13 Curve Sketching

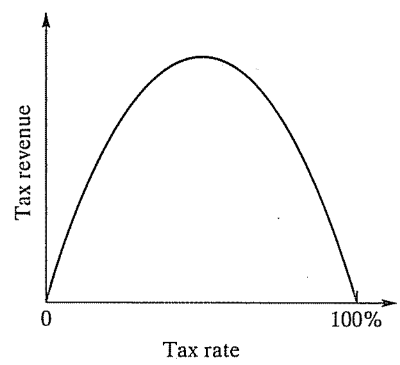
In the mid-1970s, economist Arthur Laffer was explaining his views on taxes to a politician. To illustrate his argument, Laffer grabbed a paper napkin and sketched the graph that now bears his name: the Laffer curve.

The Laffer curve describes total government tax revenue as a function of the tax rate. Obviously, if the tax rate is zero, the government gets nothing. But if the tax rate is 100%, revenue would again equal zero, because there is no incentive to earn money if it will all be taken away. Since tax rates between 0% and 100% do generate revenue, Laffer reasoned, the curve relating revenue to tax rate must look, qualitatively, more or less as shown in the figure below.

Laffer's argument was not meant to show that the optimal tax rate was 50%. It was meant to show that under some circumstances, namely when the tax rate is to the right of the peak of the curve, it is possible to *raise government revenue by lowering taxes.* This was a key argument made for the tax cuts passed by Congress during the first term of the Reagan presidency.

Because the Laffer curve is only a qualitative picture, it does not actually give an optimal tax rate. Revenue-based arguments for tax cuts involve the claim that the point of peak revenue lies to the left of the current taxation scheme on the horizontal axis. By the same token, those who urge raising taxes to raise government income are assuming either a different relationship between rates and revenues or a different location of the curve's peak.

By itself, then, the Laffer curve is too abstract to be of much help in determining the optimal tax rate. But even very simple sketched curves, like supply and demand curves and the Laffer curve, can help economists describe the causal factors that drive an economy. In this chapter, we will discuss techniques for sketching and interpreting curves.



Objective

To find when a function is increasing or decreasing, to find critical values, to locate relative maxima and relative minima, and to state the first-derivative test. Also, to sketch the graph of a function by using the information obtained from the first derivative.

13.1 Relative Extrema

Increasing or Decreasing Nature of a Function

Examining the graphical behavior of functions is a basic part of mathematics and has applications to many areas of study. When we sketch a curve, just plotting points may not give enough information about its shape. For example, the points $(-1, 0)$, $(0, -1)$, and $(1, 0)$ satisfy the equation given by $y = (x+1)^3(x-1)$. On the basis of these points, we might hastily conclude that the graph should appear as in Figure 13.1(a), but in fact the true shape is given in Figure 13.1(b). In this chapter we will explore the powerful role that differentiation plays in analyzing a function so that we can determine the true shape and behavior of its graph.

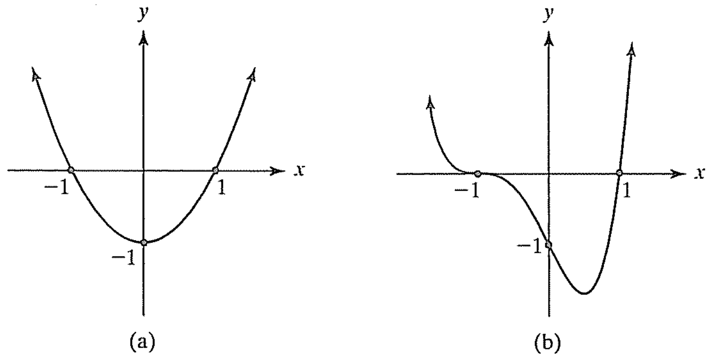


FIGURE 13.1 Curves passing through $(-1, 0)$, $(0, -1)$, and $(1, 0)$.

We begin by analyzing the graph of the function $y = f(x)$ in Figure 13.2. Notice that as x increases (goes from left to right) on the interval I_1, between a and b, the values of $f(x)$ increase and the curve is rising. Mathematically, this observation means that if x_1 and x_2 are any two points in I_1 such that $x_1 < x_2$, then $f(x_1) < f(x_2)$. Here f is said to be an *increasing function* on I_1. On the other hand, as x increases on the interval I_2 between c and d, the curve is falling. On this interval, $x_3 < x_4$ implies that $f(x_3) > f(x_4)$, and f is said to be a *decreasing function* on I_2. We summarize these observations in the following definition.

> **Definition**
> A function f is said to be ***increasing*** on an interval I when, for any two numbers x_1, x_2 in I, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. A function f is ***decreasing*** on an interval I when, for any two numbers x_1, x_2 in I, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

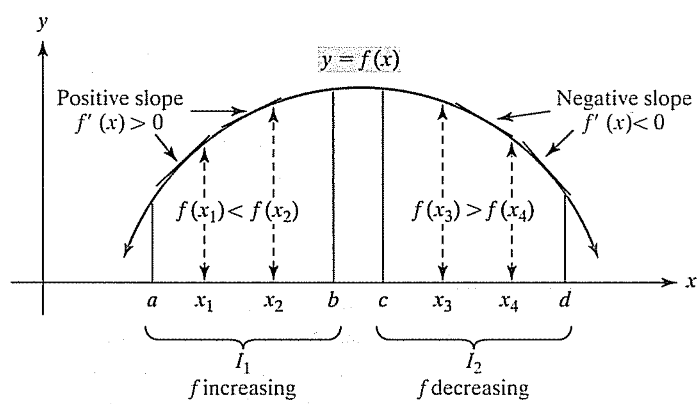


FIGURE 13.2 Increasing or decreasing nature of function.

