$

51. $\int \frac{\sqrt{s}}{e^{\sqrt{s^3}}}\,ds$

52. $\int \frac{\ln^3 x}{3x}\,dx$

53. $\int e^{\ln (x^2+1)}\,dx$

54. $\int dx$

55. $\int \frac{\ln\left(\frac{e^x}{x}\right)}{x}\,dx$

56. $\int e^{f(x)+\ln(f'(x))}\,dx$ assuming $f' > 0$

In Problems 57 and 58, dr/dq is a marginal-revenue function. Find the demand function.

57. $\frac{dr}{dq} = \frac{200}{(q+2)^2}$

58. $\frac{dr}{dq} = \frac{900}{(2q+3)^3}$

In Problems 59 and 60, dc/dq is a marginal-cost function. Find the total-cost function if fixed costs in each case are 2000.

59. $\frac{dc}{dq} = \frac{20}{q+5}$

60. $\frac{dc}{dq} = 4e^{0.005q}$

In Problems 61–63, dC/dI represents the marginal propensity to consume. Find the consumption function subject to the given condition.

61. $\frac{dC}{dI} = \frac{1}{\sqrt{I}}$; $C(9) = 8$

62. $\frac{dC}{dI} = \frac{1}{2} - \frac{1}{2\sqrt{2I}}$; $C(2) = \frac{3}{4}$

63. $\frac{dC}{dI} = \frac{3}{4} - \frac{1}{6\sqrt{I}}$; $C(25) = 23$

64. Cost Function The marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = 10 - \frac{100}{q+10}$$

where c is the total cost in dollars when q units are produced. When 100 units are produced, the average cost is \$50 per unit. To the nearest dollar, determine the manufacturer's fixed cost.

65. Cost Function Suppose the marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = \frac{100q^2 - 3998q + 60}{q^2 - 40q + 1}$$

where c is the total cost in dollars when q units are produced.

(a) Determine the marginal cost when 40 units are produced.
(b) If fixed costs are \$10,000, find the total cost of producing 40 units.
(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 42 units.

66. Cost Function The marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = \frac{9}{10}\sqrt{q}\sqrt{0.04q^{3/4} + 4}$$

where c is the total cost in dollars when q units are produced. Fixed costs are \$360.

(a) Determine the marginal cost when 25 units are produced.
(b) Find the total cost of producing 25 units.
(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 23 units.

67. Value of Land It is estimated that t years from now the value V (in dollars) of an acre of land near the ghost town of Cherokee, California, will be increasing at the rate of $\frac{8t^3}{\sqrt{0.2t^4 + 8000}}$ dollars per year. If the land is currently worth \$500 per acre, how much will it be worth in 10 years? Express your answer to the nearest dollar.

68. Revenue Function The marginal-revenue function for a manufacturer's product is of the form

$$\frac{dr}{dq} = \frac{a}{e^q + b}$$

for constants a and b, where r is the total revenue received (in dollars) when q units are produced and sold. Find the demand function, and express it in the form $p = f(q)$. (*Hint:* Rewrite dr/dq by multiplying both numerator and denominator by e^{-q}.)

69. Savings A certain country's marginal propensity to save is given by

$$\frac{dS}{dI} = \frac{5}{(I+2)^2}$$

where S and I represent total national savings and income, respectively, and are measured in billions of dollars. If total national consumption is \$7.5 billion when total national income is \$8 billion, for what value(s) of I is total national savings equal to zero?

70. Consumption Function A certain country's marginal propensity to save is given by

$$\frac{dS}{dI} = \frac{2}{5} - \frac{1.6}{\sqrt[3]{2I^2}}$$

where S and I represent total national savings and income, respectively, and are measured in billions of dollars.

(a) Determine the marginal propensity to consume when total national income is \$16 billion.
(b) Determine the consumption function, given that savings are \$10 billion when total national income is \$54 billion.
(c) Use the result in part (b) to show that consumption is $\$\frac{82}{5} = 16.4$ billion when total national income is \$16 billion (a deficit situation).
(d) Use differentials and the results in parts (a) and (c) to approximate consumption when total national income is \$18 billion.

14.6 The Definite Integral

Objective

To motivate, by means of the concept of area, the definite integral as a limit of a special sum; to evaluate simple definite integrals by using a limiting process.

Figure 14.2 shows the region R bounded by the lines $y = f(x) = 2x$, $y = 0$ (the x-axis), and $x = 1$. The region is simply a right triangle. If b and h are the lengths of the base and the height, respectively, then, from geometry, the area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(1)(2) = 1$ square unit. (Henceforth, we will treat areas as pure numbers and write *square unit* only if it seems necessary for emphasis.) We will now find this area by another method, which, as we will see later, applies to more complex regions. This method involves the summation of areas of rectangles.

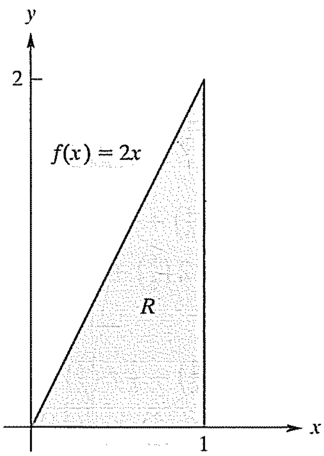


FIGURE 14.2 Region bounded by $f(x) = 2x, y = 0$, and $x = 1$.

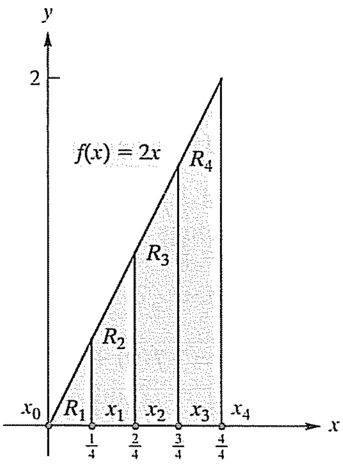


FIGURE 14.3 Four subregions of R.

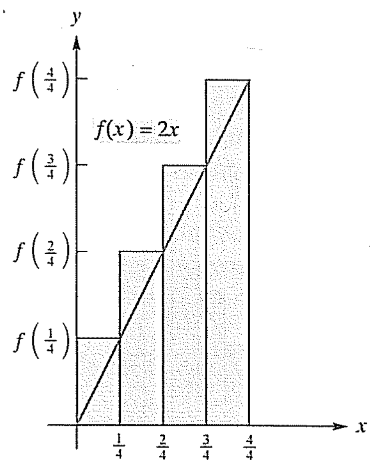


FIGURE 14.4 Four circumscribed rectangles.

Let us divide the interval [0, 1] on the x-axis into four subintervals of equal length by means of the equally spaced points $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4}$, and $x_4 = \frac{4}{4} = 1$. (See Figure 14.3.) Each subinterval has length $\Delta x = \frac{1}{4}$. These subintervals determine four subregions of R: R_1, R_2, R_3, and R_4, as indicated.

With each subregion, we can associate a *circumscribed* rectangle (Figure 14.4)—that is, a rectangle whose base is the corresponding subinterval and whose height is the *maximum* value of $f(x)$ on that subinterval. Since f is an increasing function, the maximum value of $f(x)$ on each subinterval occurs when x is the right-hand endpoint. Thus, the areas of the circumscribed rectangles associated with regions R_1, R_2, R_3, and R_4 are $\frac{1}{4}f(\frac{1}{4}), \frac{1}{4}f(\frac{2}{4}), \frac{1}{4}f(\frac{3}{4})$, and $\frac{1}{4}f(\frac{4}{4})$, respectively. The area of each rectangle is an approximation to the area of its corresponding subregion. Hence, the sum of the areas of these rectangles, denoted by $\overline{S}_4$ (read "S upper bar sub 4" or "the fourth upper sum"), approximates the area A of the triangle. We have

$$\begin{aligned}\overline{S}_4 &= \tfrac{1}{4}f\left(\tfrac{1}{4}\right) + \tfrac{1}{4}f\left(\tfrac{2}{4}\right) + \tfrac{1}{4}f\left(\tfrac{3}{4}\right) + \tfrac{1}{4}f\left(\tfrac{4}{4}\right)\\ &= \tfrac{1}{4}\left(2\left(\tfrac{1}{4}\right) + 2\left(\tfrac{2}{4}\right) + 2\left(\tfrac{3}{4}\right) + 2\left(\tfrac{4}{4}\right)\right) = \tfrac{5}{4}\end{aligned}$$

You can verify that $\overline{S}_4 = \sum_{i=1}^{4} f(x_i)\Delta x$. The fact that $\overline{S}_4$ is greater than the actual area of the triangle might have been expected, since $\overline{S}_4$ includes areas of shaded regions that are not in the triangle. (See Figure 14.4.)

On the other hand, with each subregion we can also associate an *inscribed* rectangle (Figure 14.5)—that is, a rectangle whose base is the corresponding subinterval, but whose height is the *minimum* value of $f(x)$ on that subinterval. Since f is an increasing function, the minimum value of $f(x)$ on each subinterval will occur when x is the left-hand endpoint. Thus, the areas of the four inscribed rectangles associated with R_1, R_2, R_3, and R_4 are $\frac{1}{4}f(0), \frac{1}{4}f(\frac{1}{4}), \frac{1}{4}f(\frac{2}{4})$, and $\frac{1}{4}f(\frac{3}{4})$, respectively. Their sum, denoted $\underline{S}_4$ (read "S lower bar sub 4" or "the fourth lower sum"), is also an approximation to the area A of the triangle. We have

$$\begin{aligned}\underline{S}_4 &= \tfrac{1}{4}f(0) + \tfrac{1}{4}f\left(\tfrac{1}{4}\right) + \tfrac{1}{4}f\left(\tfrac{2}{4}\right) + \tfrac{1}{4}f\left(\tfrac{3}{4}\right)\\ &= \tfrac{1}{4}\left(2(0) + 2\left(\tfrac{1}{4}\right) + 2\left(\tfrac{2}{4}\right) + 2\left(\tfrac{3}{4}\right)\right) = \tfrac{3}{4}\end{aligned}$$

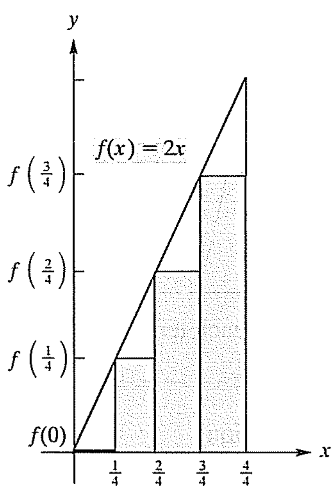


FIGURE 14.5 Four inscribed rectangles.

Using summation notation, we can write $\underline{S}_4 = \sum_{i=0}^{3} f(x_i)\Delta x$. Note that $\underline{S}_4$ is less than the area of the triangle, because the rectangles do not account for the portion of the triangle that is not shaded in Figure 14.5.

Since

$$\frac{3}{4} = \underline{S}_4 \leq A \leq \overline{S}_4 = \frac{5}{4}$$

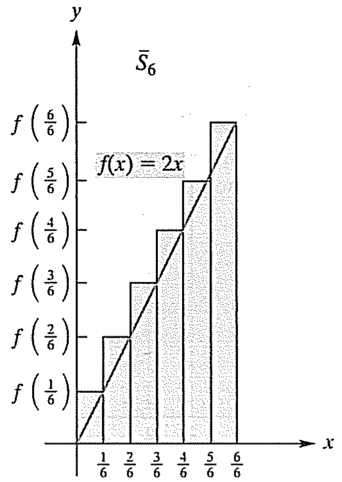


FIGURE 14.6 Six circumscribed rectangles.

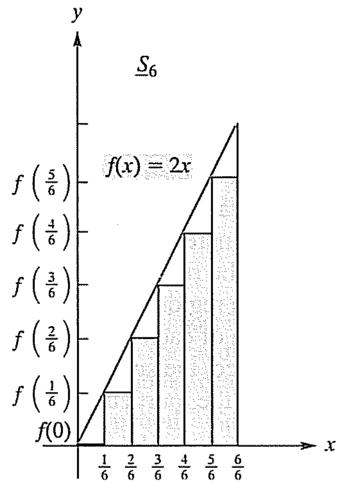


FIGURE 14.7 Six inscribed rectangles.

TO REVIEW summation notation, refer to Section 1.5.

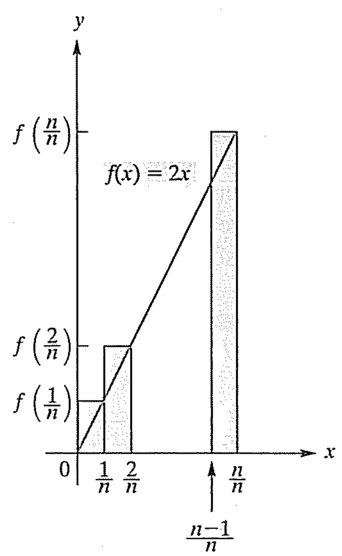


FIGURE 14.8 n circumscribed rectangles.

we say that $\underline{S}_4$ is an approximation to A from *below* and $\overline{S}_4$ is an approximation to A from *above*.

If [0, 1] is divided into more subintervals, we expect that better approximations to A will occur. To test this, let us use six subintervals of equal length $\Delta x = \frac{1}{6}$. Then $\overline{S}_6$, the total area of six circumscribed rectangles (see Figure 14.6), and $\underline{S}_6$, the total area of six inscribed rectangles (see Figure 14.7), are

$$\overline{S}_6 = \tfrac{1}{6}f\left(\tfrac{1}{6}\right) + \tfrac{1}{6}f\left(\tfrac{2}{6}\right) + \tfrac{1}{6}f\left(\tfrac{3}{6}\right) + \tfrac{1}{6}f\left(\tfrac{4}{6}\right) + \tfrac{1}{6}f\left(\tfrac{5}{6}\right) + \tfrac{1}{6}f\left(\tfrac{6}{6}\right)$$
$$= \tfrac{1}{6}\left(2\left(\tfrac{1}{6}\right) + 2\left(\tfrac{2}{6}\right) + 2\left(\tfrac{3}{6}\right) + 2\left(\tfrac{4}{6}\right) + 2\left(\tfrac{5}{6}\right) + 2\left(\tfrac{6}{6}\right)\right) = \tfrac{7}{6}$$

and

$$\underline{S}_6 = \tfrac{1}{6}f(0) + \tfrac{1}{6}f\left(\tfrac{1}{6}\right) + \tfrac{1}{6}f\left(\tfrac{2}{6}\right) + \tfrac{1}{6}f\left(\tfrac{3}{6}\right) + \tfrac{1}{6}f\left(\tfrac{4}{6}\right) + \tfrac{1}{6}f\left(\tfrac{5}{6}\right)$$
$$= \tfrac{1}{6}\left(2(0) + 2\left(\tfrac{1}{6}\right) + 2\left(\tfrac{2}{6}\right) + 2\left(\tfrac{3}{6}\right) + 2\left(\tfrac{4}{6}\right) + 2\left(\tfrac{5}{6}\right)\right) = \tfrac{5}{6}$$

Note that $\underline{S}_6 \le A \le \overline{S}_6$, and, with appropriate labeling, both $\overline{S}_6$ and $\underline{S}_6$ will be of the form $\Sigma f(x)\,\Delta x$. Clearly, using six subintervals gives better approximations to the area than does four subintervals, as expected.

More generally, if we divide [0, 1] into n subintervals of equal length Δx, then $\Delta x = 1/n$, and the endpoints of the subintervals are $x = 0, 1/n, 2/n, \ldots, (n-1)/n$, and $n/n = 1$. (See Figure 14.8.) The endpoints of the kth subinterval, for $k = 1, \ldots n$, are $(k-1)/n$ and k/n and the maximum value of f occurs at the right-hand endpoint k/n. It follows that the area of the kth circumscribed rectangle is $1/n \cdot f(k/n) = 1/n \cdot 2(k/n) = 2k/n^2$, for $k = 1, \ldots, n$. The total area of *all n circumscribed* rectangles is

$$\overline{S}_n = \sum_{k=1}^{n} f(k/n)\Delta x = \sum_{k=1}^{n} \frac{2k}{n^2} \tag{1}$$

$$= \frac{2}{n^2}\sum_{k=1}^{n} k \qquad \text{by factoring } \frac{2}{n^2} \text{ from each term}$$

$$= \frac{2}{n^2} \cdot \frac{n(n+1)}{2} \qquad \text{from Section 1.5}$$

$$= \frac{n+1}{n}$$

(We recall that $\sum_{k=1}^{n} k = 1+2+\cdots+n$ is the sum of the first n positive integers and the formula used above was derived in Section 1.5 in anticipation of its application here.)

For *inscribed* rectangles, we note that the minimum value of f occurs at the left-hand endpoint, $(k-1)/n$, of $[(k-1)/n, k/n]$, so that the area of the kth inscribed rectangle is $1/n \cdot f(k-1/n) = 1/n \cdot 2((k-1)/n) = 2(k-1)/n^2$, for $k = 1, \ldots n$. The total area determined of *all n inscribed* rectangles (see Figure 14.9) is

$$\underline{S}_n = \sum_{k=1}^{n} f((k-1)/n)\,\Delta x = \sum_{k=1}^{n} \frac{2(k-1)}{n^2} \tag{2}$$

$$= \frac{2}{n^2}\sum_{k=1}^{n} k-1 \qquad \text{by factoring } \frac{2}{n^2} \text{ from each term}$$

$$= \frac{2}{n^2}\sum_{k=0}^{n-1} k \qquad \text{adjusting the summation}$$

$$= \frac{2}{n^2} \cdot \frac{(n-1)n}{2} \qquad \text{adapted from Section 1.5}$$

$$= \frac{n-1}{n}$$

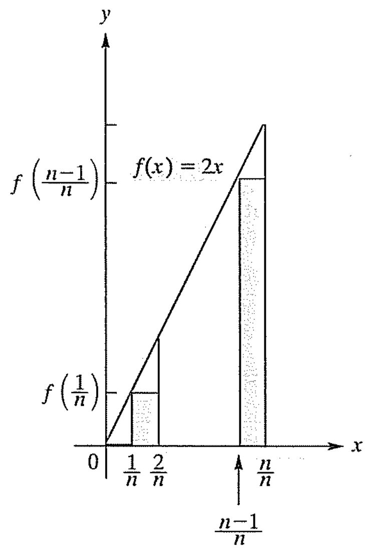


FIGURE 14.9 n inscribed rectangles.

From Equations (1) and (2), we again see that both $\overline{S}_n$ and $\underline{S}_n$ are sums of the form $\sum f(x)\Delta x$, namely, $\overline{S}_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\Delta x$ and $\underline{S}_n = \sum_{k=1}^{n} f\left(\frac{k-1}{n}\right)\Delta x$.

From the nature of $\overline{S}_n$ and $\underline{S}_n$, it seems reasonable—and it is indeed true—that

$$\underline{S}_n \leq A \leq \overline{S}_n$$

As n becomes larger, $\underline{S}_n$ and $\overline{S}_n$ become better approximations to A. In fact, let us take the limits of $\underline{S}_n$ and $\overline{S}_n$ as n approaches ∞ through positive integral values:

$$\lim_{n\to\infty} \underline{S}_n = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1$$

$$\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$$

Since $\overline{S}_n$ and $\underline{S}_n$ have the same limit, namely,

$$\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \underline{S}_n = 1 \tag{3}$$

and since

$$\underline{S}_n \leq A \leq \overline{S}_n$$

we will take this limit to be the area of the triangle. Thus $A = 1$, which agrees with our prior finding. It is important to understand that here we developed a *definition of the notion of area* that is applicable to many different regions.

We call the common limit of $\overline{S}_n$ and $\underline{S}_n$, namely, 1, the *definite integral* of $f(x) = 2x$ on the interval from $x = 0$ to $x = 1$, and we denote this quantity by writing

$$\int_0^1 2x\,dx = 1 \tag{4}$$

The reason for using the term *definite integral* and the symbolism in Equation (4) will become apparent in the next section. The numbers 0 and 1 appearing with the integral sign $\int$ in Equation (4) are called the *limits of integration;* 0 is the *lower limit* and 1 is the *upper limit.*

In general, for a function f defined on the interval from $x = a$ to $x = b$, where $a < b$, we can form the sums $\overline{S}_n$ and $\underline{S}_n$, which are obtained by considering the maximum and minimum values, respectively, on each of n subintervals of equal length Δx.[6] We can now state the following:

> The common limit of $\overline{S}_n$ and $\underline{S}_n$ as $n \to \infty$, if it exists, is called the **definite integral** of f over $[a, b]$ and is written
>
> $$\int_a^b f(x)\,dx$$
>
> The numbers a and b are called **limits of integration;** a is the **lower limit** and b is the **upper limit.** The symbol x is called the **variable of integration** and $f(x)$ is the **integrand.**

In terms of a limiting process, we have

$$\sum f(x)\,\Delta x \to \int_a^b f(x)\,dx$$

The definite integral is the limit of sums of the form $\sum f(x)\,\Delta x$. This definition will be useful in later sections.

Two points must be made about the definite integral. First, the definite integral is the limit of a sum of the form $\sum f(x)\,\Delta x$. In fact, we can think of the integral sign as an elongated "*S*," the first letter of "*S*ummation." Second, for an arbitrary function f

[6]Here we assume that the maximum and minimum values exist.

defined on an interval, we may be able to calculate the sums $\overline{S}_n$ and $\underline{S}_n$ and determine their common limit if it exists. However, some terms in the sums may be negative if $f(x)$ is negative at points in the interval. These terms are not areas of rectangles (an area is never negative), so the common limit may not represent an area. Thus, **the definite integral is nothing more than a real number; it may or may not represent an area.**

As we saw in Equation (3), $\lim_{n\to\infty} \underline{S}_n$ is equal to $\lim_{n\to\infty} \overline{S}_n$. For an arbitrary function, this is not always true. However, for the functions that we will consider, these limits will be equal, and the definite integral will always exist. To save time, we will just use the **right-hand endpoint** of each subinterval in computing a sum. For the functions in this section, this sum will be denoted S_n.

APPLY IT ▷

10. A company has determined that its marginal-revenue function is given by $R'(x) = 600 - 0.5x$, where R is the revenue (in dollars) received when x units are sold. Find the total revenue received for selling 10 units by finding the area in the first quadrant bounded by $y = R'(x) = 600 - 0.5x$ and the lines $y = 0, x = 0$, and $x = 10$.

In general, over $[a, b]$, we have

$$\Delta x = \frac{b-a}{n}$$

EXAMPLE 1 Computing an Area by Using Right-Hand Endpoints

Find the area of the region in the first quadrant bounded by $f(x) = 4 - x^2$ and the lines $x = 0$ and $y = 0$.

Solution: A sketch of the region appears in Figure 14.10. The interval over which x varies in this region is seen to be $[0, 2]$, which we divide into n subintervals of equal length Δx. Since the length of $[0, 2]$ is 2, we take $\Delta x = 2/n$. The endpoints of the subintervals are $x = 0, 2/n, 2(2/n), \ldots, (n-1)(2/n)$, and $n(2/n) = 2$, which are shown in Figure 14.11. The diagram also shows the corresponding rectangles obtained by using the right-hand endpoint of each subinterval. The area of the kth rectangle, for $k = 1, \ldots n$, is the product of its width, $2/n$, and its height, $f(k(2/n)) = 4 - (2k/n)^2$, which is the function value at the right-hand endpoint of its base. Summing these areas, we get

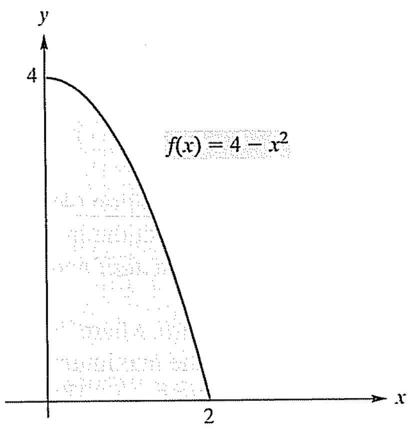


FIGURE 14.10 Region of Example 1.

$$\begin{aligned} S_n &= \sum_{k=1}^{n} f\left(k \cdot \left(\frac{2}{n}\right)\right) \Delta x = \sum_{k=1}^{n} \left(4 - \left(\frac{2k}{n}\right)^2\right) \frac{2}{n} \\ &= \sum_{k=1}^{n} \left(\frac{8}{n} - \frac{8k^2}{n^3}\right) = \sum_{k=1}^{n} \frac{8}{n} - \sum_{k=1}^{n} \frac{8k^2}{n^3} = \frac{8}{n}\sum_{k=1}^{n} 1 - \frac{8}{n^3}\sum_{k=1}^{n} k^2 \\ &= \frac{8}{n}n - \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} \\ &= 8 - \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right) \end{aligned}$$

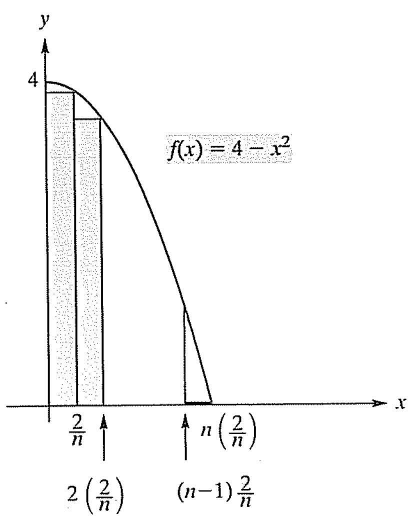


FIGURE 14.11 n subintervals and corresponding rectangles for Example 1.

The second line of the preceding computations uses basic summation manipulations as discussed in Section 1.5. The third line uses two specific summation formulas, also from Section 1.5: The sum of n copies of 1 is n and the sum of the first n squares is $\frac{n(n+1)(2n+1)}{6}$.

Finally, we take the limit of the S_n as $n \to \infty$:

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(8 - \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)\right) \\ &= 8 - \frac{4}{3}\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{n^2}\right) \\ &= 8 - \frac{4}{3}\lim_{n\to\infty}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\ &= 8 - \frac{8}{3} = \frac{16}{3} \end{aligned}$$

Hence, the area of the region is $\frac{16}{3}$.