In terms of the graph of the function, f is increasing on I if the curve rises to the right and f is decreasing on I if the curve falls to the right. Recall that a straight line with positive slope rises to the right while a straight line with negative slope falls to the right.

Turning again to Figure 13.2, we note that over the interval I_1, tangent lines to the curve have positive slopes, so $f'(x)$ must be positive for all x in I_1. A positive derivative implies that the curve is rising. Over the interval I_2, the tangent lines have negative slopes, so $f'(x) < 0$ for all x in I_2. The curve is falling where the derivative is negative. We thus have the following rule, which allows us to use the derivative to determine when a function is increasing or decreasing:

Rule 1 Criteria for Increasing or Decreasing Function

Let f be differentiable on the interval (a, b). If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b). If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

To illustrate these ideas, we will use Rule 1 to find the intervals on which $y = 18x - \frac{2}{3}x^3$ is increasing and the intervals on which y is decreasing. Letting $y = f(x)$, we must determine when $f'(x)$ is positive and when $f'(x)$ is negative. We have

$$f'(x) = 18 - 2x^2 = 2(9 - x^2) = 2(3 + x)(3 - x)$$

Using the technique of Section 10.4, we can find the sign of $f'(x)$ by testing the intervals determined by the roots of $2(3 + x)(3 - x) = 0$, namely, -3 and 3. These should be arranged in increasing order on the top of a sign chart for f' so as to divide the domain of f into intervals. (See Figure 13.3.) In each interval, the sign of $f'(x)$ is determined by the signs of its factors:

	$-\infty$	-3		3		∞
$3 + x$	$-$	0	$+$		$+$	
$3 - x$	$+$		$+$	0	$-$	
$f'(x)$	$-$	0	$+$	0	$-$	
$f(x)$	↘		↗		↘	

FIGURE 13.3 Sign chart for $f'(x) = 18 - 9x^2$ and its interpretation for $f(x)$.

If $x < -3$, then $\text{sign}(f'(x)) = 2(-)(+) = -$, so f is *decreasing*.

If $-3 < x < 3$, then $\text{sign}(f'(x)) = 2(+)(+) = +$, so f is *increasing*.

If $x > 3$, then $\text{sign}(f'(x)) = 2(+)(-) = -$, so f is *decreasing*.

These results are indicated in the sign chart given by Figure 13.3, where the bottom line is a schematic version of what the signs of f' say about f itself. Notice that the horizontal line segments in the bottom row indicate horizontal tangents for f at -3 and at 3. Thus, f is decreasing on $(-\infty, -3)$ and $(3, \infty)$ and is increasing on $(-3, 3)$. This corresponds to the rising and falling nature of the graph of f shown in Figure 13.4. Indeed, the point of a well-constructed sign chart is to provide a schematic for subsequent construction of the graph itself.

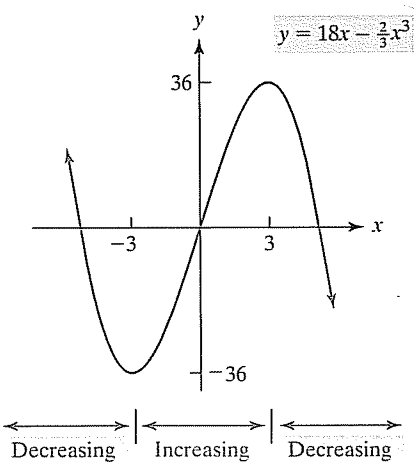


FIGURE 13.4 Increasing/decreasing for $y = 18x - \frac{2}{3}x^3$.

Extrema

Look now at the graph of $y = f(x)$ in Figure 13.5. Some observations can be made. First, there is something special about the points P, Q, and R. Notice that P is *higher* than any other "nearby" point on the curve—and likewise for R. The point Q is *lower* than any other "nearby" point on the curve. Since P, Q, and R may not necessarily be the highest or lowest points on the *entire* curve, we say that the graph

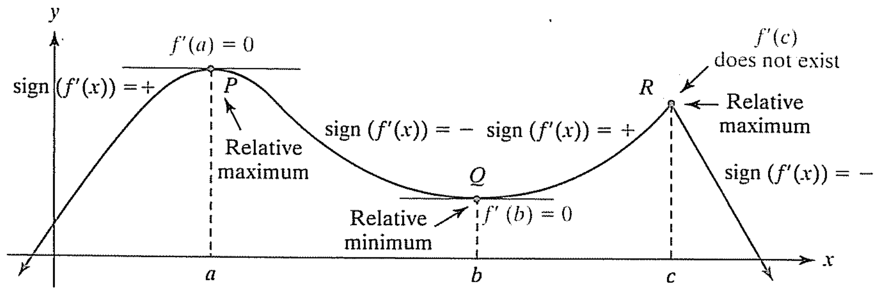


FIGURE 13.5 Relative maxima and relative minima.

CAUTION!

Be sure to note the difference between relative extreme *values* and *where* they occur.

of f *has relative maxima at a and at c; and has a relative minimum at b.* The function f *has relative maximum values of* $f(a)$ *at a and* $f(c)$ *at c; and has a relative minimum value of* $f(b)$ *at b.* We also say that $(a, f(a))$ *and* $(c, f(c))$ *are relative maximum points and* $(b, f(b))$ *is a relative minimum point on the graph of* f.

Turning back to the graph, we see that there is an *absolute maximum* (highest point on the entire curve) at a, but there is no *absolute minimum* (lowest point on the entire curve) because the curve is assumed to extend downward indefinitely. More precisely, we define these new terms as follows:

Definition

A function f has a ***relative maximum*** at a if there is an open interval containing a on which $f(a) \geq f(x)$ for all x in the interval. The relative maximum value is $f(a)$. A function f has a ***relative minimum*** at a if there is an open interval containing a on which $f(a) \leq f(x)$ for all x in the interval. The relative minimum value is $f(a)$.

If it exists, an absolute maximum value is unique; however, it may occur at more than one value of x. A similar statement is true for an absolute minimum.

Definition

A function f has an ***absolute maximum*** at a if $f(a) \geq f(x)$ for all x in the domain of f. The absolute maximum value is $f(a)$. A function f has an ***absolute minimum*** at a if $f(a) \leq f(x)$ for all x in the domain of f. The absolute minimum value is $f(a)$.

We refer to either a relative maximum or a relative minimum as a **relative extremum** (plural: *relative extrema*). Similarly, we speak of **absolute extrema.**

When dealing with relative extrema, we compare the function value at a point with values of nearby points; however, when dealing with absolute extrema, we compare the function value at a point with all other values determined by the domain. Thus, relative extrema are *local* in nature, whereas absolute extrema are *global* in nature.

Referring to Figure 13.5, we notice that at a relative extremum the derivative may not be defined (as when $x = c$). But whenever it is defined at a relative extremum, it is 0 (as when $x = a$ and when $x = b$), and hence the tangent line is horizontal. We can state the following:

Rule 2 A Necessary Condition for Relative Extrema

If f has a relative extremum at a, then $f'(a) = 0$ or $f'(a)$ does not exist.

The implication in Rule 2 goes in only one direction:

$$\left.\begin{array}{c}\text{relative extremum}\\ \text{at } a\end{array}\right\} \quad \text{implies} \quad \left\{\begin{array}{c}f'(a) = 0\\ \text{or}\\ f'(a) \text{ does not exist}\end{array}\right.$$

Rule 2 does *not* say that if $f'(a)$ is 0 or $f'(a)$ does not exist, then there must be a relative extremum at a. In fact, there may not be one at all. For example, in Figure 13.6(a), $f'(a)$

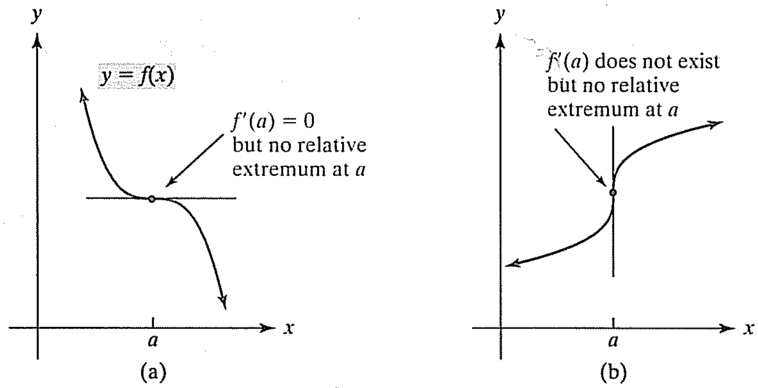


FIGURE 13.6 No relative extremum at a.

is 0 because the tangent line is horizontal at a, but there is no relative extremum there. In Figure 13.6(b), $f'(a)$ does not exist because the tangent line is vertical at a, but again there is no relative extremum there.

But if we want to find all relative extrema of a function—and this is an important task—what Rule 2 *does* tell us is that we can limit our search to those values of x in the domain of f for which *either* $f'(x) = 0$ *or* $f'(x)$ does not exist. Typically, in applications, this cuts down our search for relative extrema from the infinitely many x for which f is defined to a small finite number of *possibilities*. Because these values of x are so important for locating the relative extrema of f, they are called the *critical values* for f, and if a is a critical value for f, then we also say that $(a, f(a))$ is a *critical point* on the graph of f. Thus, in Figure 13.5, the numbers a, b, and c are critical values, and P, Q, and R are critical points.

Definition

For a in the domain of f, if either $f'(a) = 0$ or $f'(a)$ does not exist, then a is called a ***critical value*** for f. If a is a critical value, then the point $(a, f(a))$ is called a ***critical point*** for f.

At a critical point, there may be a relative maximum, a relative minimum, or neither. Moreover, from Figure 13.5, we observe that each relative extremum occurs at a point around which the sign of $f'(x)$ is changing. For the relative maximum at a, the sign of $f'(x)$ goes from $+$ for $x < a$ to $-$ for $x > a$, *as long as x is near a*. For the relative minimum at b, the sign of $f'(x)$ goes from $-$ to $+$, and for the relative maximum at c, it again goes from $+$ to $-$. Thus, *around relative maxima, f is increasing and then decreasing, and the reverse holds for relative minima.* More precisely, we have the following rule:

Rule 3 Criteria for Relative Extrema

Suppose f is continuous on an open interval I that contains the critical value a and f is differentiable on I, except possibly at a.

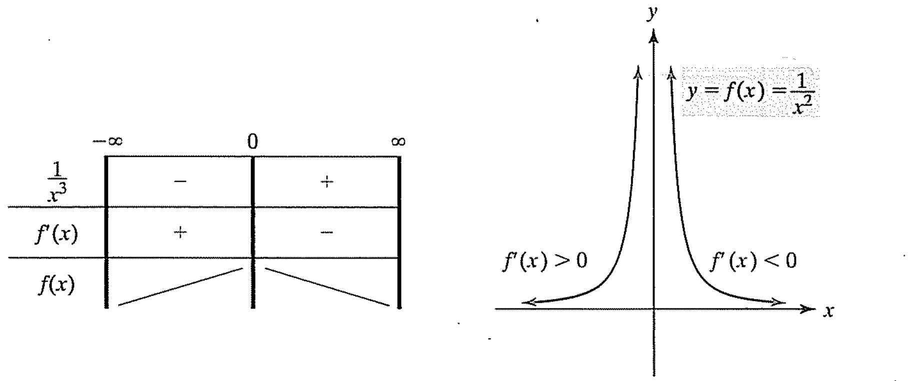
1. If $f'(x)$ changes from positive to negative as x increases through a, then f has a relative maximum at a.