Now Work Problem 7 ◁

EXAMPLE 2 Evaluating a Definite Integral

Evaluate $\int_0^2 (4 - x^2)\,dx$.

Solution: We want to find the definite integral of $f(x) = 4 - x^2$ over the interval [0, 2]. Thus, we must compute $\lim_{n\to\infty} S_n$. But this limit is precisely the limit $\frac{16}{3}$ found in Example 1, so we conclude that

No units are attached to the answer, since a definite integral is simply a number.

$$\int_0^2 (4 - x^2)\,dx = \frac{16}{3}$$

Now Work Problem 19 ◁

EXAMPLE 3 Integrating a Function over an Interval

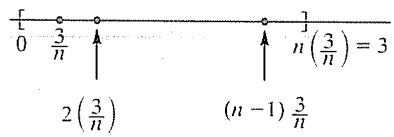


FIGURE 14.12 Dividing [0, 3] into n subintervals.

Integrate $f(x) = x - 5$ from $x = 0$ to $x = 3$; that is, evaluate $\int_0^3 (x - 5)\,dx$.

Solution: We first divide [0, 3] into n subintervals of equal length $\Delta x = 3/n$. The endpoints are $0, 3/n, 2(3/n), \ldots, (n-1)(3/n), n(3/n) = 3$. (See Figure 14.12.) Using right-hand endpoints, we form the sum and simplify

$$S_n = \sum_{k=1}^{n} f\left(k\frac{3}{n}\right)\frac{3}{n}$$

$$= \sum_{k=1}^{n}\left(\left(k\frac{3}{n} - 5\right)\frac{3}{n}\right) = \sum_{k=1}^{n}\left(\frac{9}{n^2}k - \frac{15}{n}\right) = \frac{9}{n^2}\sum_{k=1}^{n} k - \frac{15}{n}\sum_{k=1}^{n} 1$$

$$= \frac{9}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{15}{n}(n)$$

$$= \frac{9}{2}\frac{n+1}{n} - 15 = \frac{9}{2}\left(1 + \frac{1}{n}\right) - 15$$

Taking the limit, we obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{9}{2}\left(1 + \frac{1}{n}\right) - 15\right) = \frac{9}{2} - 15 = -\frac{21}{2}$$

Thus,

$$\int_0^3 (x - 5)\,dx = -\frac{21}{2}$$

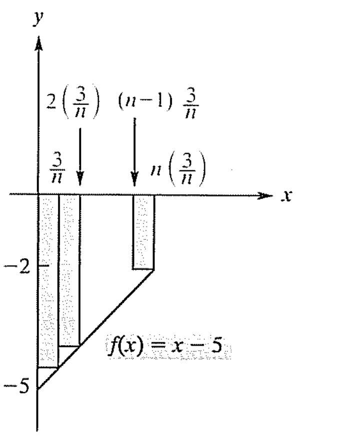


FIGURE 14.13 $f(x)$ is negative at each right-hand endpoint.

Note that the definite integral here is a *negative* number. The reason is clear from the graph of $f(x) = x - 5$ over the interval [0, 3]. (See Figure 14.13.) Since the value of $f(x)$ is negative at each right-hand endpoint, each term in S_n must also be negative. Hence, $\lim_{n\to\infty} S_n$, which is the definite integral, is negative.

Geometrically, each term in S_n is the negative of the area of a rectangle. (Refer again to Figure 14.13.) Although the definite integral is simply a number, here we can interpret it as representing the negative of the area of the region bounded by $f(x) = x - 5, x = 0, x = 3$, and the x-axis ($y = 0$).

Now Work Problem 17 ◁

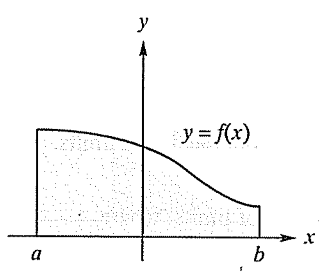


FIGURE 14.14 If f is continuous and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ represents the area under the curve.

In Example 3, it was shown that *the definite integral does not have to represent an area*. In fact, there the definite integral was negative. However, if f is continuous and $f(x) \geq 0$ on $[a, b]$, then $S_n \geq 0$ for all values of n. Therefore, $\lim_{n\to\infty} S_n \geq 0$, so $\int_a^b f(x)\,dx \geq 0$. Furthermore, this definite integral gives the area of the region bounded by $y = f(x), y = 0, x = a$, and $x = b$. (See Figure 14.14.)

Although the approach that we took to discuss the definite integral is sufficient for our purposes, it is by no means rigorous. **The important thing to remember about the definite integral is that it is the limit of a special sum.**

TECHNOLOGY

Here is a program for the TI-83 Plus graphing calculator that will estimate the limit of S_n as $n \to \infty$ for a function f defined on $[a, b]$.

```
PROGRAM:RIGHTSUM
Lbl 1
Input "SUBINTV",N
(B − A)/N → H
Ø → S
A + H → X
1 → I
Lbl 2
Y1 + S → S
X + H → X
I + 1 → I
If I ≤ N
Goto 2
H*S → S
Disp S
Pause
Goto 1
```

RIGHTSUM will compute S_n for a given number n of subintervals. Before executing the program, store $f(x)$, a, and b as Y_1, A, and B, respectively. Upon execution of the program, you will be prompted to enter the number of subintervals. Then the program proceeds to display the value of S_n. Each time ENTER is pressed, the program repeats. In this way, a display of values of S_n for various numbers of subintervals may be obtained. Figure 14.15 shows values of $S_n (n = 100, 1000, \text{and } 2000)$ for the function $f(x) = x - 5$ on the interval $[0, 3]$. As $n \to \infty$, it appears that $S_n \to -10.5$. Thus, we estimate that

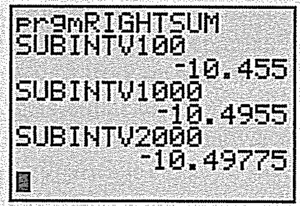


FIGURE 14.15 Values of S_n for $f(x) = x - 5$ on $[0, 3]$.

$$\lim_{n\to\infty} S_n \approx -10.5$$

Equivalently,

$$\int_0^3 (x-5)\,dx \approx -10.5$$

which agrees with our result in Example 3.

It is interesting to note that the time required for an older calculator to compute S_{2000} in Figure 14.15 was in excess of 1.5 minutes. The time required on a TI-84 Plus is less than 1 minute.

PROBLEMS 14.6

In Problems 1–4, sketch the region in the first quadrant that is bounded by the given curves. Approximate the area of the region by the indicated sum. Use the right-hand endpoint of each subinterval.

1. $f(x) = x + 1, y = 0, x = 0, x = 1;\quad S_4$

2. $f(x) = 3x, y = 0, x = 1;\quad S_5$

3. $f(x) = x^2, y = 0, x = 1;\quad S_4$

4. $f(x) = x^2 + 1, y = 0, x = 0, x = 1;\quad S_2$

In Problems 5 and 6, by dividing the indicated interval into n subintervals of equal length, find S_n for the given function. Use the right-hand endpoint of each subinterval. Do not find $\lim_{n\to\infty} S_n$.

5. $f(x) = 4x;\quad [0, 1]$

6. $f(x) = 2x + 1;\quad [0, 2]$

In Problems 7 and 8, (a) simplify S_n and (b) find $\lim_{n\to\infty} S_n$.

7. $S_n = \frac{1}{n}\left[\left(\frac{1}{n}+1\right)+\left(\frac{2}{n}+1\right)+\cdots+\left(\frac{n}{n}+1\right)\right]$

8. $S_n = \frac{2}{n}\left[\left(\frac{2}{n}\right)^2+\left(2\cdot\frac{2}{n}\right)^2+\cdots+\left(n\cdot\frac{2}{n}\right)^2\right]$

In Problems 9–14, sketch the region in the first quadrant that is bounded by the given curves. Determine the exact area of the region by considering the limit of S_n as $n \to \infty$. Use the right-hand endpoint of each subinterval.

9. Region as described in Problem 1

10. Region as described in Problem 2

11. Region as described in Problem 3

12. $y = x^2, y = 0, x = 1, x = 2$

13. $f(x) = 3x^2, y = 0, x = 1$

14. $f(x) = 9 - x^2, y = 0, x = 0$

In Problems 15–20, evaluate the given definite integral by taking the limit of S_n. Use the right-hand endpoint of each subinterval. Sketch the graph, over the given interval, of the function to be integrated.

15. $\int_1^3 5x\,dx$

16. $\int_0^a b\,dx$

17. $\int_0^3 -4x\,dx$

18. $\int_1^4 (2x+1)\,dx$

19. $\int_0^1 (x^2+x)\,dx$

20. $\int_1^2 (x+2)\,dx$

21. Find $\frac{d}{dx}\left(\int_0^1 \sqrt{1-x^2}\,dx\right)$ without the use of limits.

22. Find $\int_0^3 f(x)\,dx$ without the use of limits, where

$$f(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 4 - 2x & \text{if } 1 \le x < 2 \\ 5x - 10 & \text{if } 2 \le x \le 3 \end{cases}$$

23. Find $\int_{-1}^{3} f(x)\,dx$ without the use of limits, where

$$f(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ 2 - x & \text{if } 1 \leq x \leq 2 \\ -1 + \dfrac{x}{2} & \text{if } x > 2 \end{cases}$$

In each of Problems 24–26, use a program, such as ***RIGHTSUM,*** *to estimate the area of the region in the first quadrant bounded by the given curves. Round your answer to one decimal place.*

24. $f(x) = x^3 + 1, y = 0, x = 2, x = 3.7$

25. $f(x) = 4 - \sqrt{x}, y = 0, x = 1, x = 9$

26. $f(x) = \ln x, y = 0, x = 1, x = 2$

In each of Problems 27–30, use a program, such as ***RIGHTSUM,*** *to estimate the value of the definite integral. Round your answer to one decimal place.*

27. $\int_{2}^{5} \frac{x+1}{x+2}\,dx$

28. $\int_{-3}^{-1} \frac{1}{x^2}\,dx$

29. $\int_{-1}^{2} (4x^2 + x - 13)\,dx$

30. $\int_{1}^{2} \ln x\,dx$

Objective

To develop informally the Fundamental Theorem of Integral Calculus and to use it to compute definite integrals.

14.7 The Fundamental Theorem of Integral Calculus

The Fundamental Theorem

Thus far, the limiting processes of both the derivative and definite integral have been considered separately. We will now bring these fundamental ideas together and develop the important relationship that exists between them. As a result, we will be able to evaluate definite integrals more efficiently.

The graph of a function f is given in Figure 14.16. Assume that f is continuous on the interval $[a, b]$ and that its graph does not fall below the x-axis. That is, $f(x) \geq 0$. From the preceding section, the area of the region below the graph and above the x-axis from $x = a$ to $x = b$ is given by $\int_a^b f(x)\,dx$. We will now consider another way to determine this area.

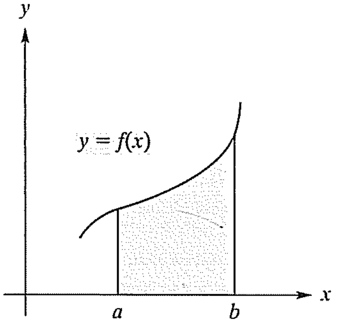


FIGURE 14.16 On $[a, b]$, f is continuous and $f(x) \geq 0$.

Suppose that there is a function $A = A(x)$, which we will refer to as an area function, that gives the area of the region below the graph of f and above the x-axis from a to x, where $a \leq x \leq b$. This region is shaded in Figure 14.17. Do not confuse $A(x)$, which is an area, with $f(x)$, which is the height of the graph at x.

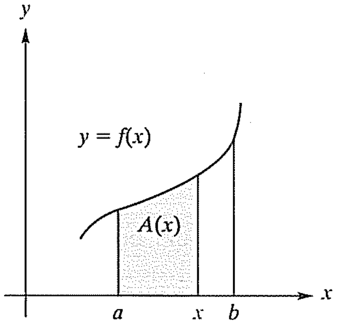


FIGURE 14.17 $A(x)$ is an area function.

From its definition, we can state two properties of A immediately:

1. $A(a) = 0$, since there is "no area" from a to a
2. $A(b)$ is the area from a to b; that is,

$$A(b) = \int_a^b f(x)\,dx$$

If x is increased by h units, then $A(x + h)$ is the area of the shaded region in Figure 14.18. Hence, $A(x+h) - A(x)$ is the difference of the areas in Figures 14.18 and 14.17, namely, the area of the shaded region in Figure 14.19. For h sufficiently close to zero, the area of this region is the same as the area of a rectangle (Figure 14.20) whose base is h and whose height is some value $\bar{y}$ between $f(x)$ and $f(x + h)$. Here $\bar{y}$ is a function of h. Thus, on the one hand, the area of the rectangle is $A(x + h) - A(x)$, and, on the other hand, it is $h\bar{y}$, so

$$A(x + h) - A(x) = h\bar{y}$$

Equivalently,

$$\frac{A(x + h) - A(x)}{h} = \bar{y} \qquad \text{dividing by } h$$

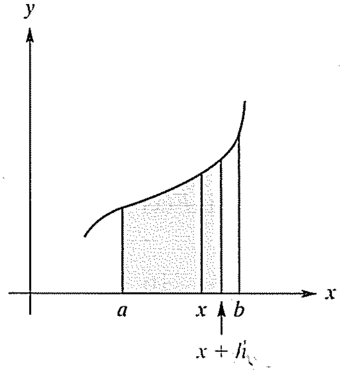


FIGURE 14.18 $A(x + h)$ gives the area of the shaded region.

Since $\bar{y}$ is between $f(x)$ and $f(x + h)$, it follows that as $h \to 0$, $\bar{y}$ approaches $f(x)$, so

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x) \tag{1}$$

But the left side is merely the derivative of A. Thus, Equation (1) becomes

$$A'(x) = f(x)$$

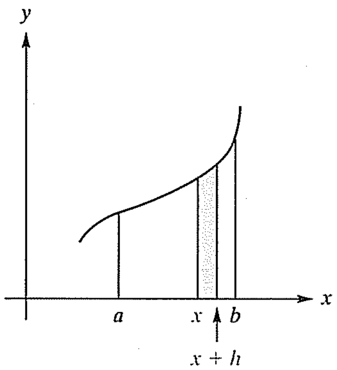


FIGURE 14.19 Area of shaded region is $A(x+h) - A(x)$.

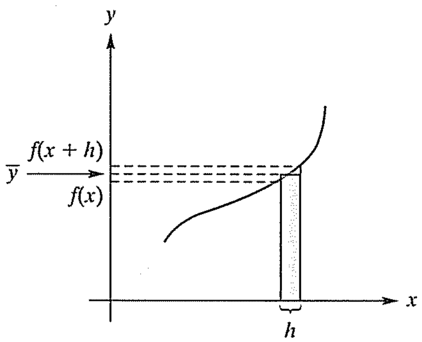


FIGURE 14.20 Area of rectangle is the same as area of shaded region in Figure 14.19.

We conclude that the area function A has the additional property that its derivative A' is f. That is, A is an antiderivative of f. Now, suppose that F is *any* antiderivative of f. Then, since both A and F are antiderivatives of the same function, they differ at most by a constant C:

$$A(x) = F(x) + C. \tag{2}$$

Recall that $A(a) = 0$. So, evaluating both sides of Equation (2) when $x = a$ gives

$$0 = F(a) + C$$

so that

$$C = -F(a)$$

Thus, Equation (2) becomes

$$A(x) = F(x) - F(a) \tag{3}$$

If $x = b$, then, from Equation (3),

$$A(b) = F(b) - F(a) \tag{4}$$

But recall that

$$A(b) = \int_a^b f(x)\,dx \tag{5}$$

From Equations (4) and (5), we get

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

A relationship between a definite integral and antidifferentiation has now become clear. To find $\int_a^b f(x)\,dx$, it suffices to find an antiderivative of f, say, F, and subtract the value of F at the lower limit a from its value at the upper limit b. We assumed here that f was continuous and $f(x) \geq 0$ so that we could appeal to the concept of an area. However, our result is true for any continuous function[7] and is known as the *Fundamental Theorem of Integral Calculus*.

Fundamental Theorem of Integral Calculus

If f is continuous on the interval $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The definite integral is a number, and an indefinite integral is a function.

It is important that you understand the difference between a definite integral and an indefinite integral. The **definite integral** $\int_a^b f(x)\,dx$ is a **number** defined to be the limit of a sum. The Fundamental Theorem states that the **indefinite integral** $\int f(x)\,dx$ (the most general antiderivative of f), which is a **function** of x related to the differentiation process, can be used to determine this limit.

Suppose we apply the Fundamental Theorem to evaluate $\int_0^2 (4 - x^2)\,dx$. Here $f(x) = 4 - x^2$, $a = 0$, and $b = 2$. Since an antiderivative of $4 - x^2$ is $F(x) = 4x - (x^3/3)$, it follows that

$$\int_0^2 (4 - x^2)\,dx = F(2) - F(0) = \left(8 - \frac{8}{3}\right) - (0) = \frac{16}{3}$$

[7] If f is continuous on $[a, b]$, it can be shown that $\int_a^b f(x)\,dx$ does indeed exist.

This confirms our result in Example 2 of Section 14.6. If we had chosen $F(x)$ to be $4x - (x^3/3) + C$, then we would have

$$F(2) - F(0) = \left[\left(8 - \frac{8}{3}\right) + C\right] - [0 + C] = \frac{16}{3}$$

as before. Since the choice of the value of C is immaterial, for convenience we will always choose it to be 0, as originally done. Usually, $F(b) - F(a)$ is abbreviated by writing

$$F(b) - F(a) = F(x)\Big|_a^b$$

Since F in the Fundamental Theorem of Calculus is *any* antiderivative of f and $\int f(x)\,dx$ is the most general antiderivative of f, it showcases the notation to write

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$$

Using the $\Big|_a^b$ notation, we have

$$\int_0^2 (4 - x^2)\,dx = \left(4x - \frac{x^3}{3}\right)\Big|_0^2 = \left(8 - \frac{8}{3}\right) - 0 = \frac{16}{3}$$

APPLY IT ▷

11. The income (in dollars) from a fast-food chain is increasing at a rate of $f(t) = 10{,}000e^{0.02t}$, where t is in years. Find $\int_3^6 10{,}000e^{0.02t}\,dt$, which gives the total income for the chain between the third and sixth years.

EXAMPLE 1 **Applying the Fundamental Theorem**

Find $\int_{-1}^{3} (3x^2 - x + 6)\,dx$.

Solution: An antiderivative of $3x^2 - x + 6$ is

$$x^3 - \frac{x^2}{2} + 6x$$

Thus,

$$\begin{aligned}
&\int_{-1}^{3} (3x^2 - x + 6)\,dx \\
&= \left(x^3 - \frac{x^2}{2} + 6x\right)\Big|_{-1}^{3} \\
&= \left[3^3 - \frac{3^2}{2} + 6(3)\right] - \left[(-1)^3 - \frac{(-1)^2}{2} + 6(-1)\right] \\
&= \left(\frac{81}{2}\right) - \left(-\frac{15}{2}\right) = 48
\end{aligned}$$

Now Work Problem 1 ◁

Properties of the Definite Integral

For $\int_a^b f(x)\,dx$, we have assumed that $a < b$. We now define the cases in which $a > b$ or $a = b$. First,

$$\text{If } a > b, \quad \text{then} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

That is, interchanging the limits of integration changes the integral's sign. For example,

$$\int_2^0 (4 - x^2)\,dx = -\int_0^2 (4 - x^2)\,dx$$

If the limits of integration are equal, we have

$$\int_a^a f(x)\,dx = 0$$

Some properties of the definite integral deserve mention. The first of the properties that follow restates more formally our comment from the preceding section concerning area.

Properties of the Definite Integral

1. If f is continuous and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ can be interpreted as the area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

2. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is a constant

3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Properties 2 and 3 are similar to rules for indefinite integrals because a definite integral may be evaluated by the Fundamental Theorem in terms of an antiderivative. Two more properties of definite integrals are as follows.

4. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

The variable of integration is a "dummy variable" in the sense that any other variable produces the same result—that is, the same number.

To illustrate property 4, you can verify, for example, that

$$\int_0^2 x^2\,dx = \int_0^2 t^2\,dt$$

5. If f is continuous on an interval I and a, b, and c are in I, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Property 5 means that the definite integral over an interval can be expressed in terms of definite integrals over subintervals. Thus,

$$\int_0^2 (4 - x^2)\,dx = \int_0^1 (4 - x^2)\,dx + \int_1^2 (4 - x^2)\,dx$$

We will look at some examples of definite integration now and compute some areas in Section 14.9.

EXAMPLE 2 Using the Fundamental Theorem

Find $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1+x^4}}\,dx$.

Solution: To find an antiderivative of the integrand, we will apply the power rule for integration:

$$\int_0^1 \frac{x^3}{\sqrt{1+x^4}}\,dx = \int_0^1 x^3(1+x^4)^{-1/2}\,dx$$

$$= \frac{1}{4}\int_0^1 (1+x^4)^{-1/2}\,d(1+x^4) = \left(\frac{1}{4}\right)\frac{(1+x^4)^{1/2}}{\frac{1}{2}}\Bigg|_0^1$$

CAUTION!

In Example 2, the value of the antiderivative $\frac{1}{2}(1+x^4)^{1/2}$ at the lower limit 0 is $\frac{1}{2}(1)^{1/2}$. **Do not** assume that an evaluation at the limit zero will yield 0.

$$= \frac{1}{2}(1+x^4)^{1/2}\Big|_0^1 = \frac{1}{2}\left((2)^{1/2} - (1)^{1/2}\right)$$

$$= \frac{1}{2}(\sqrt{2} - 1)$$

Now Work Problem 13 ◁

EXAMPLE 3 Evaluating Definite Integrals

a. Find $\int_1^2 [4t^{1/3} + t(t^2+1)^3]\,dt$.

Solution:

$$\int_1^2 [4t^{1/3} + t(t^2+1)^3]\,dt = 4\int_1^2 t^{1/3}\,dt + \frac{1}{2}\int_1^2 (t^2+1)^3\,d(t^2+1)$$

$$= (4)\,\frac{t^{4/3}}{\frac{4}{3}}\Bigg|_1^2 + \left(\frac{1}{2}\right)\frac{(t^2+1)^4}{4}\Bigg|_1^2$$

$$= 3(2^{4/3} - 1) + \frac{1}{8}(5^4 - 2^4)$$

$$= 3\cdot 2^{4/3} - 3 + \frac{609}{8}$$

$$= 6\sqrt[3]{2} + \frac{585}{8}$$

b. Find $\int_0^1 e^{3t}\,dt$.

Solution:

$$\int_0^1 e^{3t}\,dt = \frac{1}{3}\int_0^1 e^{3t}\,d(3t)$$

$$= \left(\frac{1}{3}\right)e^{3t}\Big|_0^1 = \frac{1}{3}(e^3 - e^0) = \frac{1}{3}(e^3 - 1)$$

Now Work Problem 15 ◁

EXAMPLE 4 Finding and Interpreting a Definite Integral

Evaluate $\int_{-2}^1 x^3\,dx$.

Solution:

$$\int_{-2}^1 x^3\,dx = \frac{x^4}{4}\Big|_{-2}^1 = \frac{1^4}{4} - \frac{(-2)^4}{4} = \frac{1}{4} - \frac{16}{4} = -\frac{15}{4}$$

The reason the result is negative is clear from the graph of $y = x^3$ on the interval $[-2, 1]$. (See Figure 14.21.) For $-2 \le x < 0$, $f(x)$ is negative. Since a definite integral is a limit of a sum of the form $\Sigma f(x)\,\Delta x$, it follows that $\int_{-2}^0 x^3\,dx$ is not only a negative number, but also the negative of the area of the shaded region in the third quadrant. On the other hand, $\int_0^1 x^3\,dx$ is the area of the shaded region in the first quadrant, since $f(x) \ge 0$ on $[0, 1]$. The definite integral over the entire interval $[-2, 1]$ is the *algebraic* sum of these numbers, because, from property 5,

$$\int_{-2}^1 x^3\,dx = \int_{-2}^0 x^3\,dx + \int_0^1 x^3\,dx$$

Thus, $\int_{-2}^1 x^3\,dx$ does not represent the area between the curve and the x-axis. However, if area is desired, it can be given by

$$\left|\int_{-2}^0 x^3\,dx\right| + \int_0^1 x^3\,dx$$

Now Work Problem 25 ◁

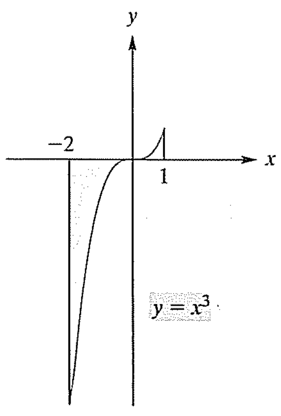


FIGURE 14.21 Graph of $y = x^3$ on the interval $[-2, 1]$.

CAUTION!

Remember that $\int_a^b f(x)\,dx$ is a limit of a sum. In some cases this limit represents an area. In others it does not. When $f(x) \ge 0$ on $[a, b]$, the integral represents the area between the graph of f and the x-axis from $x = a$ to $x = b$.

The Definite Integral of a Derivative

Since a function f is an antiderivative of f', by the Fundamental Theorem we have

$$\int_a^b f'(x)\,dx = f(b) - f(a) \tag{6}$$

But $f'(x)$ is the rate of change of f with respect to x. Hence, if we know the rate of change of f and want to find the difference in function values $f(b) - f(a)$, it suffices to evaluate $\int_a^b f'(x)\,dx$.

APPLY IT ▷

12. A managerial service determines that the rate of increase in maintenance costs (in dollars per year) for a particular apartment complex is given by $M'(x) = 90x^2 + 5000$, where x is the age of the apartment complex in years and $M(x)$ is the total (accumulated) cost of maintenance for x years. Find the total cost for the first five years.

EXAMPLE 5 Finding a Change in Function Values by Definite Integration

A manufacturer's marginal-cost function is

$$\frac{dc}{dq} = 0.6q + 2$$

If production is presently set at $q = 80$ units per week, how much more would it cost to increase production to 100 units per week?