2. If $f'(x)$ changes from negative to positive as x increases through a, then f has a relative minimum at a.

To illustrate Rule 3 with a concrete example, refer again to Figure 13.3, the sign chart for $f'(x) = 18 - 2x^2$. The row labeled by $f'(x)$ shows clearly that $f(x) = 18x - \frac{2}{3}x^3$ has a relative minimum at -3 and a relative maximum at 3. The row providing the interpretation of the chart for f, labeled $f(x)$, is immediately deduced

CAUTION!

We point out again that not every critical value corresponds to a relative extremum. For example, if $y = f(x) = x^3$, then $f'(x) = 3x^2$. Since $f'(0) = 0$, 0 is a critical value. But if $x < 0$, then $3x^2 > 0$, and if $x > 0$, then $3x^2 > 0$. Since $f'(x)$ does not change sign at 0, there is no relative extremum at 0. Indeed, since $f'(x) \geq 0$ for all x, the graph of f never falls, and f is said to be *nondecreasing*. (See Figure 13.8.)



(a) (b)

FIGURE 13.7 $f'(0)$ is not defined, but 0 is not a critical value because 0 is not in the domain of f.

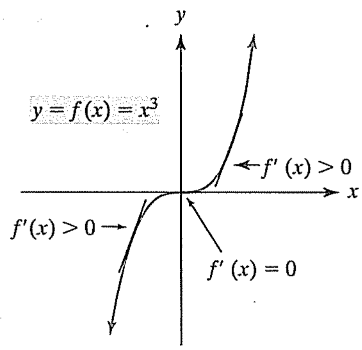


FIGURE 13.8 Zero is a critical value, but does not give a relative extremum.

from the row above it. The significance of the $f(x)$ row is that it provides an intermediate step in actually sketching the graph of f. In this row it stands out, visually, that f has a relative minimum at -3 and a relative maximum at 3.

When searching for extrema of a function f, care must be paid to those a that are not in the domain of f but that are near values in the domain of f. Consider the following example. If

$$y = f(x) = \frac{1}{x^2}, \quad \text{then} \quad f'(x) = -\frac{2}{x^3}$$

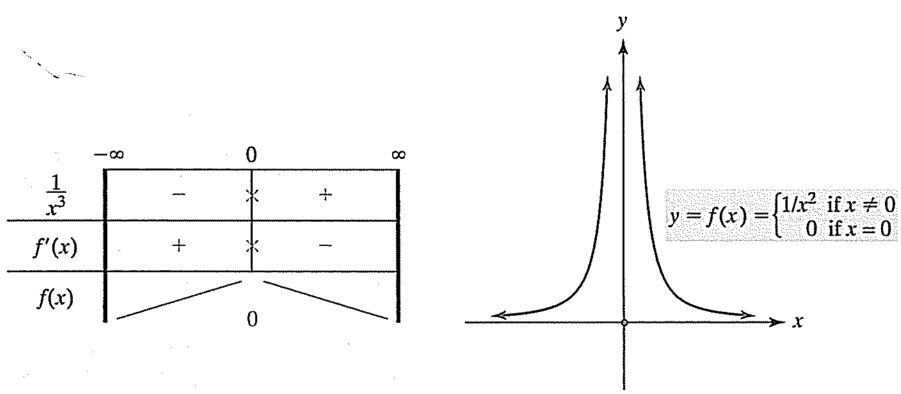
Although $f'(x)$ does not exist at 0, 0 is not a critical value, because 0 is not in the domain of f. Thus, a relative extremum cannot occur at 0. Nevertheless, the derivative may change sign around any x-value where $f'(x)$ is not defined, so such values are important in determining intervals over which f is increasing or decreasing. In particular, such values should be included in a sign chart for f'. See Figure 13.7(a) and the accompanying graph in Figure 13.7(b).

Observe that the thick vertical rule at 0 on the chart serves to indicate that 0 is not in the domain of f. Here there are no extrema of any kind.

In Rule 3 the hypotheses must be satisfied, or the conclusion need not hold. For example, consider the case-defined function

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here, 0 is explicitly in the domain of f but f is not continuous at 0. We recall from Section 11.1 that if a function f is not continuous at a, then f is not differentiable at a, meaning that $f'(a)$ does not exist. Thus $f'(0)$ does not exist and 0 is a critical value that must be included in the sign chart for f' shown in Figure 13.9(a). We extend our sign



(a) (b)

FIGURE 13.9 Zero is a critical value, but Rule 3 does not apply.

chart conventions by indicating with a $\times$ symbol those values for which f' does not exist. We see in this example that $f'(x)$ changes from positive to negative as x increases through 0 but f does *not* have a relative maximum at 0. Here Rule 3 does not apply because its continuity hypothesis is not met. In Figure 13.9(b), 0 is displayed in the domain of f. It is clear that f has an absolute *minimum* at 0 because $f(0) = 0$ and, for all $x \neq 0, f(x) > 0$.

Summarizing the results of this section, we have the *first-derivative test* for the relative extrema of $y = f(x)$:

First-Derivative Test for Relative Extrema

Step 1. Find $f'(x)$.

Step 2. Determine all critical values of f [those a where $f'(a) = 0$ or $f'(a)$ does not exist] and any a that are not in the domain of f but that are near values in the domain of f, and construct a sign chart that shows for each of the intervals determined by these values whether f is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$).

Step 3. For each critical value a at which f is continuous, determine whether $f'(x)$ changes sign as x increases through a. There is a relative maximum at a if $f'(x)$ changes from $+$ to $-$ going from left to right and a relative minimum if $f'(x)$ changes from $-$ to $+$ going from left to right. If $f'(x)$ does not change sign, there is no relative extremum at a.