Solution: The total-cost function is $c = c(q)$, and we want to find the difference $c(100) - c(80)$. The rate of change of c is dc/dq, so, by Equation (6),

$$\begin{aligned} c(100) - c(80) &= \int_{80}^{100} \frac{dc}{dq}dq = \int_{80}^{100} (0.6q + 2)dq \\ &= \left[\frac{0.6q^2}{2} + 2q\right]\Bigg|_{80}^{100} = [0.3q^2 + 2q]\Big|_{80}^{100} \\ &= [0.3(100)^2 + 2(100)] - [0.3(80)^2 + 2(80)] \\ &= 3200 - 2080 = 1120 \end{aligned}$$

If c is in dollars, then the cost of increasing production from 80 units to 100 units is \$1120.

Now Work Problem 59 ◁

TECHNOLOGY

Many graphing calculators have the capability to estimate the value of a definite integral. On a TI-83 Plus, to estimate

$$\int_{80}^{100} (0.6q + 2)\,dq$$

we use the "fnInt(" command, as indicated in Figure 14.22. The four parameters that must be entered with this command are

function to be integrated	variable of integration	lower limit	upper limit

We see that the value of the definite integral is approximately 1120, which agrees with the result in Example 5.

Similarly, to estimate

$$\int_{-2}^{1} x^3\,dx$$

we enter

$$\text{fnInt}(X^3, X, -2, 1)$$

or, alternatively, if we first store x^3 as Y_1, we can enter

$$\text{fnInt}(Y_1, X, -2, 1)$$

In each case we obtain -3.75, which agrees with the result in Example 4.

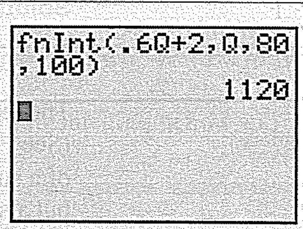


FIGURE 14.22 Estimating $\int_{80}^{100} (0.6q + 2)\,dq$.

PROBLEMS 14.7

In Problems 1–43, evaluate the definite integral.

1. $\int_0^3 5\,dx$

2. $\int_1^5 (e+3e)\,dx$

3. $\int_1^2 5x\,dx$

4. $\int_2^8 -5x\,dx$

5. $\int_{-3}^1 (2x-3)\,dx$

6. $\int_{-1}^1 (4-9y)\,dy$

7. $\int_1^4 (y^2+4y+4)\,dy$

8. $\int_4^1 (2t-3t^2)\,dt$

9. $\int_{-2}^{-1} (3w^2-w-1)\,dw$

10. $\int_8^9 dt$

11. $\int_1^3 3t^{-3}\,dt$

12. $\int_2^3 \frac{3}{x^2}\,dx$

13. $\int_{-8}^8 \sqrt[3]{x^4}\,dx$

14. $\int_{1/2}^{3/2} (x^2+x+1)\,dx$

15. $\int_{1/2}^3 \frac{1}{x^2}\,dx$

16. $\int_9^{36} (\sqrt{x}-2)\,dx$

17. $\int_{-2}^2 (z+1)^4\,dz$

18. $\int_1^8 (x^{1/3}-x^{-1/3})\,dx$

19. $\int_0^1 2x^2(x^3-1)^3\,dx$

20. $\int_2^3 (x+2)^3\,dx$

21. $\int_1^8 \frac{4}{y}\,dy$

22. $\int_{-e^2}^{-1} \frac{2}{x}\,dx$

23. $\int_0^1 e^5\,dx$

24. $\int_2^{e+1} \frac{1}{x-1}\,dx$

25. $\int_0^1 5x^2e^{x^3}\,dx$

26. $\int_0^1 (3x^2+4x)(x^3+2x^2)^4\,dx$

27. $\int_3^4 \frac{3}{(x+3)^2}\,dx$

28. $\int_{-1/3}^{20/3} \sqrt{3x+5}\,dx$

29. $\int_{1/3}^2 \sqrt{10-3p}\,dp$

30. $\int_{-1}^1 q\sqrt{q^2+3}\,dq$

31. $\int_0^1 x^2\sqrt[3]{7x^3+1}\,dx$

32. $\int_0^{\sqrt{2}} \left(2x - \frac{x}{(x^2+1)^{2/3}}\right)\,dx$

33. $\int_0^1 \frac{2x^3+x}{x^2+x^4+1}\,dx$

34. $\int_a^b (m+ny)\,dy$

35. $\int_0^1 \frac{e^x-e^{-x}}{2}\,dx$

36. $\int_{-2}^1 8|x|\,dx$

37. $\int_e^{\sqrt{2}} 3(x^{-2}+x^{-3}-x^{-4})\,dx$

38. $\int_1^2 \left(6\sqrt{x} - \frac{1}{\sqrt{2x}}\right)\,dx$

39. $\int_1^3 (x+1)e^{x^2+2x}\,dx$

40. $\int_1^{95} \frac{x}{\ln e^x}\,dx$

41. $\int_0^2 \frac{x^6+6x^4+x^3+8x^2+x+5}{x^3+5x+1}\,dx$

42. $\int_1^2 \frac{1}{1+e^x}\,dx$ (*Hint:* Multiply the integrand by $\frac{e^{-x}}{e^{-x}}$.)

43. $\int_0^2 f(x)\,dx$, where $f(x) = \begin{cases} 4x^2 & \text{if } 0 \le x < \frac{1}{2} \\ 2x & \text{if } \frac{1}{2} \le x \le 2 \end{cases}$

44. Evaluate $\left(\int_1^3 x\,dx\right)^3 - \int_1^3 x^3\,dx$.

45. Suppose $f(x) = \int_1^x 3\frac{1}{t^2}\,dt$. Evaluate $\int_e^1 f(x)\,dx$.

46. Evaluate $\int_7^7 e^{x^2}\,dx + \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}\,dx$.

47. If $\int_1^2 f(x)\,dx = 5$ and $\int_3^1 f(x)\,dx = 2$, find $\int_2^3 f(x)\,dx$.

48. If $\int_1^4 f(x)\,dx = 6$, $\int_2^4 f(x)\,dx = 5$, and $\int_1^3 f(x)\,dx = 2$, find $\int_2^3 f(x)\,dx$.

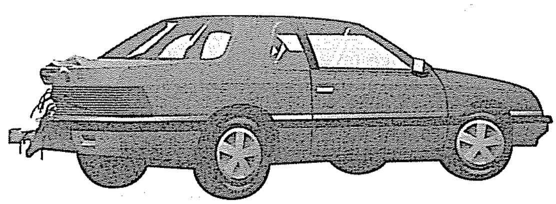
49. Evaluate $\int_2^3 \left(\frac{d}{dx}\int_2^3 e^{x^3}\,dx\right)\,dx$
(*Hint:* It is not necessary to find $\int_2^3 e^{x^3}\,dx$.)

50. Suppose that $f(x) = \int_e^x \frac{e^t-e^{-t}}{e^t+e^{-t}}\,dt$ where $x > e$. Find $f'(x)$.

51. **Severity Index** In discussing traffic safety, Shonle[8] considers how much acceleration a person can tolerate in a crash so that there is no major injury. The *severity index* is defined as

$$\text{S.I.} = \int_0^T \alpha^{5/2}\,dt$$

where α (a Greek letter read "alpha") is considered a constant involved with a weighted average acceleration, and T is the duration of the crash. Find the severity index.



52. **Statistics** In statistics, the mean μ (a Greek letter read "mu") of the continuous probability density function f defined on the interval $[a, b]$ is given by

$$\mu = \int_a^b xf(x)\,dx$$

and the variance σ^2 (σ is a Greek letter read "sigma") is given by

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x)\,dx$$

Compute μ and then σ^2 if $a = 0$, $b = 1$, and $f(x) = 6(x - x^2)$.

53. **Distribution of Incomes** The economist Pareto[9] has stated an empirical law of distribution of higher incomes that gives the number N of persons receiving x or more dollars. If

$$\frac{dN}{dx} = -Ax^{-B}$$

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[8] J. I. Shonle, *Environmental Applications of General Physics* (Reading, MA: Addison-Wesley Publishing Company, Inc., 1975).

[9] G. Tintner, *Methodology of Mathematical Economics and Econometrics* (Chicago: University of Chicago Press, 1967), p. 16.

where A and B are constants, set up a definite integral that gives the total number of persons with incomes between a and b, where $a < b$.

54. Biology In a discussion of gene mutation,[10] the following integral occurs:

$$\int_0^{10^{-4}} x^{-1/2}\,dx$$

Evaluate this integral.

55. Continuous Income Flow The present value (in dollars) of a continuous flow of income of \$2000 a year for five years at 6% compounded continuously is given by

$$\int_0^5 2000e^{-0.06t}\,dt$$

Evaluate the present value to the nearest dollar.

56. Biology In biology, problems frequently arise involving the transfer of a substance between compartments. An example is a transfer from the bloodstream to tissue. Evaluate the following integral, which occurs in a two-compartment diffusion problem:[11]

$$\int_0^t (e^{-a\tau} - e^{-b\tau})d\tau$$

Here, τ (read "tau") is a Greek letter; a and b are constants.

57. Demography For a certain small population, suppose l is a function such that $l(x)$ is the number of persons who reach the age of x in any year of time. This function is called a *life table function*. Under appropriate conditions, the integral

$$\int_a^b l(t)\,dt$$

gives the expected number of people in the population between the exact ages of a and b, inclusive. If

$$l(x) = 1000\sqrt{110 - x} \quad \text{for } 0 \le x \le 110$$

determine the number of people between the exact ages of 10 and 29, inclusive. Give your answer to the nearest integer, since fractional answers make no sense. What is the size of the population?

58. Mineral Consumption If C is the yearly consumption of a mineral at time $t = 0$, then, under continuous consumption, the total amount of the mineral used in the interval $[0, t]$ is

$$\int_0^t Ce^{k\tau}\,d\tau$$

where k is the rate of consumption. For a rare-earth mineral, it has been determined that $C = 3000$ units and $k = 0.05$. Evaluate the integral for these data.

59. Marginal Cost A manufacturer's marginal-cost function is

$$\frac{dc}{dq} = 0.2q + 8$$

If c is in dollars, determine the cost involved to increase production from 65 to 75 units.

60. Marginal Cost Repeat Problem 59 if

$$\frac{dc}{dq} = 0.004q^2 - 0.5q + 50$$

and production increases from 90 to 180 units.

61. Marginal Revenue A manufacturer's marginal-revenue function is

$$\frac{dr}{dq} = \frac{2000}{\sqrt{300q}}$$

If r is in dollars, find the change in the manufacturer's total revenue if production is increased from 500 to 800 units.

62. Marginal Revenue Repeat Problem 61 if

$$\frac{dr}{dq} = 100 + 50q - 3q^2$$

and production is increased from 5 to 10 units.

63. Crime Rate A sociologist is studying the crime rate in a certain city. She estimates that t months after the beginning of next year, the total number of crimes committed will increase at the rate of $8t + 10$ crimes per month. Determine the total number of crimes that can be expected to be committed next year. How many crimes can be expected to be committed during the last six months of that year?

64. Hospital Discharges For a group of hospitalized individuals, suppose the discharge rate is given by

$$f(t) = \frac{81 \times 10^6}{(300 + t)^4}$$

where $f(t)$ is the proportion of the group discharged per day at the end of t days. What proportion has been discharged by the end of 700 days?

65. Production Imagine a one-dimensional country of length $2R$. (See Figure 14.23.[12]) Suppose the production of goods for this country is continuously distributed from border to border. If the amount produced each year per unit of distance is $f(x)$, then the country's total yearly production is given by

$$G = \int_{-R}^{R} f(x)\,dx$$

Evaluate G if $f(x) = i$, where i is constant.

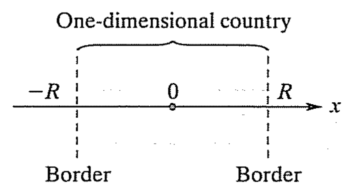

FIGURE 14.23

66. Exports For the one-dimensional country of Problem 65, under certain conditions the amount of the country's exports is given by

$$E = \int_{-R}^{R} \frac{i}{2}[e^{-k(R-x)} + e^{-k(R+x)}]\,dx$$

where i and k are constants ($k \neq 0$). Evaluate E.

[10] W. J. Ewens, *Population Genetics* (London: Methuen & Company Ltd., 1969).

[11] W. Simon, *Mathematical Techniques for Physiology and Medicine* (New York: Academic Press, Inc., 1972).

[12] R. Taagepera, "Why the Trade/GNP Ratio Decreases with Country Size," *Social Science Research*, 5 (1976), 385–404.

67. Average Delivered Price In a discussion of a delivered price of a good from a mill to a customer, DeCanio[13] claims that the average delivered price paid by consumers is given by

$$A = \frac{\int_0^R (m+x)[1-(m+x)]\,dx}{\int_0^R [1-(m+x)]\,dx}$$

where m is mill price, and x is the maximum distance to the point of sale. DeCanio determines that

$$A = \frac{m + \frac{R}{2} - m^2 - mR - \frac{R^2}{3}}{1 - m - \frac{R}{2}}$$

Verify this.

In Problems 68–70, use the Fundamental Theorem of Integral Calculus to determine the value of the definite integral. Confirm your result with your calculator.

68. $\int_{2.5}^{3.5} (1 + 2x + 3x^2)\,dx$ **69.** $\int_0^4 \frac{1}{(4x+4)^2}\,dx$

70. $\int_0^1 e^{3t}\,dt$ Round your answer to two decimal places.

In Problems 71–74, estimate the value of the definite integral. Round your answer to two decimal places.

71. $\int_{-1}^{5} \frac{x^2+1}{x^2+4}\,dx$ **72.** $\int_3^4 \frac{1}{x \ln x}\,dx$

73. $\int_0^3 2\sqrt{t^2+3}\,dt$ **74.** $\int_{-1}^{1} \frac{6\sqrt{q+1}}{q+3}\,dq$

Objective

To estimate the value of a definite integral by using either the trapezoidal rule or Simpson's rule.

14.8 Approximate Integration

Trapezoidal Rule

Any function f constructed from polynomials, exponentials, and logarithms using algebraic operations and composition can be differentiated and the resulting function f' is again of the same kind—one that can be constructed from polynomials, exponentials, and logarithms using algebraic operations and composition. Let us call such functions *elementary* (although the term usually has a slightly different meaning). In this terminology, the derivative of an elementary function is also elementary. Integration is more complicated. If an elementary function f has F as an antiderivative, then F may fail to be elementary. In other words, even for a fairly simple-looking function f it is sometimes impossible to find $\int f(x)\,dx$ in terms of the functions that we consider in this book. For example, there is no elementary function whose derivative is e^{x^2} so that you cannot expect to "do" the integral $\int e^{x^2}\,dx$.

On the other hand, consider a function f that is continuous on a closed interval $[a, b]$ with $f(x) \geq 0$ for all x in $[a, b]$. Then $\int_a^b f(x)\,dx$ is simply the *number* that gives the area of the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$. It is unsatisfying, and perhaps impractical, not to say anything about the number $\int_a^b f(x)\,dx$ because of an inability to "do" the integral $\int f(x)\,dx$. This also applies when the integral $\int f(x)\,dx$ is merely too difficult for the person who needs to find the number $\int_a^b f(x)\,dx$.

Since $\int_a^b f(x)\,dx$ is defined as a limit of sums of the form $\sum f(x)\,\Delta x$, any particular well-formed sum of the form $\sum f(x)\,\Delta x$ can be regarded as an approximation of $\int_a^b f(x)\,dx$. At least for nonnegative f such sums can be regarded as sums of areas of thin rectagles. Consider, for example, Figure 14.11 in Section 14.6, in which two rectangles are explicitly shown. It is clear that the error that arises from such rectangles is associated with the small side at the top. The error would be reduced if we replaced the rectangles by shapes that have a top side that is closer to the shape of the curve. We will consider two possibilities: using thin trapezoids rather than thin rectangles, the *trapezoidal rule;* and using thin regions surmounted by parabolic arcs, *Simpson's rule*. In each case only a finite number of numerical values of $f(x)$ needs be known and the calculations involved are especially suitable for computers or calculators. In both cases, we assume that f is continuous on $[a, b]$.

In developing the trapezoidal rule, for convenience we will also assume that $f(x) \geq 0$ on $[a, b]$, so that we can think in terms of area. This rule involves approximating the graph of f by straight-line segments.

[13]S. J. DeCanio, "Delivered Pricing and Multiple Basing Point Equationilibria: A Reevaluation," *The Quarterly Journal of Economics*, XCIX, no. 2 (1984), 329–49.

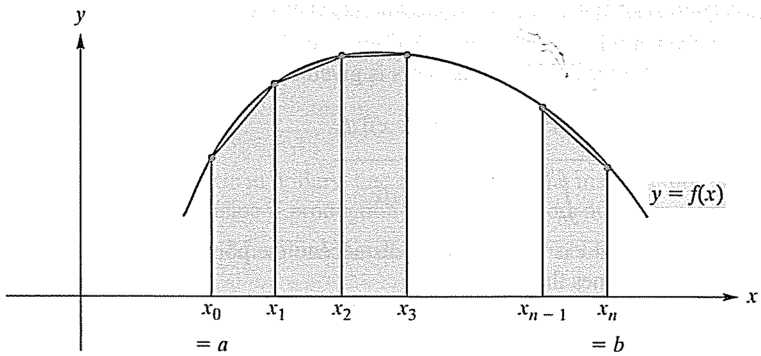

FIGURE 14.24 Approximating an area by using trapezoids.

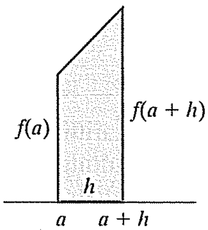

FIGURE 14.25 First trapezoid.

In Figure 14.24, the interval $[a, b]$ is divided into n subintervals of equal length by the points $a = x_0, x_1, x_2, \ldots,$ and $x_n = b$. Since the length of $[a, b]$ is $b - a$, the length of each subinterval is $(b - a)/n$, which we will call h.

Clearly,

$$x_1 = a + h, x_2 = a + 2h, \ldots, x_n = a + nh = b$$

With each subinterval, we can associate a trapezoid (a four-sided figure with two parallel sides). The area A of the region bounded by the curve, the x-axis, and the lines $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$ and can be approximated by the sum of the areas of the trapezoids determined by the subintervals.

Consider the first trapezoid, which is redrawn in Figure 14.25. Since the area of a trapezoid is equal to one-half the base times the sum of the lengths of the parallel sides, this trapezoid has area

$$\tfrac{1}{2}h[f(a) + f(a + h)]$$

Similarly, the second trapezoid has area

$$\tfrac{1}{2}h[f(a + h) + f(a + 2h)]$$

The area A under the curve is approximated by the sum of the areas of n trapezoids:

$$\begin{aligned} A &\approx \tfrac{1}{2}h[f(a) + f(a + h)] + \tfrac{1}{2}h[f(a + h) + f(a + 2h)] \\ &\quad + \tfrac{1}{2}h[f(a + 2h) + f(a + 3h)] + \cdots + \tfrac{1}{2}h[f(a + (n - 1)h) + f(b)] \end{aligned}$$

Since $A = \int_a^b f(x)\,dx$, by simplifying the preceding formula we have the trapezoidal rule:

The Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + 2f(a + h) + 2f(a + 2h) + \cdots + 2f(a + (n - 1)h) + f(b)]$$

where $h = (b - a)/n$.

The pattern of the coefficients inside the braces is 1, 2, 2, ..., 2, 1. Usually, the more subintervals, the better is the approximation. In our development, we assumed for convenience that $f(x) \geq 0$ on $[a, b]$. However, the trapezoidal rule is valid without this restriction.

EXAMPLE 1 Trapezoidal Rule

Use the trapezoidal rule to estimate the value of

$$\int_0^1 \frac{1}{1 + x^2}\,dx$$

APPLY IT ▷

13. An oil tanker is losing oil at a rate of $R'(t) = \dfrac{60}{\sqrt{t^2+9}}$, where t is the time in minutes and $R(t)$ is the radius of the oil slick in feet. Use the trapezoidal rule with $n=5$ to approximate $\int_0^5 \dfrac{60}{\sqrt{t^2+9}}\,dt$, the size of the radius after five seconds.

for $n = 5$. Compute each term to four decimal places, and round the answer to three decimal places.

Solution: Here $f(x) = 1/(1+x^2)$, $n = 5$, $a = 0$, and $b = 1$. Thus,

$$h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5} = 0.2$$

The terms to be added are

$$\begin{aligned} f(a) &= f(0) = 1.0000 \\ 2f(a+h) &= 2f(0.2) = 1.9231 \\ 2f(a+2h) &= 2f(0.4) = 1.7241 \\ 2f(a+3h) &= 2f(0.6) = 1.4706 \\ 2f(a+4h) &= 2f(0.8) = 1.2195 \\ f(b) &= f(1) = \underline{0.5000} \qquad a+nh=b \\ & \qquad 7.8373 = \text{sum} \end{aligned}$$

Hence, our estimate for the integral is

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{0.2}{2}(7.8373) \approx 0.784$$

The actual value of the integral is approximately 0.785.

Now Work Problem 1 ◁

Simpson's Rule

Another method for estimating $\int_a^b f(x)\,dx$ is given by Simpson's rule, which involves approximating the graph of f by parabolic segments. We will omit the derivation.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 4f(a+(n-1)h) + f(b)]$$

where $h = (b-a)/n$ and n is even.

The pattern of coefficients inside the braces is $1, 4, 2, 4, 2, \ldots, 2, 4, 1$, which requires that n **be even.** Let us use this rule for the integral in Example 1.

APPLY IT ▷

14. A yeast culture is growing at the rate of $A'(t) = 0.3e^{0.2t^2}$, where t is the time in hours and $A(t)$ is the amount in grams. Use Simpson's rule with $n = 8$ to approximate $\int_0^4 0.3e^{0.2t^2}\,dt$, the amount the culture grew over the first four hours.

EXAMPLE 2 Simpson's Rule

Use Simpson's rule to estimate the value of $\int_0^1 \dfrac{1}{1+x^2}\,dx$ for $n = 4$. Compute each term to four decimal places, and round the answer to three decimal places.

Solution: Here $f(x) = 1/(1+x^2)$, $n = 4$, $a = 0$, and $b = 1$. Thus, $h = (b-a)/n = 1/4 = 0.25$. The terms to be added are

$$\begin{aligned} f(a) &= f(0) = 1.0000 \\ 4f(a+h) &= 4f(0.25) = 3.7647 \\ 2f(a+2h) &= 2f(0.5) = 1.6000 \\ 4f(a+3h) &= 4f(0.75) = 2.5600 \\ f(b) &= f(1) = \underline{0.5000} \\ & \qquad 9.4247 = \text{sum} \end{aligned}$$

Therefore, by Simpson's rule,

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{0.25}{3}(9.4247) \approx 0.785$$

This is a better approximation than that which we obtained in Example 1 by using the trapezoidal rule.

Now Work Problem 5 ◁

Both Simpson's rule and the trapezoidal rule can be used if we know only $f(a)$, $f(a+h)$, and so on; we do not need to know $f(x)$ for all x in $[a, b]$. Example 3 will illustrate.

In Example 3, a definite integral is estimated from data points; the function itself is not known.

EXAMPLE 3 Demography

A function often used in demography (the study of births, marriages, mortality, etc., in a population) is the **life-table function,** denoted l. In a population having 100,000 births in any year of time, $l(x)$ represents the number of persons who reach the age of x in any year of time. For example, if $l(20) = 98{,}857$, then the number of persons who attain age 20 in any year of time is 98,857. Suppose that the function l applies to all people born over an extended period of time. It can be shown that, at any time, the expected number of persons in the population between the exact ages of x and $x+m$, inclusive, is given by

$$\int_x^{x+m} l(t)\,dt$$

The following table gives values of $l(x)$ for males and females in the United States.[14] Approximate the number of women in the 20–35 age group by using the trapezoidal rule with $n = 3$.

Life Table

	$l(x)$			$l(x)$	
Age $= x$	Males	Females	Age $= x$	Males	Females
0	100,000	100,000	45	93,717	96,582
5	99,066	99,220	50	91,616	95,392
10	98,967	99,144	55	88,646	93,562
15	98,834	99,059	60	84,188	90,700
20	98,346	98,857	65	77,547	86,288
25	97,648	98,627	70	68,375	79,926
30	96,970	98,350	75	56,288	70,761
35	96,184	97,964	80	42,127	58,573
40	95,163	97,398			

Solution: We want to estimate

$$\int_{20}^{35} l(t)\,dt$$

We have $h = \dfrac{b-a}{n} = \dfrac{35-20}{3} = 5$. The terms to be added under the trapezoidal rule are

$$\begin{aligned} l(20) &= 98{,}857 \\ 2l(25) = 2(98{,}627) &= 197{,}254 \\ 2l(30) = 2(98{,}350) &= 196{,}700 \\ l(35) &= \underline{97{,}964} \\ & \quad 590{,}775 = \text{sum} \end{aligned}$$

By the trapezoidal rule,

$$\int_{20}^{35} l(t)\,dt \approx \frac{5}{2}(590{,}775) = 1{,}476{,}937.5$$

Now Work Problem 17 ◁

Formulas used to determine the accuracy of answers obtained with the trapezoidal or Simpson's rule can be found in standard texts on numerical analysis.

[14] *National Vital Statistics Report*, vol. 48, no. 18, February 7, 2001.

PROBLEMS 14.8

In Problems 1 and 2, use the trapezoidal rule or Simpson's rule (as indicated) and the given value of n to estimate the integral.

1. $\int_{-2}^{4} \frac{170}{1+x^2}\,dx$; trapezoidal rule, $n = 6$

2. $\int_{-2}^{4} \frac{170}{1+x^2}\,dx$; Simpson's rule, $n = 6$

In Problems 3–8, use the trapezoidal rule or Simpson's rule (as indicated) and the given value of n to estimate the integral. Compute each term to four decimal places, and round the answer to three decimal places. In Problems 3–6, also evaluate the integral by antidifferentiation (the Fundamental Theorem of Integral Calculus).

3. $\int_{0}^{1} x^3\,dx$; trapezoidal rule, $n = 5$

4. $\int_{0}^{1} x^2\,dx$; Simpson's rule, $n = 4$

5. $\int_{1}^{4} \frac{dx}{x^2}$; Simpson's rule, $n = 4$

6. $\int_{1}^{4} \frac{dx}{x}$; trapezoidal rule, $n = 6$

7. $\int_{0}^{2} \frac{x\,dx}{x+1}$; trapezoidal rule, $n = 4$

8. $\int_{1}^{4} \frac{dx}{x}$; Simpson's rule, $n = 6$

In Problems 9 and 10, use the life table in Example 3 to estimate the given integrals by the trapezoidal rule.

9. $\int_{45}^{70} l(t)\,dt$, males, $n = 5$ **10.** $\int_{35}^{55} l(t)\,dt$, females, $n = 4$

In Problems 11 and 12, suppose the graph of a continuous function f, where $f(x) \geq 0$, contains the given points. Use Simpson's rule and all of the points to approximate the area between the graph and the x-axis on the given interval. Round the answer to one decimal place.

11. $(1, 0.4), (2, 0.6), (3, 1.2), (4, 0.8), (5, 0.5)$; $[1,5]$

12. $(2, 0), (2.5, 6), (3, 10), (3.5, 11), (4, 14), (4.5, 15), (5, 16)$; $[2,5]$

13. Using all the information given in Figure 14.26, estimate $\int_1^3 f(x)\,dx$ by using Simpson's rule. Give your answer in fractional form.

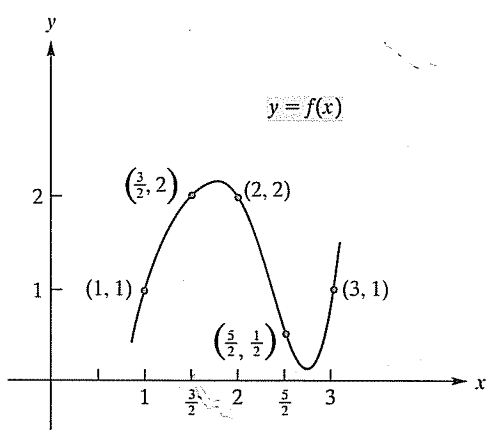

FIGURE 14.26

In Problems 14 and 15, use Simpson's rule and the given value of n to estimate the integral. Compute each term to four decimal places, and round the answer to three decimal places.

14. $\int_{1}^{3} \frac{2}{\sqrt{1+x}}\,dx$; $n = 4$ Also, evaluate the integral by the Fundamental Theorem of Integral Calculus.

15. $\int_{0}^{1} \sqrt{1-x^2}\,dx$; $n = 4$

16. Revenue Use Simpson's rule to approximate the total revenue received from the production and sale of 80 units of a product if the values of the marginal-revenue function dr/dq are as follows:

q (units)	0	10	20	30	40	50	60	70	80
$\frac{dr}{dq}$ ($ per unit)	10	9	8.5	8	8.5	7.5	7	6.5	7

17. Area of Pool Lesley Griffith, who has taken a commerce mathematics class, would like to determine the surface area of her curved, irregularly shaped swimming pool. There is a straight fence that runs along the side of the pool. Lesley marks off points a and b on the fence as shown in Figure 14.27. She notes that the distance from a to b is 8 m and subdivides the interval into eight equal subintervals, naming the resulting points on the fence $x_1, x_2, x_3, x_4, x_5, x_6$, and x_7. Lesley (L) stands at point x_1, holds a tape measure, and has a friend Chester (C) take the free end of the tape measure to the point P_1 on the far side of the pool. She asks her other friend Willamina (W) to stand at point Q_1 on the near side of the pool and note the distance on the tape measure. See Figure 14.27.

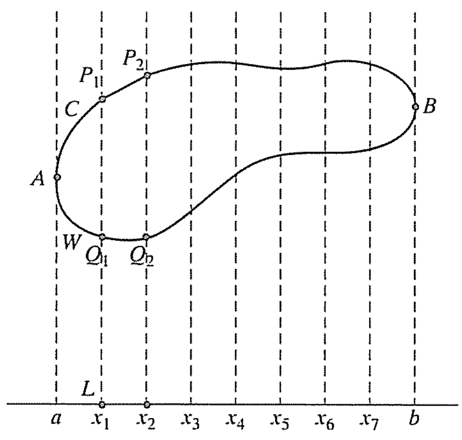

FIGURE 14.27

Lesley then moves to point x_2 and the three friends repeat the procedure. They do this for each of the remaining points x_3 to x_7. Lesley tabulates their measurements in the following table:

Distance along fence (m)	0	1	2	3	4	5	6	7	8
Distance across pool (m)	0	3	4	3	3	2	2	2	0

Lesley says that Simpson's rule now allows them to approximate the area of the pool as

$$\frac{1}{3}(4(3) + 2(4) + 4(3) + 2(3) + 4(2) + 2(2) + 4(2)) = \frac{58}{3}$$

square meters. Chester says that this is not how he remembers Simpson's rule. Willamina thinks that some terms are missing, but Chester gets bored and goes for a swim. Is Lesley's calculation correct? Explain.

18. Manufacturing A manufacturer estimated both marginal cost (MC) and marginal revenue (MR) at various levels of output (q). These estimates are given in the following table:

q(units)	0	20	40	60	80	100
MC ($ per unit)	260	250	240	200	240	250
MR ($ per unit)	410	350	300	250	270	250

(a) Using the trapezoidal rule, estimate the total variable costs of production for 100 units.
(b) Using the trapezoidal rule, estimate the total revenue from the sale of 100 units.
(c) If we assume that maximum profit occurs when MR = MC (that is, when $q = 100$), estimate the maximum profit if fixed costs are $2000.

Objective

To find the area of a region bounded by curves using integration over both vertical and horizontal strips.

14.9 Area between Curves

In Sections 14.6 and 14.7 we saw that the area of a region bounded by the lines $x = a$, $x = b$, $y = 0$, and a curve $y = f(x)$ with $f(x) \geq 0$ for $a \leq x \leq b$ can be found by evaluating the definite integral $\int_a^b f(x)\,dx$. Similarly, for a function $f(x) \leq 0$ on an interval $[a, b]$, the area of the region bounded by $x = a$, $x = b$, $y = 0$, and $y = f(x)$ is given by $-\int_a^b f(x)\,dx = \int_a^b -f(x)\,dx$. Most of the functions f we have encountered, and will encounter, are continuous and have a finite number of roots of $f(x) = 0$. For such functions, the roots of $f(x) = 0$ partition the domain of f into a finite number of intervals on each of which we have either $f(x) \geq 0$ or $f(x) \leq 0$. For such a function we can determine the area bounded by $y = f(x)$, $y = 0$ and *any* pair of vertical lines $x = a$ and $x = b$, with a and b in the domain of f. We have only to find all the roots $c_1 < c_2 < \cdots < c_k$ with $a < c_1$ and $c_k < b$; calculate the integrals $\int_a^{c_1} f(x)\,dx$, $\int_{c_1}^{c_2} f(x)\,dx, \cdots, \int_{c_k}^{b} f(x)\,dx$; attach to each integral the correct sign to correspond to an area; and add the results. Example 1 will provide a modest example of this idea.

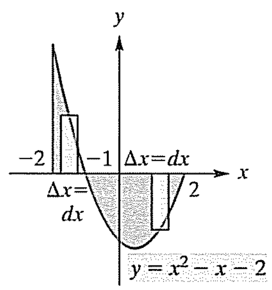

FIGURE 14.28 Diagram for Example 1.

For such an area determination, a rough sketch of the region involved is extremely valuable. To set up the integrals needed, a sample rectangle should be included in the sketch for each individual integral as in Figure 14.28. The area of the region is a limit of sums of areas of rectangles. A sketch helps to understand the integration process and it is indispensable when setting up integrals to find areas of complicated regions. Such a rectangle (see Figure 14.28) is called a **vertical strip**. In the diagram, the width of the vertical strip is Δx. We know from our work on differentials in Section 14.1 that we can consistently write $\Delta x = dx$, for x the independent variable. The height of the vertical strip is the y-value of the curve. Hence, the rectangle has area $y\,\Delta x = f(x)\,dx$. The area of the entire region is found by summing the areas of all such vertical strips between $x = a$ and $x = b$ and finding the limit of this sum, which is the definite integral. Symbolically, we have

$$\Sigma y \Delta x \to \int_a^b f(x)\,dx$$

For $f(x) \geq 0$ it is helpful to think of dx as a length differential and $f(x)dx$ as an area differential dA. Then, as we saw in Section 14.7, we have $\frac{dA}{dx} = f(x)$ for some area function A and

$$\int_a^b f(x)\,dx = \int_a^b dA = A(b) - A(a)$$

[If our area function A measures area starting at the line $x = a$, as it did in Section 14.7, then $A(a) = 0$ and the area under f (and over 0) from a to b is just $A(b)$.] It is important to understand here that we need $f(x) \geq 0$ in order to think of $f(x)$ as a length and hence $f(x)dx$ as a differential area. But if $f(x) \leq 0$ then $-f(x) \geq 0$ so that $-f(x)$ becomes a length and $-f(x)dx$ becomes a differential area.

EXAMPLE 1 An Area Requiring Two Definite Integrals

CAUTION!

It is wrong to write hastily that the area is $\int_{-2}^{2} y\,dx$, for the following reason: For the left rectangle, the height is y. However, for the rectangle on the right, y is negative, so its height is the positive number $-y$. This points out the importance of sketching the region.

Find the area of the region bounded by the curve

$$y = x^2 - x - 2$$

and the line $y = 0$ (the x-axis) from $x = -2$ to $x = 2$.

Solution: A sketch of the region is given in Figure 14.28. Notice that the x-intercepts are $(-1, 0)$ and $(2, 0)$.

On the interval $[-2, -1]$, the area of the vertical strip is

$$y dx = (x^2 - x - 2)dx$$

On the interval $[-1, 2]$, the area of the vertical strip is

$$(-y)dx = -(x^2 - x - 2)dx$$

Thus,

$$\begin{aligned}
\text{area} &= \int_{-2}^{-1} (x^2 - x - 2)\,dx + \int_{-1}^{2} -(x^2 - x - 2)\,dx \\
&= \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Bigg|_{-2}^{-1} - \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Bigg|_{-1}^{2} \\
&= \left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{8}{3} - \frac{4}{2} + 4\right)\right] \\
&\quad - \left[\left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(-\frac{1}{3} - \frac{1}{2} + 2\right)\right] \\
&= \frac{19}{3}
\end{aligned}$$

Now Work Problem 22 ◁

Before embarking on more complicated area problems, we motivate the further study of area by seeing the use of area as a probability in statistics.

EXAMPLE 2 Statistics Application

In statistics, a (probability) **density function** f of a variable x, where x assumes all values in the interval $[a, b]$, has the following properties:

(i) $f(x) \geq 0$

(ii) $\int_a^b f(x)\,dx = 1$

The probability that x assumes a value between c and d, which is written $P(c \leq x \leq d)$, where $a \leq c \leq d \leq b$, is represented by the area of the region bounded by the graph of f and the x-axis between $x = c$ and $x = d$. Hence (see Figure 14.29),

$$P(c \leq x \leq d) = \int_c^d f(x)\,dx$$

[In the terminology of Chapters 8 and 9, the condition $c \leq x \leq d$ defines an *event* and $P(c \leq x \leq d)$ is consistent with the notation of the earlier chapters. Note too that the hypothesis **(ii)** above ensures that $a \leq x \leq b$ is the *certain event*.]

For the density function $f(x) = 6(x - x^2)$, where $0 \leq x \leq 1$, find each of the following probabilities.

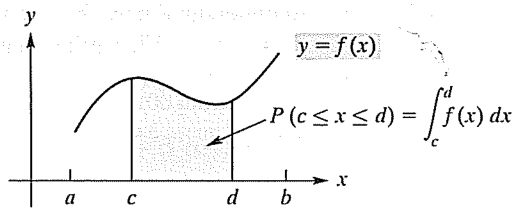

FIGURE 14.29 Probability as an area.

a. $P(0 \leq x \leq \frac{1}{4})$

Solution: Here $[a, b]$ is $[0, 1]$, c is 0, and d is $\frac{1}{4}$. We have

$$P\left(0 \leq x \leq \tfrac{1}{4}\right) = \int_0^{1/4} 6(x - x^2)\,dx = 6\int_0^{1/4} (x - x^2)\,dx$$

$$= 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^{1/4} = (3x^2 - 2x^3)\Big|_0^{1/4}$$

$$= \left(3\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right)^3\right) - 0 = \frac{5}{32}$$

b. $P\left(x \geq \frac{1}{2}\right)$

Solution: Since the domain of f is $0 \leq x \leq 1$, to say that $x \geq \frac{1}{2}$ means that $\frac{1}{2} \leq x \leq 1$. Thus,

$$P\left(x \geq \frac{1}{2}\right) = \int_{1/2}^{1} 6(x - x^2)\,dx = 6\int_{1/2}^{1} (x - x^2)\,dx$$

$$= 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{1/2}^{1} = (3x^2 - 2x^3)\Big|_{1/2}^{1} = \frac{1}{2}$$

Now Work Problem 27 ◁

Vertical Strips

We will now find the area of a region enclosed by several curves. As before, our procedure will be to draw a sample strip of area and use the definite integral to "add together" the areas of all such strips.

For example, consider the area of the region in Figure 14.30 that is bounded on the top and bottom by the curves $y = f(x)$ and $y = g(x)$ and on the sides by the lines $x = a$ and $x = b$. The width of the indicated vertical strip is dx, and the height is the y-value of the upper curve minus the y-value of the lower curve, which we will write as $y_{\text{upper}} - y_{\text{lower}}$. Thus, the area of the strip is

$$(y_{\text{upper}} - y_{\text{lower}})\,dx$$

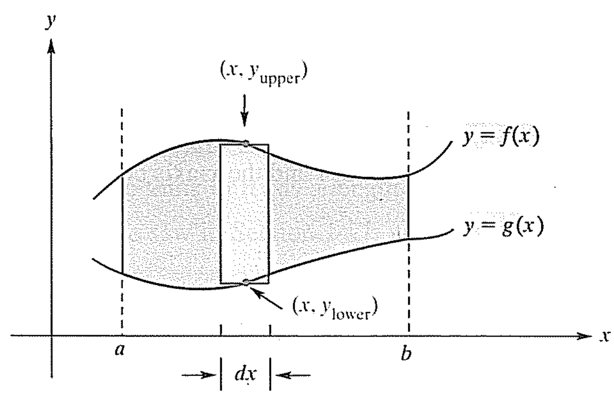

FIGURE 14.30 Region between curves.

which is

$$(f(x) - g(x))\,dx$$

Summing the areas of all such strips from $x = a$ to $x = b$ by the definite integral gives the area of the region:

$$\sum (f(x) - g(x))\,dx \to \int_a^b (f(x) - g(x))\,dx = \text{area}$$

We remark that there is another way to view this area problem. In Figure 14.30 both f and g are above $y = 0$ and it is clear that the area we seek is also the area under f minus the area under g. That approach tells us that the required area is

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$$

However, our first approach does not require that either f or g lie above 0. Our usage of y_{upper} and y_{lower} is really just a way of saying that $f \geq g$ on $[a, b]$. This is equivalent to saying that $f - g \geq 0$ on $[a, b]$ so that each differential $(f(x) - g(x))\,dx$ is meaningful as an area.

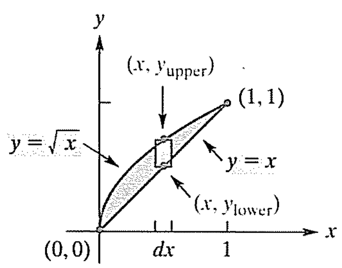

FIGURE 14.31 Diagram for Example 3.

EXAMPLE 3 Finding an Area between Two Curves

Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

Solution: A sketch of the region appears in Figure 14.31. To determine where the curves intersect, we solve the system formed by the equations $y = \sqrt{x}$ and $y = x$. Eliminating y by substitution, we obtain

$$\begin{aligned} \sqrt{x} &= x \\ x &= x^2 \qquad \text{squaring both sides} \\ 0 &= x^2 - x = x(x-1) \\ x &= 0 \quad \text{or} \quad x = 1 \end{aligned}$$

It should be obvious that knowing the points of intersection is important in determining the limits of integration.

Since we squared both sides, we must check the solutions found with respect to the *original* equation. It is easily determined that both $x = 0$ and $x = 1$ are solutions of $\sqrt{x} = x$. If $x = 0$, then $y = 0$; if $x = 1$, then $y = 1$. Thus, the curves intersect at (0, 0) and (1, 1). The width of the indicated strip of area is dx. The height is the y-value on the upper curve minus the y-value on the lower curve:

$$y_{\text{upper}} - y_{\text{lower}} = \sqrt{x} - x$$

Hence, the area of the strip is $(\sqrt{x} - x)\,dx$. Summing the areas of all such strips from $x = 0$ to $x = 1$ by the definite integral, we get the area of the entire region:

$$\begin{aligned} \text{area} &= \int_0^1 (\sqrt{x} - x)\,dx \\ &= \int_0^1 (x^{1/2} - x)\,dx = \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^2}{2}\right)\Bigg|_0^1 \\ &= \left(\frac{2}{3} - \frac{1}{2}\right) - (0 - 0) = \frac{1}{6} \end{aligned}$$

Now Work Problem 47 ◁

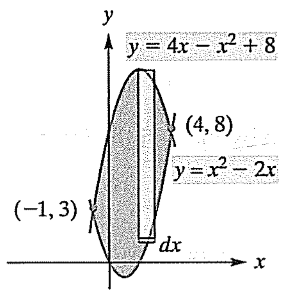

FIGURE 14.32 Diagram for Example 4.

EXAMPLE 4 Finding an Area between Two Curves

Find the area of the region bounded by the curves $y = 4x - x^2 + 8$ and $y = x^2 - 2x$.

Solution: A sketch of the region appears in Figure 14.32. To find where the curves intersect, we solve the system of equations $y = 4x - x^2 + 8$ and $y = x^2 - 2x$:

$$\begin{aligned} 4x - x^2 + 8 &= x^2 - 2x, \\ -2x^2 + 6x + 8 &= 0, \\ x^2 - 3x - 4 &= 0, \\ (x+1)(x-4) &= 0 \qquad \text{factoring} \\ x = -1 \quad &\text{or} \quad x = 4 \end{aligned}$$

When $x = -1$, then $y = 3$; when $x = 4$, then $y = 8$. Thus, the curves intersect at $(-1, 3)$ and (4, 8). The width of the indicated strip is dx. The height is the y-value on the upper curve minus the y-value on the lower curve:

$$y_{\text{upper}} - y_{\text{lower}} = (4x - x^2 + 8) - (x^2 - 2x)$$

Therefore, the area of the strip is

$$[(4x - x^2 + 8) - (x^2 - 2x)]\,dx = (-2x^2 + 6x + 8)\,dx$$

Summing all such areas from $x = -1$ to $x = 4$, we have

$$\text{area} = \int_{-1}^{4} (-2x^2 + 6x + 8)\,dx = 41\tfrac{2}{3}$$

Now Work Problem 51 ◁

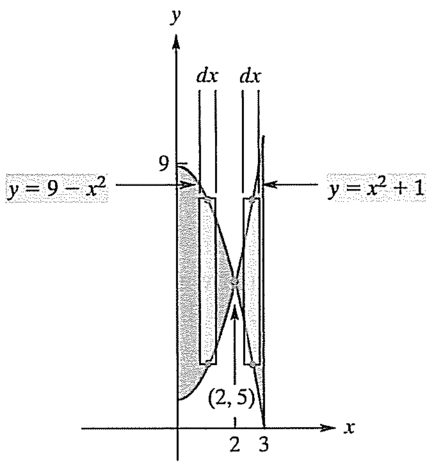

FIGURE 14.33 y_{upper} is $9 - x^2$ on [0, 2] and is $x^2 + 1$ on [2, 3].

EXAMPLE 5 Area of a Region Having Two Different Upper Curves

Find the area of the region between the curves $y = 9 - x^2$ and $y = x^2 + 1$ from $x = 0$ to $x = 3$.

Solution: The region is sketched in Figure 14.33. The curves intersect when

$$\begin{aligned} 9 - x^2 &= x^2 + 1 \\ 8 &= 2x^2 \\ 4 &= x^2 \\ x &= \pm 2 \qquad \text{two solutions} \end{aligned}$$

When $x = \pm 2$, then $y = 5$, so the points of intersection are $(\pm 2, 5)$. Because we are interested in the region from $x = 0$ to $x = 3$, the intersection point that is of concern to us is (2, 5). Notice in Figure 14.33 that in the region to the *left* of the intersection point (2, 5), a strip has

$$y_{\text{upper}} = 9 - x^2 \quad \text{and} \quad y_{\text{lower}} = x^2 + 1$$

but for a strip to the *right* of (2, 5) the reverse is true, namely,

$$y_{\text{upper}} = x^2 + 1 \quad \text{and} \quad y_{\text{lower}} = 9 - x^2$$

Thus, from $x = 0$ to $x = 2$, the area of a strip is

$$\begin{aligned} (y_{\text{upper}} - y_{\text{lower}})\,dx &= [(9 - x^2) - (x^2 + 1]\,dx \\ &= (8 - 2x^2)\,dx \end{aligned}$$

but from $x = 2$ to $x = 3$, it is

$$\begin{aligned} (y_{\text{upper}} - y_{\text{lower}})\,dx &= [(x^2 + 1) - (9 - x^2)]\,dx \\ &= (2x^2 - 8)\,dx \end{aligned}$$

Therefore, to find the area of the entire region, we need *two* integrals:

$$\begin{aligned}\text{area} &= \int_0^2 (8-2x^2)\,dx + \int_2^3 (2x^2-8)\,dx\\ &= \left(8x-\frac{2x^3}{3}\right)\Bigg|_0^2 + \left(\frac{2x^3}{3}-8x\right)\Bigg|_2^3\\ &= \left[\left(16-\frac{16}{3}\right)-0\right]+\left[(18-24)-\left(\frac{16}{3}-16\right)\right]\\ &= \frac{46}{3}\end{aligned}$$

Now Work Problem 42 ◁

Horizontal Strips

Sometimes area can more easily be determined by summing areas of horizontal strips rather than vertical strips. In the following example, an area will be found by both methods. In each case, the strip of area determines the form of the integral.

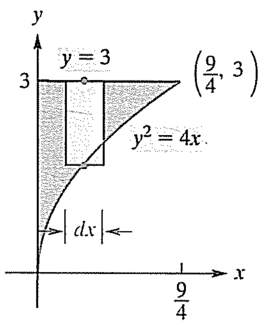

FIGURE 14.34 Vertical strip of area.

EXAMPLE 6 Vertical Strips and Horizontal Strips

Find the area of the region bounded by the curve $y^2 = 4x$ and the lines $y = 3$ and $x = 0$ (the y-axis).

Solution: The region is sketched in Figure 14.34. When the curves $y = 3$ and $y^2 = 4x$ intersect, $9 = 4x$, so $x = \frac{9}{4}$. Thus, the intersection point is $(\frac{9}{4}, 3)$. Since the width of the vertical strip is dx, we integrate with respect to the variable x. Accordingly, y_{upper} and y_{lower} must be expressed as functions of x. For the lower curve, $y^2 = 4x$, we have $y = \pm 2\sqrt{x}$. But $y \geq 0$ for the portion of this curve that bounds the region, so we use $y = 2\sqrt{x}$. The upper curve is $y = 3$. Hence, the height of the strip is

$$y_{upper} - y_{lower} = 3 - 2\sqrt{x}$$

Therefore, the strip has an area of $(3-2\sqrt{x})\,\Delta x$, and we wish to sum all such areas from $x = 0$ to $x = \frac{9}{4}$. We have

$$\begin{aligned}\text{area} &= \int_0^{9/4} (3-2\sqrt{x})\,dx = \left(3x - \frac{4x^{3/2}}{3}\right)\Bigg|_0^{9/4}\\ &= \left[3\left(\frac{9}{4}\right)-\frac{4}{3}\left(\frac{9}{4}\right)^{3/2}\right]-(0)\\ &= \frac{27}{4}-\frac{4}{3}\left[\left(\frac{9}{4}\right)^{1/2}\right]^3 = \frac{27}{4}-\frac{4}{3}\left(\frac{3}{2}\right)^3 = \frac{9}{4}\end{aligned}$$

CAUTION!

With horizontal strips, the width is dy.

Let us now approach this problem from the point of view of a **horizontal strip** as shown in Figure 14.35. The width of the strip is dy. The length of the strip is the *x-value on the rightmost curve minus the x-value on the leftmost curve*. Thus, the area of the strip is

$$(x_{right} - x_{left})\,dy$$

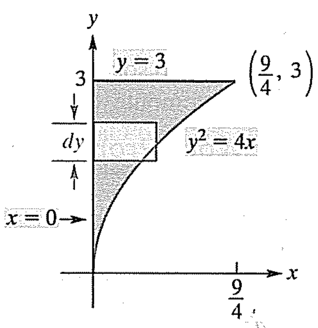

FIGURE 14.35 Horizontal strip of area.

We wish to sum all such areas from $y = 0$ to $y = 3$:

$$\sum (x_{right} - x_{left})\,dy \to \int_0^3 (x_{right} - x_{left})\,dy$$

Since the variable of integration is y, we must express x_{right} and x_{left} as functions of y. The rightmost curve is $y^2 = 4x$ so that $x = y^2/4$. The left curve is $x = 0$. Thus,

$$\text{area} = \int_0^3 (x_{\text{right}} - x_{\text{left}})\, dy$$

$$= \int_0^3 \left(\frac{y^2}{4} - 0\right) dy = \frac{y^3}{12}\bigg|_0^3 = \frac{9}{4}$$

Note that for this region, horizontal strips make the definite integral easier to evaluate (and set up) than an integral with vertical strips. In any case, remember that **the limits of integration are limits for the variable of integration.**

Now Work Problem 56 ◁

EXAMPLE 7 Advantage of Horizontal Elements

Find the area of the region bounded by the graphs of $y^2 = x$ and $x - y = 2$.