Step 4. For critical values a at which f is not continuous, analyze the situation by using the definitions of extrema directly.

APPLY IT ▷

1. The cost equation for a hot dog stand is given by $c(q) = 2q^3 - 21q^2 + 60q + 500$, where q is the number of hot dogs sold, and $c(q)$ is the cost in dollars. Use the first-derivative test to find where relative extrema occur.

EXAMPLE 1 First-Derivative Test

If $y = f(x) = x + \dfrac{4}{x+1}$, for $x \neq -1$ use the first-derivative test to find where relative extrema occur.

Solution:

Step 1. $f(x) = x + 4(x+1)^{-1}$, so

$$f'(x) = 1 + 4(-1)(x+1)^{-2} = 1 - \frac{4}{(x+1)^2}$$

$$= \frac{(x+1)^2 - 4}{(x+1)^2} = \frac{x^2 + 2x - 3}{(x+1)^2}$$

$$= \frac{(x+3)(x-1)}{(x+1)^2} \quad \text{for } x \neq -1$$

Note that we expressed $f'(x)$ as a quotient with numerator and denominator fully factored. This enables us in Step 2 to determine easily where $f'(x)$ is 0 or does not exist and the signs of f'.

Step 2. Setting $f'(x) = 0$ gives $x = -3, 1$. The denominator of $f'(x)$ is 0 when x is -1. We note that -1 is not in the domain of f but all values near -1 are in the domain of f. We construct a sign chart, headed by the values -3, -1, and 1 (which we have placed in increasing order). See Figure 13.10.

The three values lead us to test four intervals as shown in our sign chart. On each of these intervals, f is differentiable and is not zero. We determine the sign of f' on each interval by first determining the sign of each of its factors on each iterval. For example, considering first the interval $(-\infty, -3)$, it is not easy to see immediately that $f'(x) > 0$ there; but it is easy to see that $x+3 < 0$ for $x < -3$, while $(x+1)^{-2} > 0$ for all $x \neq -1$, and $x-1 < 0$ for $x < 1$. These observations account for the signs of the factors in the $(-\infty, -3)$ column of the chart. The sign of $f'(x)$ in that column is obtained by "multiplying signs" (downward): $(-)(+)(-) = +$. We repeat these considerations for the other three intervals. Note that the thick vertical line at -1 in the chart indicates that

	$-\infty$	-3		-1		1	∞
$x+3$	$-$	0	$+$		$+$		$+$
$(x+1)^{-2}$	$+$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$f'(x)$	$+$	0	$-$		$-$	0	$+$
$f(x)$							

FIGURE 13.10 Sign chart for $f'(x) = \dfrac{(x+3)(x-1)}{(x+1)^2}$.

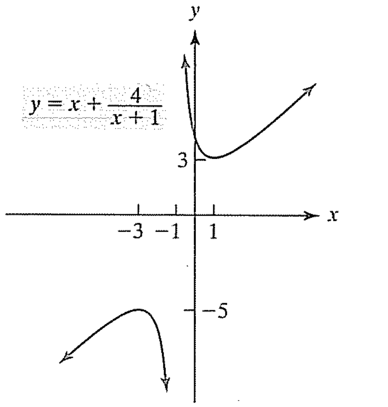


FIGURE 13.11 Graph of $y = x + \dfrac{4}{x+1}$.

-1 is not in the domain of f and hence cannot give rise to any extrema. In the bottom row of the sign chart we record, graphically, the nature of tangent lines to $f(x)$ in each interval and at the values where f' is 0.

Step 3. From the sign chart alone we conclude that at -3 there is a relative maximum (since $f'(x)$ changes from $+$ to $-$ at -3). Going beyond the chart, we compute $f(-3) = -3 + (4/-2) = -5$, and this gives the relative maximum value of -5 at -3. We also conclude from the chart that there is a relative minimum at 1 [because $f'(x)$ changes from $-$ to $+$ at 1]. From $f(1) = 1 + 4/2 = 3$ we see that at 1 the relative minimum value is 3.

Step 4. There are no critical values at which f is not continuous, so our considerations above provide the whole story about the relative extrema of $f(x)$, whose graph is given in Figure 13.11. Note that the general shape of the graph was indeed forecast by the bottom row of the sign chart (Figure 13.10).

Now Work Problem 37 ◁

EXAMPLE 2 A Relative Extremum where $f'(x)$ Does Not Exist

Test $y = f(x) = x^{2/3}$ for relative extrema.

Solution: We have

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

	$-\infty$	0	∞
$(x)^{-1/3}$	$-$	$\times$	$+$
$f'(x)$	$-$	$\times$	$+$
$f(x)$			

FIGURE 13.12 Sign chart for $f'(x) = \dfrac{2}{3\sqrt[3]{x}}$.

and the sign chart is given in Figure 13.12. Again, we use the symbol $\times$ on the vertical line at 0 to indicate that the factor $x^{-1/3}$ does not exist at 0. Hence $f'(0)$ does not exist. Since f is continuous at 0, we conclude from Rule 3 that f has a relative minimum at 0 of $f(0) = 0$, and there are no other relative extrema. We note further, by inspection of the sign chart, that f has an *absolute* minimum at 0. The graph of f follows as Figure 13.13. Note that we could have predicted its shape from the bottom line of the sign chart in Figure 13.12, which shows there can be no tangent with a slope at 0. (Of course, the tangent does exist at 0 but it is a vertical line.)

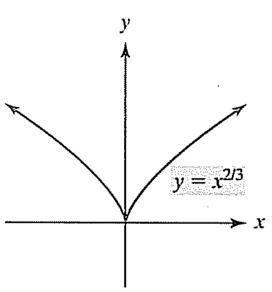


FIGURE 13.13 Derivative does not exist at 0 and there is a minimum at 0.