Solution: The region is sketched in Figure 14.36. The curves intersect when $y^2 - y = 2$. Thus, $y^2 - y - 2 = 0$; equivalently, $(y + 1)(y - 2) = 0$, from which it follows that $y = -1$ or $y = 2$. This gives the intersection points $(1, -1)$ and $(4, 2)$. Let us try vertical strips of area. [See Figure 14.36(a).] Solving $y^2 = x$ for y gives $y = \pm\sqrt{x}$. As seen in Figure 14.36(a), to the *left* of $x = 1$, the upper end of the strip lies on $y = \sqrt{x}$ and the lower end lies on $y = -\sqrt{x}$. To the *right* of $x = 1$, the upper curve is $y = \sqrt{x}$ and the lower curve is $x - y = 2$ (or $y = x - 2$). Thus, with vertical strips, *two* integrals are needed to evaluate the area:

$$\text{area} = \int_0^1 (\sqrt{x} - (-\sqrt{x}))\, dx + \int_1^4 (\sqrt{x} - (x - 2))\, dx$$

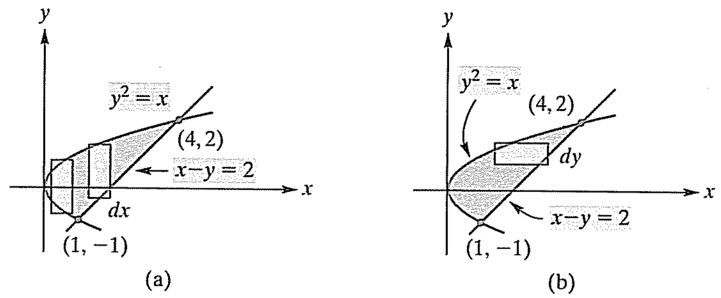

FIGURE 14.36 Region of Example 7 with vertical and horizontal strips.

Perhaps the use of horizontal strips can simplify our work. In Figure 14.36(b), the width of the strip is Δy. The rightmost curve is *always* $x - y = 2$ (or $x = y + 2$), and the leftmost curve is always $y^2 = x$ (or $x = y^2$). Therefore, the area of the horizontal strip is $[(y + 2) - y^2]\,\Delta y$, so the total area is

$$\text{area} = \int_{-1}^2 (y + 2 - y^2)\, dy = \frac{9}{2}$$

Clearly, the use of horizontal strips is the most desirable approach to solving the problem. Only a single integral is needed, and it is much simpler to compute.

Now Work Problem 57 ◁

PROBLEMS 14.9

In Problems 1–24, use a definite integral to find the area of the region bounded by the given curve, the x-axis, and the given lines. In each case, first sketch the region. Watch out for areas of regions that are below the x-axis.

1. $y = 5x + 2, \quad x = 1, \quad x = 4$

2. $y = x + 5, \quad x = 2, \quad x = 4$

3. $y = 3x^2, \quad x = 1, \quad x = 3$

4. $y = x^2, \quad x = 2, \quad x = 3$

5. $y = x + x^2 + x^3, \quad x = 1$

6. $y = x^2 - 2x, \quad x = -3, \quad x = -1$

7. $y = 3x^2 - 4x, \quad x = -2, \quad x = -1$

8. $y = 2 - x - x^2$

9. $y = \dfrac{4}{x}, \quad x = 1, \quad x = 2$

10. $y = 2 - x - x^3, \quad x = -3, \quad x = 0$

11. $y = e^x, \quad x = 1, \quad x = 3$

12. $y = \dfrac{1}{(x-1)^2}, \quad x = 2, \quad x = 3$

13. $y = \dfrac{1}{x}, \quad x = 1, \quad x = e$

14. $y = \sqrt{x+9}, \quad x = -9, \quad x = 0$

15. $y = x^2 - 4x, \quad x = 2, \quad x = 6$

16. $y = \sqrt{2x-1}, \quad x = 1, \quad x = 5$

17. $y = x^3 + 3x^2, \quad x = -2, \quad x = 2$

18. $y = \sqrt[3]{x}, \quad x = 2$

19. $y = e^x + 1, \quad x = 0, \quad x = 1$

20. $y = |x|, \quad x = -2, \quad x = 2$

21. $y = x + \dfrac{2}{x}, \quad x = 1, \quad x = 2$

22. $y = x^3, \quad x = -2, \quad x = 4$

23. $y = \sqrt{x-2}, \quad x = 2, \quad x = 6$

24. $y = x^2 + 1, \quad x = 0, \quad x = 4$

25. Given that

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x < 2 \\ 16 - 2x & \text{if } x \ge 2 \end{cases}$$

determine the area of the region bounded by the graph of $y = f(x)$, the x-axis, and the line $x = 3$. Include a sketch of the region.

26. Under conditions of a continuous uniform distribution (a topic in statistics), the proportion of persons with incomes between a and t, where $a \le t \le b$, is the area of the region between the curve $y = 1/(b-a)$ and the x-axis from $x = a$ to $x = t$. Sketch the graph of the curve and determine the area of the given region.

27. Suppose $f(x) = x/8$, where $0 \le x \le 4$. If f is a density function (refer to Example 2), find each of the following.

(a) $P(0 \le x \le 1)$
(b) $P(2 \le x \le 4)$
(c) $P(x \ge 3)$

28. Suppose $f(x) = \frac{1}{3}(1-x)^2$, where $0 \le x \le 3$. If f is a density function (refer to Example 2), find each of the following.

(a) $P(1 \le x \le 2)$
(b) $P\left(1 \le x \le \frac{5}{2}\right)$
(c) $P(x \le 1)$
(d) $P(x \ge 1)$ using your result from part (c)

29. Suppose $f(x) = 1/x$, where $e \le x \le e^2$. If f is a density function (refer to Example 2), find each of the following.

(a) $P(3 \le x \le 7)$
(b) $P(x \le 5)$
(c) $P(x \ge 4)$
(d) Verify that $P(e \le x \le e^2) = 1$.

30. **(a)** Let r be a real number, where $r > 1$. Evaluate

$$\int_1^r \frac{1}{x^2}\,dx$$

(b) Your answer to part (a) can be interpreted as the area of a certain region of the plane. Sketch this region.

(c) Evaluate $\lim_{r\to\infty}\left(\int_1^r \frac{1}{x^2}\,dx\right)$.

(d) Your answer to part (c) can be interpreted as the area of a certain region of the plane. Sketch this region.

In Problems 31–34, use definite integration to estimate the area of the region bounded by the given curve, the x-axis, and the given lines. Round your answer to two decimal places.

31. $y = \dfrac{1}{x^2+1}, \quad x = -2, \quad x = 1$

32. $y = \dfrac{x}{\sqrt{x+5}}, \quad x = 2, \quad x = 7$

33. $y = x^4 - 2x^3 - 2, \quad x = 1, \quad x = 3$

34. $y = 1 + 3x - x^4$

In Problems 35–38, express the area of the shaded region in terms of an integral (or integrals). Do not evaluate your expression.

35. See Figure 14.37.

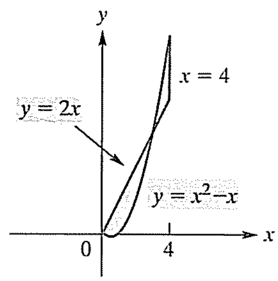

FIGURE 14.37

36. See Figure 14.38.

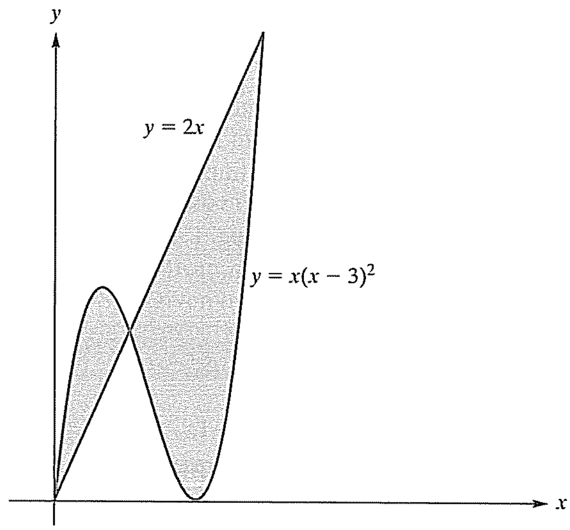

FIGURE 14.38

37. See Figure 14.39.

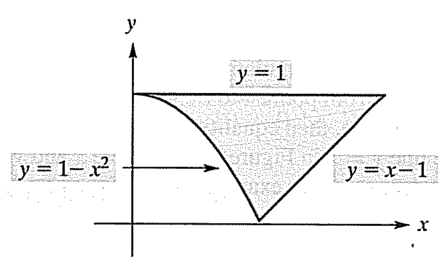

FIGURE 14.39

38. See Figure 14.40.

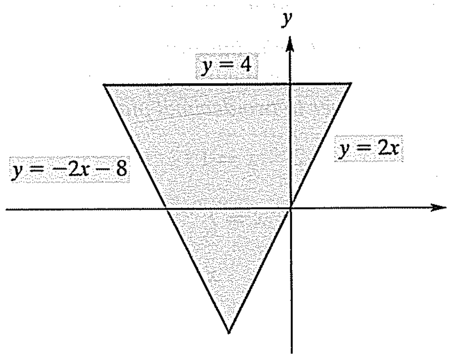

FIGURE 14.40

39. Express, in terms of a single integral, the total area of the region to the right of the line $x = 1$ that is between the curves $y = x^2 - 5$ and $y = 7 - 2x^2$. Do *not* evaluate the integral.

40. Express, in terms of a single integral, the total area of the region in the first quadrant bounded by the x-axis and the graphs of $y^2 = x$ and $2y = 3 - x$. Do *not* evaluate the integral.

In Problems 41–56, find the area of the region bounded by the graphs of the given equations. Be sure to find any needed points of intersection. Consider whether the use of horizontal strips makes the integral simpler than when vertical strips are used.

41. $y = x^2, \quad y = 2x$ **42.** $y = x, \quad y = -x + 3, \quad y = 0$

43. $y = 10 - x^2, \quad y = 4$ **44.** $y^2 = x + 1, \quad x = 1$

45. $x = 8 + 2y, \quad x = 0, \quad y = -1, \quad y = 3$

46. $y = x - 6, \quad y^2 = x$ **47.** $y^2 = 4x, \quad y = 2x - 4$

48. $y = x^3, \quad y = x + 6, \quad x = 0.$
(*Hint:* The only real root of $x^3 - x - 6 = 0$ is 2.)

49. $2y = 4x - x^2, \quad 2y = x - 4$

50. $y = \sqrt{x}, \quad y = x^2$

51. $y = 8 - x^2, \quad y = x^2, \quad x = -1, \quad x = 1$

52. $y = x^3 + x, y = 0, x = -1, x = 2$

53. $y = x^3 - 1, \quad y = x - 1$

54. $y = x^3, \quad y = \sqrt{x}$

55. $4x + 4y + 17 = 0, \quad y = \dfrac{1}{x}$

56. $y^2 = -x - 2, \quad x - y = 5, \quad y = -1, \quad y = 1$

57. Find the area of the region that is between the curves

$$y = x - 1 \quad \text{and} \quad y = 5 - 2x$$

from $x = 0$ to $x = 4$.

58. Find the area of the region that is between the curves

$$y = x^2 - 4x + 4 \quad \text{and} \quad y = 10 - x^2$$

from $x = 2$ to $x = 4$.

59. Lorenz Curve A *Lorenz curve* is used in studying income distributions. If x is the cumulative percentage of income recipients, ranked from poorest to richest, and y is the cumulative percentage of income, then equality of income distribution is given by the line $y = x$ in Figure 14.41, where x and y are expressed as decimals. For example, 10% of the people receive 10% of total income, 20% of the people receive 20% of the income, and so on. Suppose the actual distribution is given by the Lorenz curve defined by

$$y = \frac{14}{15}x^2 + \frac{1}{15}x$$

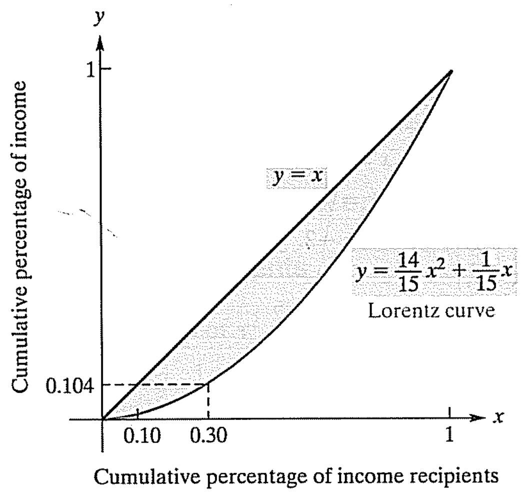

FIGURE 14.41

Note, for example, that 30% of the people receive only 10.4% of total income. The degree of deviation from equality is measured by the *coefficient of inequality*[15] for a Lorenz curve. This coefficient is defined to be the area between the curve and the diagonal, divided by the area under the diagonal:

$$\frac{\text{area between curve and diagonal}}{\text{area under diagonal}}$$

For example, when all incomes are equal, the coefficient of inequality is zero. Find the coefficient of inequality for the Lorenz curve just defined.

60. Lorenz curve Find the coefficient of inequality as in Problem 59 for the Lorenz curve defined by $y = \frac{11}{12}x^2 + \frac{1}{12}x$.

61. Find the area of the region bounded by the graphs of the equations $y^2 = 3x$ and $y = mx$, where m is a positive constant.

62. **(a)** Find the area of the region bounded by the graphs of $y = x^2 - 1$ and $y = 2x + 2$.
(b) What percentage of the area in part (a) lies above the x-axis?

63. The region bounded by the curve $y = x^2$ and the line $y = 4$ is divided into two parts of equal area by the line $y = k$, where k is a constant. Find the value of k.

In Problems 64–68, estimate the area of the region bounded by the graphs of the given equations. Round your answer to two decimal places.

64. $y = x^2 - 4x + 1, \quad y = -\dfrac{6}{x}$

65. $y = \sqrt{25 - x^2}, \quad y = 7 - 2x - x^4$

66. $y = x^3 - 8x + 1, \quad y = x^2 - 5$

67. $y = x^5 - 3x^3 + 2x, \quad y = 3x^2 - 4$

68. $y = x^4 - 3x^3 - 15x^2 + 19x + 30, \quad y = x^3 + x^2 - 20x$

[15] G. Stigler, *The Theory of Price*, 3rd ed. (New York: The Macmillan Company, 1966), pp. 293–94.

Objective

To develop the economic concepts of consumers' surplus and producers' surplus, which are represented by areas.

14.10 Consumers' and Producers' Surplus

Determining the area of a region has applications in economics. Figure 14.42 shows a supply curve for a product. The curve indicates the price p per unit at which the manufacturer will sell (or supply) q units. The diagram also shows a demand curve for the product. This curve indicates the price p per unit at which consumers will purchase (or demand) q units. The point (q_0, p_0) where the two curves intersect is called the *point of equilibrium*. Here p_0 is the price per unit at which consumers will purchase the same quantity q_0 of a product that producers wish to sell at that price. In short, p_0 is the price at which stability in the producer–consumer relationship occurs.

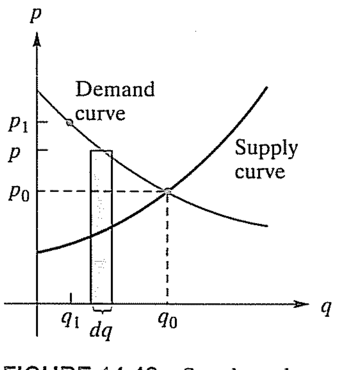

FIGURE 14.42 Supply and demand curves.

Let us assume that the market is at equilibrium and the price per unit of the product is p_0. According to the demand curve, there are consumers who would be willing to pay *more* than p_0. For example, at the price per unit of p_1, consumers would buy q_1 units. These consumers are benefiting from the lower equilibrium price p_0.

The vertical strip in Figure 14.42 has area $p\,dq$. This expression can also be thought of as the total amount of money that consumers would spend by buying dq units of the product if the price per unit were p. Since the price is actually p_0, these consumers spend only $p_0\,dq$ for the dq units and thus benefit by the amount $p dq - p_0\,dq$. This expression can be written $(p - p_0)\,dq$, which is the area of a rectangle of width dq and height $p - p_0$. (See Figure 14.43.) Summing the areas of all such rectangles from $q = 0$ to $q = q_0$ by definite integration, we have

$$\int_0^{q_0} (p - p_0)\,dq$$

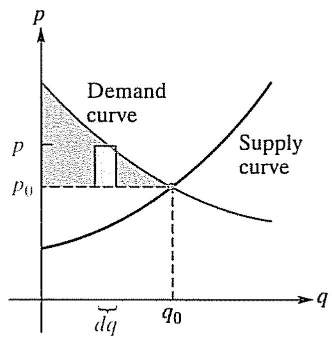

FIGURE 14.43 Benefit to consumers for dq units.

This integral, under certain conditions, represents the total gain to consumers who are willing to pay more than the equilibrium price. This total gain is called **consumers' surplus**, abbreviated CS. If the demand function is given by $p = f(q)$, then

$$\text{CS} = \int_0^{q_0} [f(q) - p_0]\,dq$$

Geometrically (see Figure 14.44), consumers' surplus is represented by the area between the line $p = p_0$ and the demand curve $p = f(q)$ from $q = 0$ to $q = q_0$.

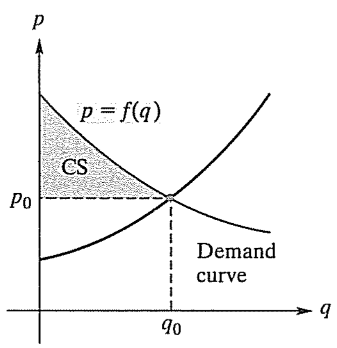

FIGURE 14.44 Consumers' surplus.

Some of the producers also benefit from the equilibrium price, since they are willing to supply the product at prices *less* than p_0. Under certain conditions, the total gain to the producers is represented geometrically in Figure 14.45 by the area between the line $p = p_0$ and the supply curve $p = g(q)$ from $q = 0$ to $q = q_0$. This gain, called **producers' surplus** and abbreviated PS, is given by

$$\text{PS} = \int_0^{q_0} [p_0 - g(q)]\,dq$$

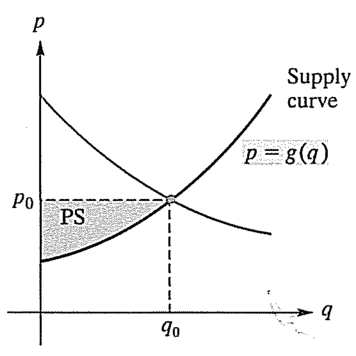

FIGURE 14.45 Producers' surplus.

EXAMPLE 1 Finding Consumers' Surplus and Producers' Surplus

The demand function for a product is

$$p = f(q) = 100 - 0.05q$$

where p is the price per unit (in dollars) for q units. The supply function is

$$p = g(q) = 10 + 0.1q$$

Determine consumers' surplus and producers' surplus under market equilibrium.

Solution: First we must find the equilibrium point (p_0, q_0) by solving the system formed by the functions $p = 100 - 0.05q$ and $p = 10 + 0.1q$. We thus equate the two expressions for p and solve:

$$10 + 0.1q = 100 - 0.05q$$
$$0.15q = 90$$
$$q = 600$$

When $q = 600$ then $p = 10 + 0.1(600) = 70$. Hence, $q_0 = 600$ and $p_0 = 70$. Consumers' surplus is

$$\text{CS} = \int_0^{q_0} [f(q) - p_0]\, dq = \int_0^{600} (100 - 0.05q - 70)\, dq$$

$$= \left(30q - 0.05\frac{q^2}{2}\right)\Bigg|_0^{600} = 9000$$

Producers' surplus is

$$\text{PS} = \int_0^{q_0} [p_0 - g(q)]\, dq = \int_0^{600} [70 - (10 + 0.1q)]\, dq$$

$$= \left(60q - 0.1\frac{q^2}{2}\right)\Bigg|_0^{600} = 18{,}000$$

Therefore, consumers' surplus is \$9000 and producers' surplus is \$18,000.

Now Work Problem 1 ◁

EXAMPLE 2 Using Horizontal Strips to Find Consumers' Surplus and Producers' Surplus

The demand equation for a product is

$$q = f(p) = \frac{90}{p} - 2$$

and the supply equation is $q = g(p) = p - 1$. Determine consumers' surplus and producers' surplus when market equilibrium has been established.

Solution: Determining the equilibrium point, we have

$$p - 1 = \frac{90}{p} - 2$$

$$p^2 + p - 90 = 0$$

$$(p + 10)(p - 9) = 0$$

Thus, $p_0 = 9$, so $q_0 = 9 - 1 = 8$. (See Figure 14.46.) Note that the demand equation expresses q as a function of p. Since consumers' surplus can be considered an area, this area can be determined by means of horizontal strips of width dp and length $q = f(p)$. The areas of these strips are summed from $p = 9$ to $p = 45$ by integrating with

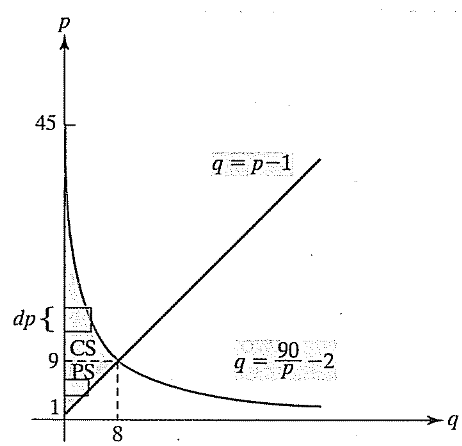

FIGURE 14.46 Diagram for Example 2.

respect to p:

$$CS = \int_9^{45} \left(\frac{90}{p} - 2\right) dp = (90 \ln |p| - 2p)\Big|_9^{45}$$
$$= 90 \ln 5 - 72 \approx 72.85$$

Using horizontal strips for producers' surplus, we have

$$PS = \int_1^9 (p-1)\, dp = \frac{(p-1)^2}{2}\Big|_1^9 = 32$$

Now Work Problem 5 ◁

PROBLEMS 14.10

In Problems 1–6, the first equation is a demand equation and the second is a supply equation of a product. In each case, determine consumers' surplus and producers' surplus under market equilibrium.

1. $p = 22 - 0.8q$
   $p = 6 + 1.2q$

2. $p = 2200 - q^2$
   $p = 400 + q^2$

3. $p = \dfrac{50}{q+5}$
   $p = \dfrac{q}{10} + 4.5$

4. $p = 900 - q^2$
   $p = 10q + 300$

5. $q = 100(10 - 2p)$
   $q = 50(2p - 1)$

6. $q = \sqrt{100 - p}$
   $q = \dfrac{p}{2} - 10$

7. The demand equation for a product is

$$q = 10\sqrt{100 - p}$$

Calculate consumers' surplus under market equilibrium, which occurs at a price of \$84.

8. The demand equation for a product is

$$q = 400 - p^2$$

and the supply equation is

$$p = \frac{q}{60} + 5$$

Find producers' surplus and consumers' surplus under market equilibrium.

9. The demand equation for a product is $p = 2^{10-q}$, and the supply equation is $p = 2^{q+2}$, where p is the price per unit (in hundreds of dollars) when q units are demanded or supplied. Determine, to the nearest thousand dollars, consumers' surplus under market equilibrium.

10. The demand equation for a product is

$$(p + 10)(q + 20) = 1000$$

and the supply equation is

$$q - 4p + 10 = 0$$

(a) Verify, by substitution, that market equilibrium occurs when $p = 10$ and $q = 30$.
(b) Determine consumers' surplus under market equilibrium.

11. The demand equation for a product is

$$p = 60 - \frac{50q}{\sqrt{q^2 + 3600}}$$

and the supply equation is

$$p = 10 \ln (q + 20) - 26$$

Determine consumers' surplus and producers' surplus under market equilibrium. Round your answers to the nearest integer.

12. **Producers' Surplus** The supply function for a product is given by the following table, where p is the price per unit (in dollars) at which q units are supplied to the market:

q	0	10	20	30	40	50
p	25	49	59	71	80	94

Use the trapezoidal rule to estimate the producers' surplus if the selling price is \$80.

Chapter 14 Review

Important Terms and Symbols

			Examples
Section 14.1	**Differentials**		
	differential, dy, dx		Ex. 1, p. 627
Section 14.2	**The Indefinite Integral**		
	antiderivative indefinite integral	$\int f(x)\,dx$ integral sign	Ex. 1, p. 633
	integrand variable of integration	constant of integration	Ex. 2, p. 633
Section 14.3	**Integration with Initial Conditions**		
	initial condition		Ex. 1, p. 638
Section 14.4	**More Integration Formulas**		
	power rule for integration		Ex. 1, p. 642

Summary

If $y = f(x)$ is a differentiable function of x, we define the differential dy by

$$dy = f'(x)\,dx$$

where $dx = \Delta x$ is a change in x and can be any real number. (Thus dy is a function of two variables, namely x and dx.) If dx is close to zero, then dy is an approximation to $\Delta y = f(x+dx) - f(x)$.

$$\Delta y \approx dy$$

Moreover, dy can be used to approximate a function value using

$$f(x+dx) \approx f(x) + dy$$

An antiderivative of a function f is a function F such that $F'(x) = f(x)$. Any two antiderivatives of f differ at most by a constant. The most general antiderivative of f is called the indefinite integral of f and is denoted $\int f(x)\,dx$. Thus,

$$\int f(x)\,dx = F(x) + C$$

where C is called the constant of integration, if and only if $F' = f$.

Some elementary integration formulas are as follows:

$$\int k\,dx = kx + C \qquad k \text{ a constant}$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \qquad a \neq -1$$

$$\int \frac{1}{x}dx = \ln x + C \qquad \text{for } x > 0$$

$$\int e^x\,dx = e^x + C$$

$$\int kf(x)\,dx = k\int f(x)\,dx \qquad k \text{ a constant}$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Another formula is the power rule for integration:

$$\int u^a du = \frac{u^{a+1}}{a+1} + C, \quad \text{if } a \neq -1$$

Here u represents a differentiable function of x, and du is its differential. In applying the power rule to a given integral, it is important that the integral be written in a form that precisely matches the power rule. Other integration formulas are

$$\int e^u du = e^u + C$$

$$\text{and} \quad \int \frac{1}{u}du = \ln|u| + C \qquad u \neq 0$$

If the rate of change of a function f is known—that is, if f' is known—then f is an antiderivative of f'. In addition, if we know that f satisfies an initial condition, then we can find the particular antiderivative. For example, if a marginal-cost function dc/dq is given to us, then by integration, we can find the most general form of c. That form involves a constant of integration. However, if we are also given fixed costs (that is, costs involved when $q = 0$), then we can determine the value of the constant of integration and thus find the particular cost function c. Similarly, if we are given a marginal-revenue function dr/dq, then by integration and by using the fact that $r = 0$ when $q = 0$, we can determine the particular revenue function r. Once r is known, the corresponding demand equation can be found by using the equation $p = r/q$.

It is helpful at this point to review summation notation from Section 1.5. This notation is especially useful in determining areas. For continuous $f \geq 0$, to find the area of the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$, we divide the interval $[a, b]$ into n subintervals of equal length $dx = (b-a)/n$. If x_i is the right-hand endpoint of an arbitrary subinterval, then the product $f(x_i)\,dx$ is the area of a rectangle. Denoting the sum of all such areas of rectangles for the n subintervals by S_n, we define the limit of S_n as $n \to \infty$ as the area of the entire region:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,dx = \text{area}$$

If the restriction that $f(x) \geq 0$ is omitted, this limit is defined as the definite integral of f over $[a, b]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,dx = \int_a^b f(x)\,dx$$

Instead of evaluating definite integrals by using limits, we may be able to employ the Fundamental Theorem of Integral Calculus. Mathematically,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where F is any antiderivative of f.

Some properties of the definite integral are

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \qquad k \text{ a constant}$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

If $f(x) \geq 0$ is continuous on $[a, b]$, then the definite integral can be used to find the area of the region bounded by $y = f(x)$, the x-axis, $x = a$, and $x = b$. The definite integral can also be used to find areas of more complicated regions. In these situations, a strip of area should be drawn in the region. This allows us to set up the proper definite integral. In this regard, both vertical strips and horizontal strips have their uses.

One application of finding areas involves consumers' surplus and producers' surplus. Suppose the market for a product is at equilibrium and (q_0, p_0) is the equilibrium point (the point of intersection of the supply curve and the demand curve for the product). Then consumers' surplus, CS, corresponds to the area from $q = 0$ to $q = q_0$, bounded above by the demand curve and below by the line $p = p_0$. Thus,

$$\text{CS} = \int_0^{q_0} (f(q) - p_0)\,dq$$

where f is the demand function. Producers' surplus, PS, corresponds to the area from $q = 0$ to $q = q_0$, bounded above by the line $p = p_0$ and below by the supply curve. Therefore,

$$\text{PS} = \int_0^{q_0} (p_0 - g(q))\,dq$$

where g is the supply function.

Review Problems

In Problems 1–40, determine the integrals.

1. $\int (x^3 + 2x - 7)\,dx$

2. $\int dx$

3. $\int_0^{12} (9\sqrt{3x} + 3x^2)\,dx$

4. $\int \frac{4}{5 - 3x}\,dx$

5. $\int \frac{6}{(x+5)^3}\,dx$

6. $\int_3^9 (y - 6)^{301}\,dy$

7. $\int \frac{6x^2 - 12}{x^3 - 6x + 1}\,dx$

8. $\int_0^3 2xe^{5-x^2}\,dx$

9. $\int_0^1 \sqrt[3]{3t + 8}\,dt$

10. $\int \frac{4 - 2x}{7}\,dx$

11. $\int y(y+1)^2\,dy$

12. $\int_0^1 10^{-8}\,dx$

13. $\int \frac{\sqrt[3]{t} - \sqrt{t}}{\sqrt[3]{t}}\,dt$

14. $\int \frac{(0.5x - 0.1)^4}{0.4}\,dx$

15. $\int_1^3 \frac{2t^2}{3 + 2t^3}\,dt$

16. $\int \frac{4x^2 - x}{x}\,dx$

17. $\int x^2\sqrt{3x^3 + 2}\,dx$

18. $\int (6x^2 + 4x)(x^3 + x^2)^{3/2}\,dx$

19. $\int (e^{2y} - e^{-2y})\,dy$

20. $\int \frac{8x}{3\sqrt[3]{7 - 2x^2}}\,dx$

21. $\int \left(\frac{1}{x} + \frac{2}{x^2}\right)dx$

22. $\int_0^2 \frac{3e^{3x}}{1 + e^{3x}}\,dx$

23. $\int_{-2}^2 (y^4 + y^3 + y^2 + y)\,dy$

24. $\int_7^{70} dx$

25. $\int_1^2 5x\sqrt{5 - x^2}\,dx$

26. $\int_0^1 (2x + 1)(x^2 + x)^4\,dx$

27. $\int_0^1 \left[2x - \frac{1}{(x+1)^{2/3}}\right]dx$

28. $\int_0^{18} (2x - 3\sqrt{2x} + 1)\,dx$

29. $\int \frac{\sqrt{t} - 3}{t^2}\,dt$

30. $\int \frac{3z^3}{z - 1}\,dz$

31. $\int_{-1}^0 \frac{x^2 + 4x - 1}{x + 2}\,dx$

32. $\int \frac{(x^2 + 4)^2}{x^2}\,dx$

33. $\int \frac{e^{\sqrt{x}} + x}{2\sqrt{x}}\,dx$

34. $\int \frac{e^{\sqrt{5x}}}{\sqrt{3x}}\,dx$

35. $\int_1^e \frac{e^{\ln x}}{x^2}\,dx$

36. $\int \frac{6x^2 + 4}{e^{x^3 + 2x}}\,dx$

37. $\int \frac{(1 + e^{2x})^3}{e^{-2x}}\,dx$

38. $\int \frac{c}{e^{bx}(a + e^{-bx})^n}\,dx$ for $n \neq 1$ and $b \neq 0$

39. $\int 3\sqrt{10^{3x}}\,dx$

40. $\int \frac{5x^3 + 15x^2 + 37x + 3}{x^2 + 3x + 7}\,dx$

In Problems 41 and 42, find y, subject to the given condition.

41. $y' = e^{2x} + 3, \quad y(0) = -\frac{1}{2}$

42. $y' = \frac{x + 5}{x}, \quad y(1) = 3$

In Problems 43–50, determine the area of the region bounded by the given curve, the x-axis, and the given lines.

43. $y = x^3, \quad x = 0, \ x = 2$

44. $y = 4e^x, \quad x = 0, \ x = 3$

45. $y = \sqrt{x + 4}, \quad x = 0$

46. $y = x^2 - x - 6, \quad x = -4, \ x = 3$

47. $y = 5x - x^2$

48. $y = \sqrt[3]{x}, \quad x = 8, \ x = 16$

49. $y = \frac{1}{x} + 2, \quad x = 1, \ x = 4$

50. $y = x^3 - 1, \quad x = -1$

In Problems 51–58, find the area of the region bounded by the given curves.

51. $y^2 = 4x, \quad x = 0, \quad y = 2$ **52.** $y = 3x^2 - 5, \quad x = 0, \quad y = 4$

53. $y = -x(x - a), \quad y = 0$ for $0 < a$

54. $y = 2x^2, \quad y = x^2 + 9$ **55.** $y = x^2 - x, \quad y = 10 - x^2$

56. $y = \sqrt{x}, \quad x = 0, \quad y = 3$

57. $y = \ln x, \quad x = 0, \quad y = 0, \quad y = 1$

58. $y = 3 - x, \quad y = x - 4, \quad y = 0, \quad y = 3$

59. Marginal Revenue If marginal revenue is given by

$$\frac{dr}{dq} = 100 - \frac{3}{2}\sqrt{2q}$$

determine the corresponding demand equation.

60. Marginal Cost If marginal cost is given by

$$\frac{dc}{dq} = q^2 + 7q + 6$$

and fixed costs are 2500, determine the total cost of producing six units. Assume that costs are in dollars.

61. Marginal Revenue A manufacturer's marginal-revenue function is

$$\frac{dr}{dq} = 250 - q - 0.2q^2$$

If r is in dollars, find the increase in the manufacturer's total revenue if production is increased from 15 to 25 units.

62. Marginal Cost A manufacturer's marginal-cost function is

$$\frac{dc}{dq} = \frac{1000}{\sqrt{3q + 70}}$$

If c is in dollars, determine the cost involved to increase production from 10 to 33 units.

63. Hospital Discharges For a group of hospitalized individuals, suppose the discharge rate is given by

$$f(t) = 0.007e^{-0.007t}$$

where $f(t)$ is the proportion discharged per day at the end of t days of hospitalization. What proportion of the group is discharged at the end of 100 days?

64. Business Expenses The total expenditures (in dollars) of a business over the next five years are given by

$$\int_0^5 4000e^{0.05t}\, dt$$

Evaluate the expenditures.

65. Find the area of the region between the curves $y = 9 - 2x$ and $y = x$ from $x = 0$ to $x = 4$.

66. Find the area of the region between the curves $y = 2x^2$ and $y = 2 - 5x$ from $x = -1$ to $x = \frac{1}{3}$.

67. Consumers' and Producers' Surplus The demand equation for a product is

$$p = 0.01q^2 - 1.1q + 30$$

and the supply equation is

$$p = 0.01q^2 + 8$$

Determine consumers' surplus and producers' surplus when market equilibrium has been established.

68. Consumers' Surplus The demand equation for a product is

$$p = (q - 4)^2$$

and the supply equation is

$$p = q^2 + q + 7$$

where p (in thousands of dollars) is the price per 100 units when q hundred units are demanded or supplied. Determine consumers' surplus under market equilibrium.

69. Biology In a discussion of gene mutation,[16] the equation

$$\int_{q_0}^{q_n} \frac{dq}{q - \widehat{q}} = -(u + v)\int_0^n dt$$

occurs, where u and v are gene mutation rates, the q's are gene frequencies, and n is the number of generations. Assume that all letters represent constants, except q and t. Integrate both sides and then use your result to show that

$$n = \frac{1}{u + v}\ln\left|\frac{q_0 - \widehat{q}}{q_n - \widehat{q}}\right|$$

70. Fluid Flow In studying the flow of a fluid in a tube of constant radius R, such as blood flow in portions of the body, we can think of the tube as consisting of concentric tubes of radius r, where $0 \le r \le R$. The velocity v of the fluid is a function of r and is given by[17]

$$v = \frac{(P_1 - P_2)(R^2 - r^2)}{4\eta l}$$

where P_1 and P_2 are pressures at the ends of the tube, η (a Greek letter read "eta") is the fluid viscosity, and l is the length of the tube. The volume rate of flow through the tube, Q, is given by

$$Q = \int_0^R 2\pi r v\, dr$$

Show that $Q = \dfrac{\pi R^4(P_1 - P_2)}{8\eta l}$. Note that R occurs as a factor to the fourth power. Thus, doubling the radius of the tube has the effect of increasing the flow by a factor of 16. The formula that you derived for the volume rate of flow is called *Poiseuille's law*, after the French physiologist Jean Poiseuille.

71. Inventory In a discussion of inventory, Barbosa and Friedman[18] refer to the function

$$g(x) = \frac{1}{k}\int_1^{1/x} ku^r\, du$$

[16] W. B. Mather, *Principles of Quantitative Genetics* (Minneapolis: Burgess Publishing Company, 1964).

[17] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

[18] L. C. Barbosa and M. Friedman, "Deterministic Inventory Lot Size Models—a General Root Law," *Management Science*, 24, no. 8 (1978), 819–26.

where k and r are constants, $k > 0$ and $r > -2$, and $x > 0$. Verify the claim that

$$g'(x) = -\frac{1}{x^{r+2}}$$

(*Hint:* Consider two cases: when $r \neq -1$ and when $r = -1$.)

In Problems 72–74, estimate the area of the region bounded by the given curves. Round your answer to two decimal places.

72. $y = x^3 + 9x^2 + 14x - 24, y = 0$

73. $y = x^3 + x^2 + x + 1, y = x^2 + 2x + 1$

74. $y = x^3 + x^2 - 5x - 3, y = x^2 + 2x + 3$

75. The demand equation for a product is

$$p = \frac{200}{\sqrt{q+20}}$$

and the supply equation is

$$p = 2\ln(q + 10) + 5$$

Determine consumers' surplus and producers' surplus under market equilibrium. Round your answers to the nearest integer.

EXPLORE & EXTEND Delivered Price

Suppose that you are a manufacturer of a product whose sales occur within R miles of your mill. Assume that you charge customers for shipping at the rate s, in dollars per mile, for each unit of product sold. If m is the unit price (in dollars) at the mill, then the delivered unit price p to a customer x miles from the mill is the mill price plus the shipping charge sx:

$$p = m + sx \qquad 0 \leq x \leq R \tag{1}$$

The problem is to determine the average delivered price of the units sold.

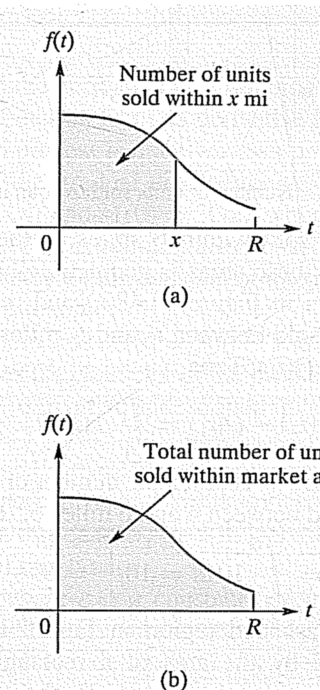

FIGURE 14.47 Number of units sold as an area.

Suppose that there is a function f such that $f(t) \geq 0$ on the interval $[0, R]$ and such that the area under the graph of f and above the t-axis from $t = 0$ to $t = x$ represents the total number of units Q sold to customers within x miles of the mill. [See Figure 14.47(a).] You can refer to f as the distribution of demand. Because Q is a function of x and is represented by area,

$$Q(x) = \int_0^x f(t)\,dt$$

In particular, the total number of units sold within the market area is

$$Q(R) = \int_0^R f(t)\,dt$$

[see Figure 14.47(b)]. For example, if $f(t) = 10$ and $R = 100$, then the total number of units sold within the market area is

$$Q(100) = \int_0^{100} 10\,dt = 10t\Big|_0^{100} = 1000 - 0 = 1000$$

The average delivered price A is given by

$$A = \frac{\text{total revenue}}{\text{total number of units sold}}$$

Because the denominator is $Q(R)$, A can be determined once the total revenue is found.

To find the total revenue, first consider the number of units sold over an interval. If $t_1 < t_2$ [see Figure 14.48(a)], then the area under the graph of f and above the t-axis from $t = 0$ to $t = t_1$ represents the number of units sold within t_1 miles of the mill. Similarly, the area under the graph of f and above the t-axis from $t = 0$ to $t = t_2$ represents the

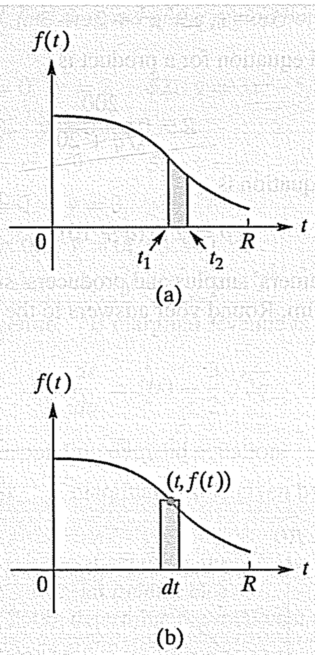

FIGURE 14.48 Number of units sold over an interval.

number of units sold within t_2 miles of the mill. Thus the difference in these areas is geometrically the area of the shaded region in Figure 14.48(a) and represents the number of units sold between t_1 and t_2 miles of the mill, which is $Q(t_2) - Q(t_1)$. Thus

$$Q(t_2) - Q(t_1) = \int_{t_1}^{t_2} f(t)\, dt$$

For example, if $f(t) = 10$, then the number of units sold to customers located between 4 and 6 miles of the mill is

$$Q(6) - Q(4) = \int_4^6 10\, dt = 10t\Big|_4^6 = 60 - 40 = 20$$

The area of the shaded region in Figure 14.48(a) can be approximated by the area of a rectangle [see Figure 14.48(b)] whose height is $f(t)$ and whose width is dt, where $dt = t_2 - t_1$. Thus the number of units sold over the interval of length dt is approximately $f(t)\, dt$. Because the price of each of these units is [from Equation (1)] approximately $m + st$, the revenue received is approximately

$$(m + st)f(t)\, dt$$

The sum of all such products from $t = 0$ to $t = R$ approximates the total revenue. Definite integration gives

$$\sum (m + st)f(t)\, dt \to \int_0^R (m + st)f(t)\, dt$$

Thus,

$$\text{total revenue} = \int_0^R (m + st)f(t)\, dt$$

Consequently, the average delivered price A is given by

$$A = \frac{\int_0^R (m + st)f(t)\, dt}{Q(R)}$$

Equivalently,

$$A = \frac{\int_0^R (m + st)f(t)\, dt}{\int_0^R f(t)\, dt}$$

For example, if $f(t) = 10, m = 200, s = 0.25$, and $R = 100$, then

$$\begin{aligned}\int_0^R (m + st)f(t)\, dt &= \int_0^{100} (200 + 0.25t) \cdot 10\, dt \\ &= 10 \int_0^{100} (200 + 0.25t)\, dt \\ &= 10 \left(200t + \frac{t^2}{8}\right)\Bigg|_0^{100} \\ &= 10 \left[\left(20{,}000 + \frac{10{,}000}{8}\right) - 0\right] \\ &= 212{,}500\end{aligned}$$

From before,

$$\int_0^R f(t)\, dt = \int_0^{100} 10\, dt = 1000$$

Thus, the average delivered price is $212{,}500/1000 = \$212.50$.

Problems

1. If $f(t) = 100 - 2t$, determine the number of units sold to customers located (a) within 5 miles of the mill, and (b) between 20 and 25 miles of the mill.

2. If $f(t) = 40 - 0.5t, m = 50, s = 0.20$, and $R = 80$, determine (a) the total revenue, (b) the total number of units sold, and (c) the average delivered price.

3. If $f(t) = 900 - t^2, m = 100, s = 1$, and $R = 30$, determine (a) the total revenue, (b) the total number of units sold, and (c) the average delivered price. Use a graphing calculator if you like.

4. How do real-world sellers of such things as books and clothing generally determine shipping charges for an order? (Visit an online retailer to find out.) How would you calculate average delivered price for their products? Is the procedure fundamentally different from the one discussed in this Explore & Extend?

15 Methods and Applications of Integration

We now know how to find the derivative of a function, and in some cases we know how to find a function from its derivative through integration. However, the integration process is not always straightforward.

Suppose we model the gradual disappearance of a chemical substance using the equations $M' = -0.004t$ and $M(0) = 3000$, where the amount M, in grams, is a function of time t in days. This initial-condition problem is easily solved by integration with respect to t and identifying the constant of integration. The result is $M = -0.002t^2 + 3000$. But what if, instead, the disappearance of the substance were modeled by the equations $M' = -0.004M$ and $M(0) = 3000$? The simple replacement of t in the first equation with M changes the character of the problem. We have not yet learned how to find a function when its derivative is described in terms of the function itself.

In the Explore & Extend in Chapter 13, there was a similar situation, involving an equation with P on one side and the derivative of P on the other. There, we used an approximation to solve the problem. In this chapter, we will learn a method that yields an exact solution for some problems of this type.

Equations of the form $y' = ky$, where k is a constant, are especially common. When y represents the amount of a radioactive substance, $y' = ky$ can represent the rate of its disappearance through radioactive decay. And if y is the temperature of a chicken just taken out of the oven or just put into a freezer, then a related formula, called Newton's law of cooling, can be used to describe the change in the chicken's internal temperature over time. Newton's law, which is discussed in this chapter, might be used to write procedures for a restaurant kitchen, so that food prone to contamination through bacterial growth does not spend too much time in the temperature danger zone (40°F to 140°F). (Bacterial growth, for that matter, also follows a $y' = ky$ law!)

Objective

To develop and apply the formula for integration by parts.

15.1 Integration by Parts[1]

Many integrals cannot be found by our previous methods. However, there are ways of changing certain integrals to forms that are easier to integrate. Of these methods, we will discuss two: *integration by parts* and (in Section 15.2) *integration using partial fractions.*

If u and v are differentiable functions of x, we have, by the product rule,

$$(uv)' = uv' + vu'$$

Rearranging gives

$$uv' = (uv)' - vu'$$

Integrating both sides with respect to x, we get

$$\int uv'\,dx = \int (uv)'\,dx - \int vu'\,dx \tag{1}$$

For $\int (uv)'\,dx$, we must find a function whose derivative with respect to x is $(uv)'$. Clearly, uv is such a function. Hence $\int (uv)'\,dx = uv + C_1$, and Equation (1) becomes

$$\int uv'\,dx = uv + C_1 - \int vu'\,dx$$

Absorbing C_1 into the constant of integration for $\int vu'\,dx$ and replacing $v'\,dx$ by dv and $u'\,dx$ by du, we have the *formula for integration by parts:*

Formula for Integration by Parts

$$\int u\,dv = uv - \int v\,du \tag{2}$$

This formula expresses an integral, $\int u\,dv$, in terms of another integral, $\int v\,du$, that may be easier to find.

To apply the formula to a given integral $\int f(x)\,dx$, we must write $f(x)\,dx$ as the product of two factors (or *parts*) by choosing a function u and a differential dv such that $f(x)\,dx = u\,dv$. However, for the formula to be useful, we must be able to integrate the part chosen for dv. To illustrate, consider

$$\int xe^x\,dx$$

This integral cannot be determined by previous integration formulas. One way to write $xe^x\,dx$ in the form $u\,dv$ is by letting

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

To apply the formula for integration by parts, we must find du and v:

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x + C_1$$

Thus,

$$\begin{aligned}\int xe^x\,dx &= \int u\,dv\\ &= uv - \int v\,du\\ &= x(e^x + C_1) - \int (e^x + C_1)\,dx\\ &= xe^x + C_1x - e^x - C_1x + C\\ &= xe^x - e^x + C\\ &= e^x(x-1) + C\end{aligned}$$

[1] This section can be omitted without loss of continuity.

The first constant, C_1, does not appear in the final answer. It is easy to prove that the constant involved in finding v from dv will always drop out, so from now on we will not write it when we find v.

When using the formula for integration by parts, sometimes the *best choice* for u and dv is not obvious. In some cases, one choice may be as good as another; in other cases, only one choice may be suitable. Insight into making a good choice (if any exists) will come only with practice and, of course, trial and error.

APPLY IT ▷

1. The monthly sales of a computer keyboard are estimated to decline at the rate of $S'(t) = -4te^{0.1t}$ keyboards per month, where t is time in months and $S(t)$ is the number of keyboards sold each month. If 5000 keyboards are sold now ($S(0) = 5000$), find $S(t)$.

EXAMPLE 1 Integration by Parts

Find $\int \frac{\ln x}{\sqrt{x}}\,dx$ by integration by parts.

Solution: We try

$$u = \ln x \quad \text{and} \quad dv = \frac{1}{\sqrt{x}}\,dx$$

Then

$$du = \frac{1}{x}\,dx \quad \text{and} \quad v = \int x^{-1/2}\,dx = 2x^{1/2}$$

Thus,

$$\begin{aligned}
\int \ln x\left(\frac{1}{\sqrt{x}}\,dx\right) &= \int u\,dv = uv - \int v\,du \\
&= (\ln x)(2\sqrt{x}) - \int (2x^{1/2})\left(\frac{1}{x}\,dx\right) \\
&= 2\sqrt{x}\ln x - 2\int x^{-1/2}\,dx \\
&= 2\sqrt{x}\ln x - 2(2\sqrt{x}) + C \qquad x^{1/2} = \sqrt{x} \\
&= 2\sqrt{x}[\ln(x) - 2] + C
\end{aligned}$$

Now Work Problem 3 ◁

Example 2 shows how a poor choice for u and dv can be made. If a choice does not work, there may be another that does.

EXAMPLE 2 Integration by Parts

Evaluate $\int_1^2 x\ln x\,dx$.

Solution: Since the integral does not fit a familiar form, we will try integration by parts. Let $u = x$ and $dv = \ln x\,dx$. Then $du = dx$, but $v = \int \ln x\,dx$ is not apparent by inspection. So we will make a different choice for u and dv. Let

$$u = \ln x \quad \text{and} \quad dv = x\,dx$$

Then

$$du = \frac{1}{x}\,dx \quad \text{and} \quad v = \int x\,dx = \frac{x^2}{2}$$

Therefore,

$$\begin{aligned}
\int_1^2 x\ln x\,dx &= (\ln x)\left(\frac{x^2}{2}\right)\Bigg|_1^2 - \int_1^2 \left(\frac{x^2}{2}\right)\frac{1}{x}\,dx \\
&= (\ln x)\left(\frac{x^2}{2}\right)\Bigg|_1^2 - \frac{1}{2}\int_1^2 x\,dx \\
&= \frac{x^2\ln x}{2}\Bigg|_1^2 - \frac{1}{2}\left(\frac{x^2}{2}\right)\Bigg|_1^2 \\
&= (2\ln 2 - 0) - \left(1 - \tfrac{1}{4}\right) = 2\ln 2 - \frac{3}{4}
\end{aligned}$$

Now Work Problem 5 ◁

EXAMPLE 3 Integration by Parts where u Is the Entire Integrand

Determine $\int \ln y \, dy$.

Solution: We cannot integrate $\ln y$ by previous methods, so we will try integration by parts. Let $u = \ln y$ and $dv = dy$. Then $du = (1/y)\,dy$ and $v = y$. So we have

$$\begin{aligned}\int \ln y \, dy &= (\ln y)(y) - \int y\left(\frac{1}{y}\,dy\right)\\ &= y\ln y - \int dy = y \ln y - y + C\\ &= y[\ln(y) - 1] + C\end{aligned}$$

Now Work Problem 37 ◁

Before trying integration by parts, see whether the technique is really needed. Sometimes the integral can be handled by a basic technique, as Example 4 shows.

EXAMPLE 4 Basic Integration Form

Determine $\int xe^{x^2}\, dx$.

Solution: This integral can be fit to the form $\int e^u \, du$.