Now Work Problem 41 ◁

EXAMPLE 3 Finding Relative Extrema

Test $y = f(x) = x^2e^x$ for relative extrema.

Solution: By the product rule,

$$f'(x) = x^2e^x + e^x(2x) = xe^x(x+2)$$

	$-\infty$	-2		0	∞
$x+2$	$-$	0	$+$		$+$
x	$-$		$-$	0	$+$
e^x	$+$		$+$		$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗		↘		↗

FIGURE 13.14 Sign chart for $f'(x) = x(x+2)e^x$.

APPLY IT ▷

2. A drug is injected into a patient's bloodstream. The concentration of the drug in the bloodstream t hours after the injection is approximated by $C(t) = \dfrac{0.14t}{t^2+4t+4}$. Find the relative extrema for $t > 0$, and use them to determine when the drug is at its greatest concentration.

Noting that e^x is always positive, we obtain the critical values 0 and -2. From the sign chart of $f'(x)$ given in Figure 13.14, we conclude that there is a relative maximum when $x = -2$ and a relative minimum when $x = 0$.

Now Work Problem 49 ◁

Curve Sketching

In the next example we show how the first-derivative test, in conjunction with the notions of intercepts and symmetry, can be used as an aid in sketching the graph of a function.

EXAMPLE 4 Curve Sketching

Sketch the graph of $y = f(x) = 2x^2 - x^4$ with the aid of intercepts, symmetry, and the first-derivative test.

Solution:

Intercepts If $x = 0$, then $f(x) = 0$ so that the y-intercept is $(0, 0)$. Next note that

$$f(x) = 2x^2 - x^4 = x^2(2 - x^2) = x^2\left(\sqrt{2}+x\right)\left(\sqrt{2}-x\right)$$

So if $y = 0$, then $x = 0, \pm\sqrt{2}$ and the x-intercepts are $(-\sqrt{2}, 0)$, $(0, 0)$, and $(\sqrt{2}, 0)$. We have the sign chart *for f itself* (Figure 13.15), which shows the intervals over which the graph of $y = f(x)$ is above the x-axis (+) and the intervals over which the graph of $y = f(x)$ is below the x-axis ($-$).

	$-\infty$	$-\sqrt{2}$		0		$\sqrt{2}$	∞
$\sqrt{2}+x$	$-$	0	$+$		$+$		$+$
x^2	$+$		$+$	0	$+$		$+$
$\sqrt{2}-x$	$+$		$+$		$+$	0	$-$
$f(x)$	$-$	0	$+$	0	$+$	0	$-$

FIGURE 13.15 Sign chart for $f(x) = (\sqrt{2}+x)x^2(\sqrt{2}-x)$.

Symmetry Testing for y-axis symmetry, we have

$$f(-x) = 2(-x)^2 - (-x)^4 = 2x^2 - x^4 = f(x)$$

So the graph is symmetric with respect to the y-axis. Because y is a function (and not the zero function), there is no x-axis symmetry and hence no symmetry about the origin.

First-Derivative Test

Step 1. $y' = 4x - 4x^3 = 4x(1 - x^2) = 4x(1+x)(1-x)$.

Step 2. Setting $y' = 0$ gives the critical values $x = 0, \pm 1$. Since f is a polynomial, it is defined and differentiable for all x. Thus the only values to head the sign chart for f' are -1, 0, 1 (in increasing order) and the sign chart is given in Figure 13.16. Since we are interested in the graph, the critical *points* are important to us. By substituting the critical values into the *original* equation, $y = 2x^2 - x^4$,

	$-\infty$ to -1	-1	-1 to 0	0	0 to 1	1	1 to ∞
$1+x$	−	0	+		+		+
$4x$	−		−	0	+		+
$1-x$	+		+		+	0	−
$f'(x)$	+	0	−	0	+	0	−
$f(x)$							

FIGURE 13.16 Sign chart of $y' = (1+x)4x(1-x)$.

we obtain the y-coordinates of these points. We find the critical points to be $(-1, 1)$, $(0, 0)$, and $(1, 1)$.

Step 3. From the sign chart and evaluations in step 2, it is clear that f has relative maxima $(-1, 1)$ and $(1, 1)$ and relative minimum $(0, 0)$. (Step 4 does not apply here.)

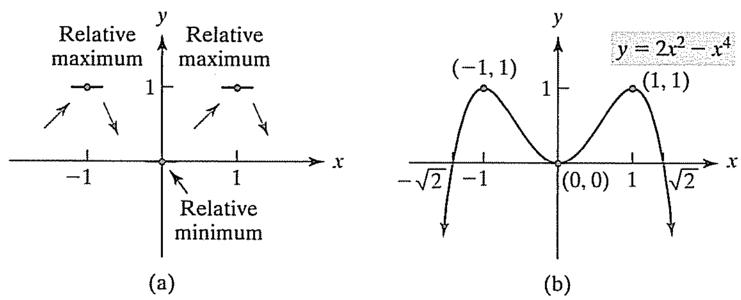


FIGURE 13.17 Putting together the graph of $y = 2x^2 - x^4$.

Discussion In Figure 13.17(a), we have indicated the horizontal tangents at the relative maximum and minimum points. We know the curve rises from the left, has a relative maximum, then falls, has a relative minimum, then rises to a relative maximum, and falls thereafter. By symmetry, it suffices to sketch the graph on one side of the y-axis and construct a mirror image on the other side. We also know, from the sign chart for f, where the graph crosses and touches the x-axis, and this adds further precision to our sketch, which is shown in Figure 13.17(b).

As a passing comment, we note that *absolute* maxima occur at $x = \pm 1$. See Figure 13.17(b). There is no absolute minimum.