CAUTION!

Remember the simpler integration forms too. Integration by parts is not needed here.

$$\begin{aligned}\int xe^{x^2}\,dx &= \frac{1}{2}\int e^{x^2}(2x\,dx)\\ &= \frac{1}{2}\int e^u\,du \qquad \text{where } u = x^2\\ &= \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C\end{aligned}$$

Now Work Problem 17 ◁

Sometimes integration by parts must be used more than once, as shown in the following example.

APPLY IT ▷

2. Suppose a population of bacteria grows at a rate of

$$P'(t) = 0.1t(\ln t)^2$$

Find the general form of $P(t)$.

EXAMPLE 5 Applying Integration by Parts Twice

Determine $\int x^2 e^{2x+1}\, dx$.

Solution: Let $u = x^2$ and $dv = e^{2x+1}\,dx$. Then $du = 2x\,dx$ and $v = e^{2x+1}/2$.

$$\begin{aligned}\int x^2e^{2x+1}\,dx &= \frac{x^2e^{2x+1}}{2} - \int \frac{e^{2x+1}}{2}(2x\,dx)\\ &= \frac{x^2e^{2x+1}}{2} - \int xe^{2x+1}\,dx\end{aligned}$$

To find $\int xe^{2x+1}\,dx$, we will again use integration by parts. Here, let $u = x$ and $dv = e^{2x+1}\,dx$. Then $du = dx$ and $v = e^{2x+1}/2$, and we have

$$\begin{aligned}\int xe^{2x+1}\,dx &= \frac{xe^{2x+1}}{2} - \int \frac{e^{2x+1}}{2}\,dx\\ &= \frac{xe^{2x+1}}{2} - \frac{e^{2x+1}}{4} + C_1\end{aligned}$$

Thus,

$$\int x^2 e^{2x+1}\,dx = \frac{x^2 e^{2x+1}}{2} - \frac{xe^{2x+1}}{2} + \frac{e^{2x+1}}{4} + C \qquad \text{where } C = -C_1$$
$$= \frac{e^{2x+1}}{2}\left(x^2 - x + \frac{1}{2}\right) + C$$

Now Work Problem 23 ◁

PROBLEMS 15.1

1. In applying integration by parts to

$$\int f(x)\,dx$$

a student found that $u = x$, $du = dx$, $dv = (x+5)^{1/2}$, and $v = \frac{2}{3}(x+5)^{3/2}$. Use this information to find $\int f(x)\,dx$.

2. Use integration by parts to find

$$\int xe^{3x+1}\,dx$$

by choosing $u = x$ and $dv = e^{3x+1}\,dx$.

In Problems 3–29, find the integrals.

3. $\int xe^{-x}\,dx$ **4.** $\int xe^{ax}\,dx$ for $a \neq 0$

5. $\int y^3 \ln y\,dy$ **6.** $\int x^2 \ln x\,dx$

7. $\int \ln(4x)\,dx$ **8.** $\int \frac{t}{e^t}\,dt$

9. $\int x\sqrt{ax+b}\,dx$ **10.** $\int \frac{12x}{\sqrt{1+4x}}\,dx$

11. $\int \frac{x}{(5x+2)^3}\,dx$ **12.** $\int \frac{\ln(x+1)}{2(x+1)}\,dx$

13. $\int \frac{\ln x}{x^2}\,dx$ **14.** $\int \frac{2x+7}{e^{3x}}\,dx$

15. $\int_1^2 4xe^{2x}\,dx$ **16.** $\int_1^2 2xe^{-3x}\,dx$

17. $\int_0^1 xe^{-x^2}\,dx$ **18.** $\int \frac{3x^3}{\sqrt{4-x^2}}\,dx$

19. $\int_5^8 \frac{4x}{\sqrt{9-x}}\,dx$ **20.** $\int (\ln x)^2\,dx$

21. $\int 3(2x-2)\ln(x-2)\,dx$ **22.** $\int \frac{xe^x}{(x+1)^2}\,dx$

23. $\int x^2 e^x\,dx$ **24.** $\int_1^4 \sqrt{x}\ln(x^9)\,dx$

25. $\int (x - e^{-x})^2\,dx$ **26.** $\int x^2 e^{3x}\,dx$

27. $\int x^3 e^{x^2}\,dx$ **28.** $\int x^5 e^{x^2}\,dx$

29. $\int (e^x + x)^2\,dx$

30. Find $\int \ln(x + \sqrt{x^2+1})\,dx$. *Hint:* Show that

$$\frac{d}{dx}[\ln(x + \sqrt{x^2+1})] = \frac{1}{\sqrt{x^2+1}}$$

31. Find the area of the region bounded by the x-axis, the curve $y = \ln x$, and the line $x = e^3$.

32. Find the area of the region bounded by the x-axis and the curve $y = x^2 e^x$ between $x = 0$ and $x = 1$.

33. Find the area of the region bounded by the x-axis and the curve $y = x^2 \ln x$ between $x = 1$ and $x = 2$.

34. Consumers' Surplus Suppose the demand equation for a manufacturer's product is given by

$$p = 5(q+5)e^{-(q+5)/5}$$

where p is the price per unit (in dollars) when q units are demanded. Assume that market equilibrium occurs when $q = 7$. Determine the consumers' surplus at market equilibrium.

35. Revenue Suppose total revenue r and price per unit p are differentiable functions of output q.

(a) Use integration by parts to show that

$$\int p\,dq = r - \int q\frac{dp}{dq}\,dq$$

(b) Using part (a), show that

$$r = \int \left(p + q\frac{dp}{dq}\right) dq$$

(c) Using part (b), prove that

$$r(q_0) = \int_0^{q_0} \left(p + q\frac{dp}{dq}\right) dq$$

(*Hint:* Refer to Section 14.7.)

36. Suppose f is a differentiable function. Apply integration by parts to $\int f(x)e^x\,dx$ to prove that

$$\int f(x)e^x\,dx + \int f'(x)e^x\,dx = f(x)e^x + C$$

$$\left(\text{Hence, } \int [f(x) + f'(x)]e^x\,dx = f(x)e^x + C\right)$$

37. Suppose that f has an inverse and that $F' = f$. Use integration by parts to develop a useful formula for $\int f^{-1}(x)\,dx$ in terms of F and f^{-1}. [*Hint:* Review Example 3. It used the idea required here, for the special case of $f(x) = e^x$.] If $f^{-1}(a) = c$ and $f^{-1}(b) = d$, show that

$$\int_a^b f^{-1}(x)\,dx = bd - ac - \int_c^d f(x)\,dx$$

For $0 < a < b$ and $f^{-1} > 0$ on $[a, b]$, draw a diagram that illustrates the last equation.

Objective

To show how to integrate a proper rational function by first expressing it as a sum of its partial fractions.

15.2 Integration by Partial Fractions[2]

Recall that a *rational function* is a quotient of polynomials $N(x)/D(x)$ and that it is *proper* if N and D have no common polynomial factor and the degree of the numerator N is less than the degree of the denominator D. If N/D is not proper, then we can use long division to divide $N(x)$ by $D(x)$:

$$D(x)\overline{)N(x)}^{\,Q(x)} \quad \vdots \quad \overline{R(x)} \qquad \text{thus} \qquad \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

Here the quotient $Q(x)$ and the remainder $R(x)$ are also polynomials and either $R(x)$ is the constant 0-polynomial or the degree of $R(x)$ is less than that of $D(x)$. Thus R/D is a proper rational function. Since

$$\int \frac{N(x)}{D(x)}\,dx = \int \left(Q(x) + \frac{R(x)}{D(x)}\right) dx = \int Q(x)\,dx + \int \frac{R(x)}{D(x)}\,dx$$

and we already know how to integrate a polynomial, it follows that the task of integrating rational functions reduces to that of integrating *proper* rational functions. We emphasize that the technique we are about to explain requires that a rational function be proper so that the long divison step is not optional. For example,

$$\int \frac{2x^4 - 3x^3 - 4x^2 - 17x - 6}{x^3 - 2x^2 - 3x}\,dx = \int \left(2x + 1 + \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x}\right) dx$$

$$= x^2 + x + \int \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x}\,dx$$

Distinct Linear Factors

We now consider

$$\int \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x}\,dx$$

It is essential that the denominator be expressed in factored form:

$$\int \frac{4x^2 - 14x - 6}{x(x+1)(x-3)}\,dx$$

Observe that in this example the denominator consists only of **linear factors** and that each factor occurs exactly once. It can be shown that, to each such factor $x - a$, there corresponds a *partial fraction* of the form

$$\frac{A}{x-a} \qquad A \text{ a constant}$$

such that the integrand is the sum of the partial fractions. If there are n such *distinct* linear factors, there will be n such partial fractions, each of which is easily integrated. Applying these facts, we can write

$$\frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3} \tag{1}$$

To determine the constants A, B, and C, we first combine the terms on the right side:

$$\frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{A(x+1)(x-3) + Bx(x-3) + Cx(x+1)}{x(x+1)(x-3)}$$

[2]This section can be omitted without loss of continuity.

Since the denominators of both sides are equal, we can equate their numerators:

$$4x^2 - 14x - 6 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1) \tag{2}$$

Although Equation (1) is not defined for $x = 0, x = -1$, and $x = 3$, we want to find values for A, B, and C that will make Equation (2) true for all values of x, so that the two sides of the equality provide equal functions. By successively setting x in Equation (2) equal to any three different numbers, we can obtain a system of equations that can be solved for A, B, and C. In particular, the work can be simplified by letting x be the roots of $D(x) = 0$; in our case, $x = 0, x = -1$, and $x = 3$. Using Equation (2), we have, for $x = 0$,

$$-6 = A(1)(-3) + B(0) + C(0) = -3A, \quad \text{so } A = 2$$

If $x = -1$,

$$12 = A(0) + B(-1)(-4) + C(0) = 4B, \quad \text{so } B = 3$$

If $x = 3$,

$$-12 = A(0) + B(0) + C(3)(4) = 12C, \quad \text{so } C = -1$$

Thus Equation (1) becomes

$$\frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{x-3}$$

Hence,

$$\begin{aligned}\int \frac{4x^2 - 14x - 6}{x(x+1)(x-3)}\,dx &= \int \left(\frac{2}{x} + \frac{3}{x+1} - \frac{1}{x-3}\right) dx \\ &= 2\int \frac{dx}{x} + 3\int \frac{dx}{x+1} - \int \frac{dx}{x-3} \\ &= 2\ln|x| + 3\ln|x+1| - \ln|x-3| + C\end{aligned}$$

For the *original* integral, we can now state that

$$\int \frac{2x^4 - 3x^3 - 4x^2 - 17x - 6}{x^3 - 2x^2 - 3x}\,dx = x^2 + x + 2\ln|x| + 3\ln|x+1| - \ln|x-3| + C$$

An alternative method of determining A, B, and C involves expanding the right side of Equation (2) and combining like terms:

$$\begin{aligned}4x^2 - 14x - 6 &= A(x^2 - 2x - 3) + B(x^2 - 3x) + C(x^2 + x) \\ &= Ax^2 - 2Ax - 3A + Bx^2 - 3Bx + Cx^2 + Cx \\ 4x^2 - 14x - 6 &= (A + B + C)x^2 + (-2A - 3B + C)x + (-3A)\end{aligned}$$

For this last equation to express an equality of functions, the coefficients of corresponding powers of x on the left and right sides must be equal:

$$\begin{cases} 4 = A + B + C \\ -14 = -2A - 3B + C \\ -6 = -3A \end{cases}$$

Solving gives $A = 2, B = 3$, and $C = -1$ as before.

APPLY IT ▷

3. The marginal revenue for a company manufacturing q radios per week is given by $r'(q) = \frac{5(q+4)}{q^2+4q+3}$, where $r(q)$ is the revenue in thousands of dollars. Find the equation for $r(q)$.

EXAMPLE 1 Distinct Linear Factors

Determine $\int \frac{2x+1}{3x^2-27}\,dx$ by using partial fractions.

Solution: Since the degree of the numerator is less than the degree of the denominator, no long division is necessary. The integral can be written as

$$\frac{1}{3}\int \frac{2x+1}{x^2-9}\,dx$$

Expressing $(2x+1)/(x^2-9)$ as a sum of partial fractions, we have

$$\frac{2x+1}{x^2-9} = \frac{2x+1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

Combining terms and equating numerators gives

$$2x+1 = A(x-3) + B(x+3)$$

If $x = 3$, then

$$7 = 6B, \quad \text{so } B = \frac{7}{6}$$

If $x = -3$, then

$$-5 = -6A, \quad \text{so } A = \frac{5}{6}$$

Thus,

$$\int \frac{2x+1}{3x^2-27}\,dx = \frac{1}{3}\left(\int \frac{\frac{5}{6}\,dx}{x+3} + \int \frac{\frac{7}{6}\,dx}{x-3}\right)$$

$$= \frac{1}{3}\left(\frac{5}{6}\ln|x+3| + \frac{7}{6}\ln|x-3|\right) + C$$

Now Work Problem 1 ◁

Repeated Linear Factors

If the denominator of $N(x)/D(x)$ contains only linear factors, some of which are repeated, then, for each factor $(x-a)^k$, where k is the maximum number of times $x-a$ occurs as a factor, there will correspond the sum of k partial fractions:

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \cdots + \frac{K}{(x-a)^k}$$

EXAMPLE 2 Repeated Linear Factors

Determine $\int \frac{6x^2+13x+6}{(x+2)(x+1)^2}\,dx$ by using partial fractions.

Solution: Since the degree of the numerator, namely 2, is less than that of the denominator, namely 3, no long division is necessary. In the denominator, the linear factor $x+2$ occurs once and the linear factor $x+1$ occurs twice. There will thus be three partial fractions and three constants to determine, and we have

$$\frac{6x^2+13x+6}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$6x^2+13x+6 = A(x+1)^2 + B(x+2)(x+1) + C(x+2)$$

Let us choose $x = -2$, $x = -1$, and, for convenience, $x = 0$. For $x = -2$, we have

$$4 = A$$

If $x = -1$, then

$$-1 = C$$

If $x = 0$, then

$$6 = A + 2B + 2C = 4 + 2B - 2 = 2 + 2B$$
$$4 = 2B$$
$$2 = B$$

Therefore,

$$\int \frac{6x^2 + 13x + 6}{(x+2)(x+1)^2}\,dx = 4\int \frac{dx}{x+2} + 2\int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2}$$
$$= 4\ln|x+2| + 2\ln|x+1| + \frac{1}{x+1} + C$$
$$= \ln[(x+2)^4(x+1)^2] + \frac{1}{x+1} + C$$

The last line above is somewhat optional (depending on what you need the integral for). It merely illustrates that in problems of this kind the logarithms that arise can often be combined.

Now Work Problem 5 ◁

Distinct Irreducible Quadratic Factors

Suppose a quadratic factor $x^2 + bx + c$ occurs in $D(x)$ and it cannot be expressed as a product of two linear factors with real coefficients. Such a factor is said to be an *irreducible quadratic factor over the real numbers*. To each distinct irreducible quadratic factor that occurs exactly once in $D(x)$, there will correspond a partial fraction of the form

$$\frac{Ax + B}{x^2 + bx + c}$$

Note that even after a rational function has been expressed in terms of partial fractions, it may still be impossible to integrate using only the basic functions we have covered in this book. For example, a very simple irreducible quadratic factor is $x^2 + 1$ and yet

$$\int \frac{1}{x^2+1}\,dx = \int \frac{dx}{x^2+1} = \tan^{-1} x + C$$

where $\tan^{-1}$ is the inverse of the trigonometric function tan when tan is restricted to $(-\pi/2, \pi/2)$. We do not discuss trigonometric functions in this book, but note that any good calculator has a $\tan^{-1}$ key.

EXAMPLE 3 **An Integral with a Distinct Irreducible Quadratic Factor**

Determine $\int \frac{-2x-4}{x^3+x^2+x}\,dx$ by using partial fractions.

Solution: Since $x^3+x^2+x = x(x^2+x+1)$, we have the linear factor x and the quadratic factor $x^2 + x + 1$, which does not seem factorable on inspection. If it were factorable as $(x - r_1)(x - r_2)$, with r_1 and r_2 real, then r_1 and r_2 would be roots of the equation $x^2 + x + 1 = 0$. By the quadratic formula, the roots are

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

Since there are no real roots, we conclude that $x^2 + x + 1$ is irreducible. Thus there will be two partial fractions and *three* constants to determine. We have

$$\frac{-2x-4}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$
$$-2x - 4 = A(x^2+x+1) + (Bx+C)x$$
$$= Ax^2 + Ax + A + Bx^2 + Cx$$
$$0x^2 - 2x - 4 = (A+B)x^2 + (A+C)x + A$$

Equating coefficients of like powers of x, we obtain

$$\begin{cases} 0 = A + B \\ -2 = A + C \\ -4 = A \end{cases}$$

Solving gives $A = -4, B = 4$, and $C = 2$. Hence,

$$\int \frac{-2x-4}{x(x^2+x+1)}\,dx = \int \left(\frac{-4}{x} + \frac{4x+2}{x^2+x+1}\right) dx$$
$$= -4\int \frac{dx}{x} + 2\int \frac{2x+1}{x^2+x+1}\,dx$$

Both integrals have the form $\int \frac{du}{u}$, so

$$\int \frac{-2x-4}{x(x^2+x+1)}\,dx = -4\ln|x| + 2\ln|x^2+x+1| + C$$
$$= \ln\left[\frac{(x^2+x+1)^2}{x^4}\right] + C$$

Now Work Problem 7 ◁

Repeated Irreducible Quadratic Factors

Suppose $D(x)$ contains factors of the form $(x^2 + bx + c)^k$, where k is the maximum number of times the irreducible factor $x^2 + bx + c$ occurs. Then, to each such factor there will correspond a sum of k partial fractions of the form

$$\frac{A + Bx}{x^2 + bx + c} + \frac{C + Dx}{(x^2 + bx + c)^2} + \cdots + \frac{M + Nx}{(x^2 + bx + c)^k}$$

EXAMPLE 4 Repeated Irreducible Quadratic Factors

Determine $\int \frac{x^5}{(x^2+4)^2}\,dx$ by using partial fractions.

Solution: Since the numerator has degree 5 and the denominator has degree 4, we first use long division, which gives

$$\frac{x^5}{x^4 + 8x^2 + 16} = x - \frac{8x^3 + 16x}{(x^2+4)^2}$$

The quadratic factor $x^2 + 4$ in the denominator of $(8x^3 + 16x)/(x^2 + 4)^2$ is irreducible and occurs as a factor twice. Thus, to $(x^2 + 4)^2$ there correspond two partial fractions and *four* coefficients to be determined. Accordingly, we set

$$\frac{8x^3 + 16x}{(x^2+4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2+4)^2}$$

and obtain

$$8x^3 + 16x = (Ax + B)(x^2 + 4) + Cx + D$$
$$8x^3 + 0x^2 + 16x + 0 = Ax^3 + Bx^2 + (4A + C)x + 4B + D$$

Equating like powers of x yields

$$\begin{cases} 8 = A \\ 0 = B \\ 16 = 4A + C \\ 0 = 4B + D \end{cases}$$

Solving gives $A = 8, B = 0, C = -16$, and $D = 0$. Therefore,

$$\int \frac{x^5}{(x^2+4)^2}\,dx = \int \left(x - \left(\frac{8x}{x^2+4} - \frac{16x}{(x^2+4)^2}\right)\right) dx$$

$$= \int x\,dx - 4\int \frac{2x}{x^2+4}\,dx + 8\int \frac{2x}{(x^2+4)^2}\,dx$$

The second integral on the preceding line has the form $\int \frac{du}{u}$, and the third integral has the form $\int \frac{du}{u^2}$. So

$$\int \frac{x^5}{(x^2+4)^2} = \frac{x^2}{2} - 4\ln(x^2+4) - \frac{8}{x^2+4} + C$$

Now Work Problem 27 ◁

CAUTION!

Be on the lookout for simple solutions too.

From our examples, you may have deduced that the number of constants needed to express $N(x)/D(x)$ by partial fractions is equal to the degree of $D(x)$, if it is assumed that $N(x)/D(x)$ defines a proper rational function. This is indeed the case. Note also that the representation of a proper rational function by partial fractions is unique; that is, there is only one choice of constants that can be made. Furthermore, regardless of the complexity of the polynomial $D(x)$, it can always (theoretically) be expressed as a product of linear and irreducible quadratic factors with real coefficients.

APPLY IT ▷

4. The rate of change of the voting population of a city with respect to time t (in years) is estimated to be $V'(t) = \frac{300t^3}{t^2+6}$. Find the general form of $V(t)$.

EXAMPLE 5 An Integral Not Requiring Partial Fractions

Find $\int \frac{2x+3}{x^2+3x+1}\,dx$.

Solution: This integral has the form $\int \frac{1}{u}\,du$. Thus,

$$\int \frac{2x+3}{x^2+3x+1}\,dx = \ln|x^2+3x+1| + C$$

Now Work Problem 17 ◁

PROBLEMS 15.2

In Problems 1–8, express the given rational function in terms of partial fractions. Watch out for any preliminary divisions.

1. $f(x) = \frac{10x}{x^2+7x+6}$

2. $f(x) = \frac{x+5}{x^2-1}$

3. $f(x) = \frac{2x^2}{x^2+5x+6}$

4. $f(x) = \frac{2x^2-15}{x^2+5x}$

5. $f(x) = \frac{3x-1}{x^2-2x+1}$

6. $f(x) = \frac{2x+3}{x^2(x-1)}$

7. $f(x) = \frac{x^2+3}{x^3+x}$

8. $f(x) = \frac{3x^2+5}{(x^2+4)^2}$

In Problems 9–30, determine the integrals.

9. $\int \frac{5x-2}{x^2-x}\,dx$

10. $\int \frac{15x+5}{x^2+5x}\,dx$

11. $\int \frac{x+10}{x^2-x-2}\,dx$

12. $\int \frac{2x-1}{x^2-x-12}\,dx$

13. $\int \frac{3x^3-3x+4}{4x^2-4}\,dx$

14. $\int \frac{7(4-x^2)}{(x-4)(x-2)(x+3)}\,dx$

15. $\int \frac{19x^2-5x-36}{2x^3-2x^2-12x}\,dx$

16. $\int \frac{4-x}{x^4-x^2}\,dx$

17. $\int \frac{2(3x^5+4x^3-x)}{x^6+2x^4-x^2-2}\,dx$

18. $\int \frac{x^4-2x^3+6x^2-11x+2}{x^3-3x^2+2x}\,dx$

19. $\int \frac{2x^2-5x-2}{(x-2)^2(x-1)}\,dx$

20. $\int \frac{5x^3+x^2+x-3}{x^4-x^3}\,dx$

21. $\int \frac{2(x^2+8)}{x^3+4x}\,dx$

22. $\int \frac{4x^3-3x^2+2x-3}{(x^2+3)(x+1)(x-2)}\,dx$

23. $\int \frac{-x^3+8x^2-9x+2}{(x^2+1)(x-3)^2}\,dx$

24. $\int \frac{5x^4+9x^2+3}{x(x^2+1)^2}\,dx$

25. $\int \frac{7x^3+24x}{(x^2+3)(x^2+4)}\,dx$

26. $\int \frac{12x^3+20x^2+28x+4}{3(x^2+2x+3)(x^2+1)}\,dx$

27. $\int \frac{3x^3+8x}{(x^2+2)^2}\,dx$

28. $\int \frac{3x^2-8x+4}{x^3-4x^2+4x-6}\,dx$

29. $\int_0^1 \frac{2-2x}{x^2+7x+12}\,dx$

30. $\int_0^1 \frac{x^2+5x+5}{x^2+3x+2}\,dx$

31. Find the area of the region bounded by the graph of

$$y = \frac{6(x^2+1)}{(x+2)^2}$$

and the x-axis from $x = 0$ to $x = 1$.

32. Consumers' Surplus Suppose the demand equation for a manufacturer's product is given by

$$p = \frac{200(q+3)}{q^2+7q+6}$$

where p is the price per unit (in dollars) when q units are demanded. Assume that market equilibrium occurs at the point $(q,p) = (10,325/22)$. Determine consumers' surplus at market equilibrium.

Objective

To illustrate the use of the table of integrals in Appendix B.

15.3 Integration by Tables

Certain forms of integrals that occur frequently can be found in standard tables of integration formulas.[3] A short table appears in Appendix B, and its use will be illustrated in this section.

A given integral may have to be replaced by an equivalent form before it will fit a formula in the table. The equivalent form must match the formula exactly. Consequently, the steps performed to get the equivalent form should be written carefully rather than performed mentally. Before proceeding with the exercises that use tables, we recommend studying the examples of this section carefully.

In the following examples, the formula numbers refer to the Table of Selected Integrals given in Appendix B.

EXAMPLE 1 Integration by Tables

Find $\int \frac{x\,dx}{(2+3x)^2}$.

Solution: Scanning the table, we identify the integrand with Formula (7):

$$\int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left(\ln|a+bu| + \frac{a}{a+bu}\right) + C$$

Now we see if we can exactly match the given integrand with that in the formula. If we replace x by u, 2 by a, and 3 by b, then $du = dx$, and by substitution we have

$$\int \frac{x\,dx}{(2+3x)^2} = \int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left(\ln|a+bu| + \frac{a}{a+bu}\right) + C$$

Returning to the variable x and replacing a by 2 and b by 3, we obtain

$$\int \frac{x\,dx}{(2+3x)^2} = \frac{1}{9}\left(\ln|2+3x| + \frac{2}{2+3x}\right) + C$$

Note that the answer must be given in terms of x, the *original* variable of integration.

Now Work Problem 5 ◁

EXAMPLE 2 Integration by Tables

Find $\int x^2\sqrt{x^2-1}\,dx$.