Now Work Problem 59 ◁

TECHNOLOGY

A graphing calculator is a powerful tool for investigating relative extrema. For example, consider the function

$$f(x) = 3x^4 - 4x^3 + 4$$

whose graph is shown in Figure 13.18. It appears that there is a relative minimum near $x = 1$. We can locate this minimum by either using "trace and zoom" or (on a TI-83 Plus) using the "minimum" feature. Figure 13.19 shows the latter approach. The relative minimum point is estimated to be (1.00, 3).

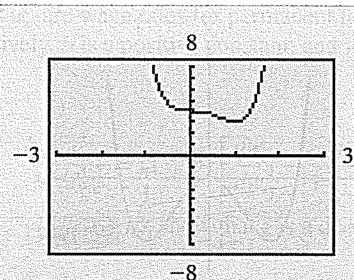


FIGURE 13.18 Graph of $f(x) = 3x^4 - 4x^3 + 4$.

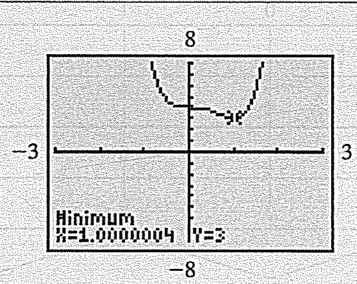


FIGURE 13.19 Relative minimum at (1.00, 3).

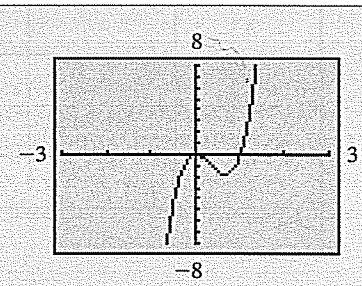


FIGURE 13.20 Graph of $f'(x) = 12x^3 - 12x^2$.

Now let us see how the graph of f' indicates when extrema occur. We have

$$f'(x) = 12x^3 - 12x^2$$

whose graph is shown in Figure 13.20. It appears that $f'(x)$ is 0 at two points. Using "trace and zoom" or the "zero" feature, we estimate the roots of $f' = 0$ (the critical values of f) to be 1 and 0. Around $x = 1$, we see that $f'(x)$ goes from negative values to positive values. (That is, the graph of f' goes from below the x-axis to above it.) Thus, we conclude that f has a relative minimum at $x = 1$, which confirms our previous result.

Around the critical value $x = 0$, the values of $f'(x)$ are negative. Since $f'(x)$ does not change sign, we conclude that there is no relative extremum at $x = 0$. This is also apparent from the graph in Figure 13.18.

It is worthwhile to note that we can approximate the graph of f' without determining $f'(x)$ itself. We make use of the "nDeriv" feature. First we enter the function f as Y_1. Then we set

$$Y_2 = \text{nDeriv}(Y_1, X, X)$$

The graph of Y_2 approximates the graph of $f'(x)$.

PROBLEMS 13.1

In Problems 1–4, the graph of a function is given (Figures 13.21–13.24). Find the open intervals on which the function is increasing, the open intervals on which the function is decreasing, and the coordinates of all relative extrema.

1.

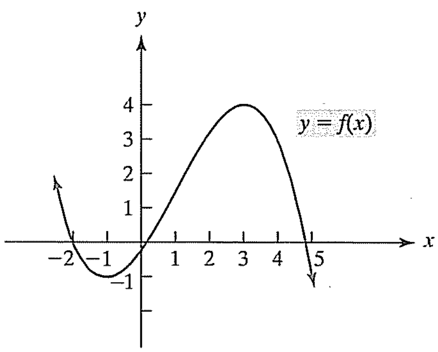


FIGURE 13.21

2.

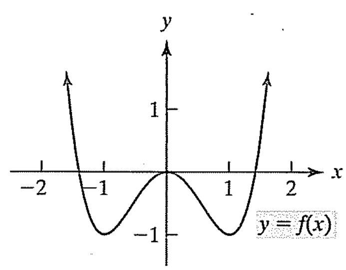


FIGURE 13.22

3.

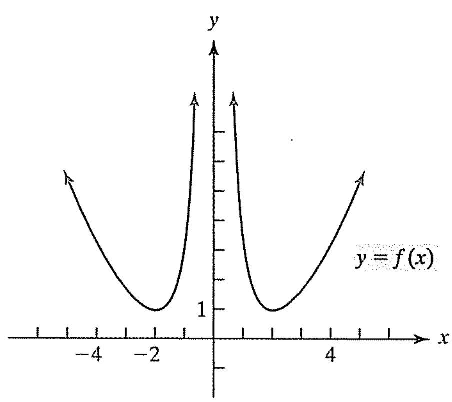


FIGURE 13.23

4.

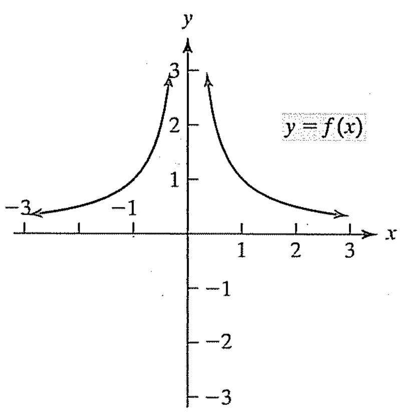


FIGURE 13.24

In Problems 5–8, the derivative of a continuous function f is given. Find the open intervals on which f is **(a)** *increasing;* **(b)** *decreasing; and* **(c)** *find the x-values of all relative extrema.*

5. $f'(x) = (x+3)(x-1)(x-2)$

6. $f'(x) = 2x(x-1)^3$

7. $f'(x) = (x+1)(x-3)^2$ **8.** $f'(x) = \dfrac{x(x+2)}{x^2+1}$

In Problems 9–52, determine where the function is **(a)** *increasing;* **(b)** *decreasing; and* **(c)** *determine where relative extrema occur. Do not sketch the graph.*