Solution: This integral is identified with Formula (24):

$$\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{u}{8}(2u^2 \pm a^2)\sqrt{u^2 \pm a^2} - \frac{a^4}{8}\ln|u + \sqrt{u^2 \pm a^2}| + C$$

In the preceding formula, if the bottommost sign in the dual symbol "±" on the left side is used, then the bottommost sign in the dual symbols on the right side must also be

[3] See, for example, W. H. Beyer (ed.), *CRC Standard Mathematical Tables and Formulae*, 30th ed. (Boca Raton, FL: CRC Press, 1996).

used. In the original integral, we let $u = x$ and $a = 1$. Then $du = dx$, and by substitution the integral becomes

$$\int x^2\sqrt{x^2-1}\,dx = \int u^2\sqrt{u^2-a^2}\,du$$

$$= \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$$

Since $u = x$ and $a = 1$,

$$\int x^2\sqrt{x^2-1}\,dx = \frac{x}{8}(2x^2-1)\sqrt{x^2-1} - \frac{1}{8}\ln|x+\sqrt{x^2-1}| + C$$

Now Work Problem 17 ◁

This example, as well as Examples 4, 5, and 7, shows how to adjust an integral so that it conforms to one in the table.

EXAMPLE 3 Integration by Tables

Find $\int \frac{dx}{x\sqrt{16x^2+3}}$.

Solution: The integrand can be identified with Formula (28):

$$\int \frac{du}{u\sqrt{u^2+a^2}} = \frac{1}{a}\ln\left|\frac{\sqrt{u^2+a^2}-a}{u}\right| + C$$

If we let $u = 4x$ and $a = \sqrt{3}$, then $du = 4\,dx$. Watch closely how, by inserting 4's in the numerator and denominator, we transform the given integral into an equivalent form that matches Formula (28):

$$\int \frac{dx}{x\sqrt{16x^2+3}} = \int \frac{(4\,dx)}{(4x)\sqrt{(4x)^2+(\sqrt{3})^2}} = \int \frac{du}{u\sqrt{u^2+a^2}}$$

$$= \frac{1}{a}\ln\left|\frac{\sqrt{u^2+a^2}-a}{u}\right| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{16x^2+3}-\sqrt{3}}{4x}\right| + C$$

Now Work Problem 7 ◁

EXAMPLE 4 Integration by Tables

Find $\int \frac{dx}{x^2(2-3x^2)^{1/2}}$.

Solution: The integrand is identified with Formula (21):

$$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$$

Letting $u = \sqrt{3}x$ and $a^2 = 2$, we have $du = \sqrt{3}\,dx$. Hence, by inserting two factors of $\sqrt{3}$ in both the numerator and denominator of the original integral, we have

$$\int \frac{dx}{x^2(2-3x^2)^{1/2}} = \sqrt{3}\int \frac{(\sqrt{3}\,dx)}{(\sqrt{3}x)^2[2-(\sqrt{3}x)^2]^{1/2}} = \sqrt{3}\int \frac{du}{u^2(a^2-u^2)^{1/2}}$$

$$= \sqrt{3}\left[-\frac{\sqrt{a^2-u^2}}{a^2u}\right] + C = \sqrt{3}\left[-\frac{\sqrt{2-3x^2}}{2(\sqrt{3}x)}\right] + C$$

$$= -\frac{\sqrt{2-3x^2}}{2x} + C$$

Now Work Problem 35 ◁

EXAMPLE 5 Integration by Tables

Find $\int 7x^2 \ln(4x)\,dx$.

Solution: This is similar to Formula (42) with $n = 2$:

$$\int u^n \ln u\,du = \frac{u^{n+1}\ln u}{n+1} - \frac{u^{n+1}}{(n+1)^2} + C$$

If we let $u = 4x$, then $du = 4\,dx$. Hence,

$$\int 7x^2 \ln(4x)\,dx = \frac{7}{4^3}\int (4x)^2 \ln(4x)(4\,dx)$$

$$= \frac{7}{64}\int u^2 \ln u\,du = \frac{7}{64}\left(\frac{u^3 \ln u}{3} - \frac{u^3}{9}\right) + C$$

$$= \frac{7}{64}\left(\frac{(4x)^3 \ln(4x)}{3} - \frac{(4x)^3}{9}\right) + C$$

$$= 7x^3\left(\frac{\ln(4x)}{3} - \frac{1}{9}\right) + C$$

$$= \frac{7x^3}{9}(3\ln(4x) - 1) + C$$

Now Work Problem 45 ◁

EXAMPLE 6 Integral Table Not Needed

Find $\int \frac{e^{2x}\,dx}{7 + e^{2x}}$.

Solution: At first glance, we do not identify the integrand with any form in the table. Perhaps rewriting the integral will help. Let $u = 7 + e^{2x}$, then $du = 2e^{2x}\,dx$. So

$$\int \frac{e^{2x}\,dx}{7+e^{2x}} = \frac{1}{2}\int \frac{(2e^{2x}\,dx)}{7+e^{2x}} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|7 + e^{2x}| + C = \frac{1}{2}\ln(7 + e^{2x}) + C$$

Thus, we had only to use our knowledge of basic integration forms. [Actually, this form appears as Formula (2) in the table, with $a = 0$ and $b = 1$.]

Now Work Problem 39 ◁

EXAMPLE 7 Finding a Definite Integral by Using Tables

Evaluate $\int_1^4 \frac{dx}{(4x^2+2)^{3/2}}$.

Solution: We will use Formula (32) to get the indefinite integral first:

$$\int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{\pm u}{a^2\sqrt{u^2 \pm a^2}} + C$$

Letting $u = 2x$ and $a^2 = 2$, we have $du = 2\,dx$. Thus,

$$\int \frac{dx}{(4x^2+2)^{3/2}} = \frac{1}{2}\int \frac{(2\,dx)}{((2x)^2+2)^{3/2}} = \frac{1}{2}\int \frac{du}{(u^2+2)^{3/2}}$$

$$= \frac{1}{2}\left(\frac{u}{2\sqrt{u^2+2}}\right) + C$$

Here we determine the limits of integration with respect to u.

Instead of substituting back to x and evaluating from $x = 1$ to $x = 4$, we can determine the corresponding limits of integration with respect to u and then evaluate the last expression between those limits. Since $u = 2x$, when $x = 1$, we have $u = 2$; when

CAUTION!

When changing the variable of integration x to the variable of integration u, be sure to change the limits of integration so that they agree with u.

$x = 4$, we have $u = 8$. Hence,

$$\int_1^4 \frac{dx}{(4x^2+2)^{3/2}} = \frac{1}{2}\int_2^8 \frac{du}{(u^2+2)^{3/2}}$$

$$= \frac{1}{2}\left(\frac{u}{2\sqrt{u^2+2}}\right)\Bigg|_2^8 = \frac{2}{\sqrt{66}} - \frac{1}{2\sqrt{6}}$$

Now Work Problem 15 ◁

Integration Applied to Annuities

Tables of integrals are useful when we deal with integrals associated with annuities. Suppose that you must pay out \$100 at the end of each year for the next two years. Recall from Chapter 5 that a series of payments over a period of time, such as this, is called an *annuity*. If you were to pay off the debt now instead, you would pay the present value of the \$100 that is due at the end of the first year, plus the present value of the \$100 that is due at the end of the second year. The sum of these present values is the present value of the annuity. (The present value of an annuity is discussed in Section 5.4.) We will now consider the present value of payments made continuously over the time interval from $t = 0$ to $t = T$, with t in years, when interest is compounded continuously at an annual rate of r.

Suppose a payment is made at time t such that on an annual basis this payment is $f(t)$. If we divide the interval $[0, T]$ into subintervals $[t_{i-1}, t_i]$ of length dt (where dt is small), then the total amount of all payments over such a subinterval is approximately $f(t_i)\,dt$. [For example, if $f(t) = 2000$ and dt were one day, the total amount of the payments would be $2000(\frac{1}{365})$.] The present value of these payments is approximately $e^{-rt_i}f(t_i)\,dt$. (See Section 5.3.) Over the interval $[0, T]$, the total of all such present values is

$$\sum e^{-rt_i}f(t_i)\,dt$$

This sum approximates the present value A of the annuity. The smaller dt is, the better the approximation. That is, as $dt \to 0$, the limit of the sum is the present value. However, this limit is also a definite integral. That is,

$$A = \int_0^T f(t)e^{-rt}\,dt \tag{1}$$

where A is the **present value of a continuous annuity** at an annual rate r (compounded continuously) for T years if a payment at time t is at the rate of $f(t)$ per year.

We say that Equation (1) gives the **present value of a continuous income stream.** Equation (1) can also be used to find the present value of future profits of a business. In this situation, $f(t)$ is the annual rate of profit at time t.

We can also consider the *future* value of an annuity rather than its present value. If a payment is made at time t, then it has a certain value at the *end* of the period of the annuity—that is, $T - t$ years later. This value is

$$\begin{pmatrix}\text{amount of}\\ \text{payment}\end{pmatrix} + \begin{pmatrix}\text{interest on this}\\ \text{payment for } T-t \text{ years}\end{pmatrix}$$

If S is the total of such values for all payments, then S is called the *accumulated amount of a continuous annuity* and is given by the formula

$$S = \int_0^T f(t)e^{r(T-t)}\,dt$$

where S is the **accumulated amount of a continuous annuity** at the end of T years at an annual rate r (compounded continuously) when a payment at time t is at the rate of $f(t)$ per year.

EXAMPLE 8 **Present Value of a Continuous Annuity**

Find the present value (to the nearest dollar) of a continuous annuity at an annual rate of 8% for 10 years if the payment at time t is at the rate of t^2 dollars per year.

Solution: The present value is given by

$$A = \int_0^T f(t)e^{-rt}\,dt = \int_0^{10} t^2 e^{-0.08t}\,dt$$

We will use Formula (39),

$$\int u^n e^{au}\,du = \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}\,du$$

This is called a *reduction formula,* since it reduces one integral to an expression that involves another integral that is easier to determine. If $u = t, n = 2$, and $a = -0.08$, then $du = dt$, and we have

$$A = \frac{t^2 e^{-0.08t}}{-0.08}\bigg|_0^{10} - \frac{2}{-0.08}\int_0^{10} te^{-0.08t}\,dt$$

In the new integral, the exponent of t has been reduced to 1. We can match this integral with Formula (38),

$$\int ue^{au}\,du = \frac{e^{au}}{a^2}(au - 1) + C$$

by letting $u = t$ and $a = -0.08$. Then $du = dt$, and

$$A = \int_0^{10} t^2 e^{-0.08t}\,dt = \frac{t^2 e^{-0.08t}}{-0.08}\bigg|_0^{10} - \frac{2}{-0.08}\left(\frac{e^{-0.08t}}{(-0.08)^2}(-0.08t - 1)\right)\bigg|_0^{10}$$

$$= \frac{100e^{-0.8}}{-0.08} - \frac{2}{-0.08}\left(\frac{e^{-0.8}}{(-0.08)^2}(-0.8 - 1) - \frac{1}{(-0.08)^2}(-1)\right)$$

$$\approx 185$$

The present value is \$185.

Now Work Problem 59 ◁

PROBLEMS 15.3

In Problems 1 and 2, use Formula (19) in Appendix B to determine the integrals.

1. $\int \frac{dx}{(6 - x^2)^{3/2}}$

2. $\int \frac{dx}{(25 - 4x^2)^{3/2}}$

In Problems 3 and 4, use Formula (30) in Appendix B to determine the integrals.

3. $\int \frac{dx}{x^2\sqrt{16x^2 + 3}}$

4. $\int \frac{3\,dx}{x^3\sqrt{x^4 - 9}}$

In Problems 5–38, find the integrals by using the table in Appendix B.

5. $\int \frac{dx}{x(6 + 7x)}$

6. $\int \frac{5x^2\,dx}{(2 + 3x)^2}$

7. $\int \frac{dx}{x\sqrt{x^2 + 9}}$

8. $\int \frac{dx}{(x^2 + 7)^{3/2}}$

9. $\int \frac{x\,dx}{(2 + 3x)(4 + 5x)}$

10. $\int 2^{5x}\,dx$

11. $\int \frac{dx}{1 + 2e^{3x}}$

12. $\int x^2\sqrt{1 + x}\,dx$

13. $\int \frac{7\,dx}{x(5 + 2x)^2}$

14. $\int \frac{dx}{x\sqrt{5 - 11x^2}}$

15. $\int_0^1 \frac{x\,dx}{2 + x}$

16. $\int \frac{-3x^2\,dx}{2 - 5x}$

17. $\int \sqrt{x^2 - 3}\,dx$

18. $\int \frac{dx}{(1 + 5x)(2x + 3)}$

19. $\int_0^{1/12} xe^{12x}\,dx$

20. $\int \sqrt{\frac{2 + 3x}{5 + 3x}}\,dx$

21. $\int x^3 e^x\,dx$

22. $\int_1^2 \frac{4\,dx}{x^2(1 + x)}$

23. $\int \frac{\sqrt{5x^2 + 1}}{2x^2}\,dx$

24. $\int \frac{dx}{x\sqrt{2 - x}}$

25. $\int \frac{x\,dx}{(1 + 3x)^2}$

26. $\int \frac{2\,dx}{\sqrt{(1 + 2x)(3 + 2x)}}$

27. $\int \frac{dx}{7 - 5x^2}$

28. $\int 7x^2\sqrt{3x^2 - 6}\,dx$

29. $\int 36x^5 \ln(3x)\,dx$

30. $\int \frac{5\,dx}{x^2(3 + 2x)^2}$

31. $\int 5x\sqrt{1+2x}\,dx$

32. $\int 9x^2 \ln x\,dx$

33. $\int \frac{dx}{\sqrt{4x^2-13}}$

34. $\int \frac{dx}{x\ln(2x)}$

35. $\int \frac{2\,dx}{x^2\sqrt{16-9x^2}}$

36. $\int \frac{\sqrt{3-x^2}}{x}\,dx$

37. $\int \frac{dx}{\sqrt{x}(\pi+7e^{4\sqrt{x}})}$

38. $\int_0^1 \frac{3x^2\,dx}{1+2x^3}$

In Problems 39–56, find the integrals by any method.

39. $\int \frac{x\,dx}{x^2+1}$

40. $\int 3x\sqrt{x}e^{x^{5/2}}\,dx$

41. $\int \frac{(\ln x)^3}{x}\,dx$

42. $\int \frac{5x^3-\sqrt{x}}{2x}\,dx$

43. $\int \frac{dx}{x^2-5x+6}$

44. $\int \frac{e^{2x}}{\sqrt{e^{2x}+3}}\,dx$

45. $\int x^3 \ln x\,dx$

46. $\int (9x-6)e^{-30x+20}\,dx$

47. $\int 4x^3e^{3x^2}\,dx$

48. $\int_1^2 35x^2\sqrt{3+2x}\,dx$

49. $\int \ln^2 x\,dx$

50. $\int_1^e 3x\ln x^2\,dx$

51. $\int_{-2}^1 \frac{x\,dx}{\sqrt{3+x}}$

52. $\int_2^3 x\sqrt{2+3x}\,dx$

53. $\int_0^1 \frac{2x\,dx}{\sqrt{8-x^2}}$

54. $\int_0^{\ln 2} x^2e^{3x}\,dx$

55. $\int_1^2 x\ln(2x)\,dx$

56. $\int_3^5 dA$

57. **Biology** In a discussion about gene frequency,[4] the integral

$$\int_{q_0}^{q_n} \frac{dq}{q(1-q)}$$

occurs, where the q's represent gene frequencies. Evaluate this integral.

58. **Biology** Under certain conditions, the number n of generations required to change the frequency of a gene from 0.3 to 0.1 is given by[5]

$$n = -\frac{1}{0.4}\int_{0.3}^{0.1} \frac{dq}{q^2(1-q)}$$

Find n (to the nearest integer).

59. **Continuous Annuity** Find the present value, to the nearest dollar, of a continuous annuity at an annual rate of r for T years if the payment at time t is at the annual rate of $f(t)$ dollars, given that

(a) $r = 0.04$ $\quad T = 9$ $\quad f(t) = 1000$
(b) $r = 0.06$ $\quad T = 10$ $\quad f(t) = 500t$

60. If $f(t) = k$, where k is a positive constant, show that the value of the integral in Equation (1) of this section is

$$k\left(\frac{1-e^{-rT}}{r}\right)$$

61. **Continuous Annuity** Find the accumulated amount, to the nearest dollar, of a continuous annuity at an annual rate of r for T years if the payment at time t is at an annual rate of $f(t)$ dollars, given that

(a) $r = 0.02$ $\quad T = 10$ $\quad f(t) = 100$
(b) $r = 0.01$ $\quad T = 10$ $\quad f(t) = 200$

62. **Value of Business** Over the next five years, the profits of a business at time t are estimated to be $50{,}000t$ dollars per year. The business is to be sold at a price equal to the present value of these future profits. To the nearest 10 dollars, at what price should the business be sold if interest is compounded continuously at the annual rate of 7%?

Objective

To develop the concept of the average value of a function.

15.4 Average Value of a Function

If we are given the three numbers 1, 2, and 9, then their average value, or *mean*, is their sum divided by 3. Denoting this average by $\bar{y}$, we have

$$\bar{y} = \frac{1+2+9}{3} = 4$$

Similarly, suppose we are given a function f defined on the interval $[a, b]$, and the points $x_1, x_2, \ldots, x_n$ are in the interval. Then the average value of the n corresponding function values $f(x_1), f(x_2), \ldots, f(x_n)$ is

$$\bar{y} = \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} = \frac{\sum_{i=1}^{n} f(x_i)}{n} \tag{1}$$

We can go a step further. Let us divide the interval $[a, b]$ into n subintervals of equal length. We will choose x_i to be the right-hand endpoint of the ith subinterval. Because

[4] W. B. Mather, *Principles of Quantitative Genetics* (Minneapolis: Burgess Publishing Company, 1964).
[5] E. O. Wilson and W. H. Bossert, *A Primer of Population Biology* (Stamford, CT: Sinauer Associates, Inc., 1971).

$[a, b]$ has length $b - a$, each subinterval has length $\frac{b-a}{n}$, which we will call dx. Thus, Equation (1) can be written

$$\bar{y} = \frac{\sum_{i=1}^{n} f(x_i)\left(\frac{dx}{dx}\right)}{n} = \frac{\frac{1}{dx}\sum_{i=1}^{n} f(x_i)\,dx}{n} = \frac{1}{n\,dx}\sum_{i=1}^{n} f(x_i)\,dx \qquad (2)$$

Since $dx = \frac{b-a}{n}$, it follows that $n\,dx = b - a$. So the expression $\frac{1}{n\,dx}$ in Equation (2) can be replaced by $\frac{1}{b-a}$. Moreover, as $n \to \infty$, the number of function values used in computing $\bar{y}$ increases, and we get the so-called *average value of the function f*, denoted by $\bar{f}$:

$$\bar{f} = \lim_{n\to\infty}\left[\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,dx\right] = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,dx$$

But the limit on the right is just the definite integral $\int_a^b f(x)\,dx$. This motivates the following definition:

Definition

The ***average value of a function*** $f(x)$ over the interval $[a, b]$ is denoted $\bar{f}$ and is given by

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

EXAMPLE 1 Average Value of a Function

Find the average value of the function $f(x) = x^2$ over the interval $[1, 2]$.

Solution:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-1}\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{7}{3}$$

Now Work Problem 1 ◁

In Example 1, we found that the average value of $y = f(x) = x^2$ over the interval $[1, 2]$ is $\frac{7}{3}$. We can interpret this value geometrically. Since

$$\frac{1}{2-1}\int_1^2 x^2\,dx = \frac{7}{3}$$

by solving for the integral we have

$$\int_1^2 x^2\,dx = \frac{7}{3}(2-1)$$

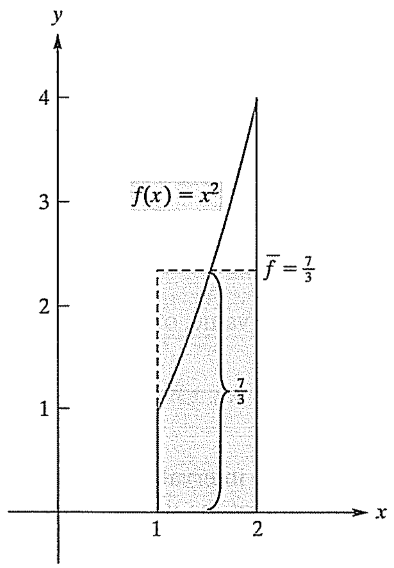

FIGURE 15.1 Geometric interpretation of the average value of a function.

However, this integral gives the area of the region bounded by $f(x) = x^2$ and the x-axis from $x = 1$ to $x = 2$. (See Figure 15.1.) From the preceding equation, this area is $\left(\frac{7}{3}\right)(2-1)$, which is the area of a rectangle whose height is the average value $\bar{f} = \frac{7}{3}$ and whose width is $b - a = 2 - 1 = 1$.

EXAMPLE 2 Average Flow of Blood

Suppose the flow of blood at time t in a system is given by

$$F(t) = \frac{F_1}{(1+\alpha t)^2} \qquad 0 \le t \le T$$

where F_1 and α (a Greek letter read "alpha") are constants.[6] Find the average flow $\overline{F}$ on the interval $[0, T]$.

Solution:

$$\overline{F} = \frac{1}{T-0}\int_0^T F(t)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{F_1}{(1+\alpha t)^2}\,dt = \frac{F_1}{\alpha T}\int_0^T (1+\alpha t)^{-2}(\alpha\,dt)$$

$$= \frac{F_1}{\alpha T}\left(\frac{(1+\alpha t)^{-1}}{-1}\right)\Bigg|_0^T = \frac{F_1}{\alpha T}\left(-\frac{1}{1+\alpha T}+1\right)$$

$$= \frac{F_1}{\alpha T}\left(\frac{-1+1+\alpha T}{1+\alpha T}\right) = \frac{F_1}{\alpha T}\left(\frac{\alpha T}{1+\alpha T}\right) = \frac{F_1}{1+\alpha T}$$

Now Work Problem 11 ◁

PROBLEMS 15.4

In Problems 1–8, find the average value of the function over the given interval.

1. $f(x) = x^2$; $[-1, 3]$

2. $f(x) = 2x + 1$; $[0, 1]$

3. $f(x) = 2 - 3x^2$; $[-1, 2]$

4. $f(x) = x^2 + x + 1$; $[1, 3]$

5. $f(t) = 2t^5$; $[-3, 3]$

6. $f(t) = t\sqrt{t^2+9}$; $[0, 4]$

7. $f(x) = \sqrt{x}$; $[0, 1]$

8. $f(x) = 5/x^2$; $[1, 3]$

9. Profit The profit (in dollars) of a business is given by

$$P = P(q) = 369q - 2.1q^2 - 400$$

where q is the number of units of the product sold. Find the average profit on the interval from $q = 0$ to $q = 100$.

10. Cost Suppose the cost (in dollars) of producing q units of a product is given by

$$c = 4000 + 10q + 0.1q^2$$

Find the average cost on the interval from $q = 100$ to $q = 500$.

11. Investment An investment of \$3000 earns interest at an annual rate of 5% compounded continuously. After t years, its value S (in dollars) is given by $S = 3000e^{0.05t}$. Find the average value of a two-year investment.

12. Medicine Suppose that colored dye is injected into the bloodstream at a constant rate R. At time t, let

$$C(t) = \frac{R}{F(t)}$$

be the concentration of dye at a location distant (distal) from the point of injection, where $F(t)$ is as given in Example 2. Show that the average concentration on $[0, T]$ is

$$\overline{C} = \frac{R\left(1 + \alpha T + \frac{1}{3}\alpha^2 T^2\right)}{F_1}$$

13. Revenue Suppose a manufacturer receives revenue r from the sale of q units of a product. Show that the average value of the marginal-revenue function over the interval $[0, q_0]$ is the price per unit when q_0 units are sold.

14. Find the average value of $f(x) = \dfrac{1}{x^2 - 4x + 5}$ over the interval $[0, 1]$ using an approximate integration technique. Round your answer to two decimal places.

15.5 Differential Equations

Objective

To solve a differential equation by using the method of separation of variables. To discuss particular solutions and general solutions. To develop interest compounded continuously in terms of a differential equation. To discuss exponential growth and decay.

Occasionally, you may have to solve an equation that involves the derivative of an unknown function. Such an equation is called a **differential equation.** An example is

$$y' = xy^2 \tag{1}$$

More precisely, Equation (1) is called a **first-order differential equation,** since it involves a derivative of the first order and none of higher order. A solution of Equation (1) is any function $y = f(x)$ that is defined on an interval and satisfies the equation for all x in the interval.

[6] W. Simon, *Mathematical Techniques for Physiology and Medicine* (New York: Academic Press, Inc., 1972).

To solve $y' = xy^2$, equivalently,

$$\frac{dy}{dx} = xy^2 \tag{2}$$

we think of dy/dx as a quotient of differentials and algebraically "separate variables" by rewriting the equation so that each side contains only one variable and a differential is not in a denominator:

$$\frac{dy}{y^2} = x\,dx$$

Integrating both sides and combining the constants of integration, we obtain

$$\int \frac{1}{y^2}\,dy = \int x\,dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C_1$$

$$-\frac{1}{y} = \frac{x^2 + 2C_1}{2}$$

Since $2C_1$ is an arbitrary constant, we can replace it by C.

$$-\frac{1}{y} = \frac{x^2 + C}{2} \tag{3}$$

Solving Equation (3) for y, we have

$$y = -\frac{2}{x^2 + C} \tag{4}$$

We can verify that y is a solution to the differential equation (2):

For if y is given by Equation (4), then

$$\frac{dy}{dx} = \frac{4x}{(x^2 + C)^2}$$

while also

$$xy^2 = x\left[-\frac{2}{x^2 + C}\right]^2 = \frac{4x}{(x^2 + C)^2}$$

showing that our y satisfies (2). Note in Equation (4) that, for *each* value of C, a different solution is obtained. We call Equation (4) the **general solution** of the differential equation. The method that we used to find it is called **separation of variables.**

In the foregoing example, suppose we are given the condition that $y = -\frac{2}{3}$ when $x = 1$; that is, $y(1) = -\frac{2}{3}$. Then the *particular* function that satisfies both Equation (2) and this condition can be found by substituting the values $x = 1$ and $y = -\frac{2}{3}$ into Equation (4) and solving for C:

$$-\frac{2}{3} = -\frac{2}{1^2 + C}$$

$$C = 2$$

Therefore, the solution of $dy/dx = xy^2$ such that $y(1) = -\frac{2}{3}$ is

$$y = -\frac{2}{x^2 + 2} \tag{5}$$

We call Equation (5) a **particular solution** of the differential equation.