9. $y = -x^3 - 1$ **10.** $y = x^2 + 4x + 3$

11. $y = x - x^2 + 2$ **12.** $y = x^3 - \frac{5}{2}x^2 - 2x + 6$

13. $y = -\dfrac{x^3}{3} - 2x^2 + 5x - 2$ **14.** $y = -\dfrac{x^4}{4} - x^3$

15. $y = x^4 - 2x^2$ **16.** $y = -3 + 12x - x^3$

17. $y = x^3 - \frac{7}{2}x^2 + 2x - 5$ **18.** $y = x^3 - 6x^2 + 12x - 6$

19. $y = 2x^3 - \frac{19}{2}x^2 + 10x + 2$ **20.** $y = -5x^3 + x^2 + x - 7$

21. $y = \dfrac{x^3}{3} - 5x^2 + 22x + 1$ **22.** $y = \frac{9}{5}x^5 - \frac{47}{3}x^3 + 10x$

23. $y = 3x^5 - 5x^3$ **24.** $y = 3x - \dfrac{x^6}{2}$ (Remark: $x^4 + x^3 + x^2 + x + 1 = 0$ has no real roots.)

25. $y = -x^5 - 5x^4 + 200$ **26.** $y = \dfrac{3x^4}{2} - 4x^3 + 17$

27. $y = 8x^4 - x^8$ **28.** $y = \frac{4}{5}x^5 - \frac{13}{3}x^3 + 3x + 4$

29. $y = (x^2 - 4)^4$ **30.** $y = \sqrt[3]{x}(x-2)$

31. $y = \dfrac{5}{x-1}$ **32.** $y = \dfrac{3}{x}$

33. $y = \dfrac{10}{\sqrt{x}}$ **34.** $y = \dfrac{ax+b}{cx+d}$
(a) for $ad - bc > 0$
(b) for $ad - bc < 0$

35. $y = \dfrac{x^2}{2-x}$ **36.** $y = 4x^2 + \dfrac{1}{x}$

37. $y = \dfrac{x^2-3}{x+2}$ **38.** $y = \dfrac{2x^2}{4x^2-25}$

39. $y = \dfrac{ax^2+b}{cx^2+d}$ for $d/c < 0$
(a) for $ad - bc > 0$
(b) for $ad - bc < 0$

40. $y = \sqrt[3]{x^3 - 9x}$

41. $y = (x-1)^{2/3}$ **42.** $y = x^2(x+3)^4$

43. $y = x^3(x-6)^4$ **44.** $y = (1-x)^{2/3}$

45. $y = e^{-\pi x} + \pi$ **46.** $y = x \ln x$

47. $y = x^2 - 9\ln x$ **48.** $y = x^{-1}e^x$

49. $y = e^x - e^{-x}$ **50.** $y = e^{-x^2/2}$

51. $y = x\ln x - x$ **52.** $y = (x^2+1)e^{-x}$

In Problems 53–64, determine intervals on which the function is increasing; intervals on which the function is decreasing; relative extrema; symmetry; and those intercepts that can be obtained conveniently. Then sketch the graph.

53. $y = x^2 - 3x - 10$ **54.** $y = 2x^2 + x - 10$

55. $y = 3x - x^3$ **56.** $y = x^4 - 16$

57. $y = 2x^3 - 9x^2 + 12x$ **58.** $y = 2x^3 - x^2 - 4x + 4$

59. $y = x^4 - 2x^2$ **60.** $y = x^6 - \frac{6}{5}x^5$

61. $y = (x-1)^2(x+2)^2$ **62.** $y = \sqrt{x}(x^2 - x - 2)$

63. $y = 2\sqrt{x} - x$ **64.** $y = x^{5/3} - 2x^{2/3}$

65. Sketch the graph of a continuous function f such that $f(2) = 2$, $f(4) = 6$, $f'(2) = f'(4) = 0$, $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $2 < x < 4$, f has a relative maximum at 4, and $\lim_{x\to\infty} f(x) = 0$.

66. Sketch the graph of a continuous function f such that $f(1) = 2$, $f(4) = 5$, $f'(1) = 0$, $f'(x) \geq 0$ for $x < 4$, f has a relative maximum when $x = 4$, and there is a vertical tangent line when $x = 4$.

67. Average Cost If $c_f = 25{,}000$ is a fixed-cost function, show that the average fixed-cost function $\bar{c}_f = c_f/q$ is a decreasing function for $q > 0$. Thus, as output q increases, each unit's portion of fixed cost declines.

68. Marginal Cost If $c = 3q - 3q^2 + q^3$ is a cost function, when is marginal cost increasing?

69. Marginal Revenue Given the demand function

$$p = 500 - 5q$$

find when marginal revenue is increasing.

70. Cost Function For the cost function $c = \sqrt{q}$, show that marginal and average costs are always decreasing for $q > 0$.

71. Revenue For a manufacturer's product, the revenue function is given by $r = 240q + 57q^2 - q^3$. Determine the output for maximum revenue.

72. Labor Markets Eswaran and Kotwal[1] consider agrarian economies in which there are two types of workers, permanent and casual. Permanent workers are employed on long-term contracts and may receive benefits such as holiday gifts and emergency aid. Casual workers are hired on a daily basis and perform routine and menial tasks such as weeding, harvesting, and threshing. The difference z in the present-value cost of hiring a permanent worker over that of hiring a casual worker is given by

$$z = (1+b)w_p - bw_c$$

where w_p and w_c are wage rates for permanent labor and casual labor, respectively, b is a positive constant, and w_p is a function of w_c.

(a) Show that

$$\frac{dz}{dw_c} = (1+b)\left[\frac{dw_p}{dw_c} - \frac{b}{1+b}\right]$$

(b) If $dw_p/dw_c < b/(1+b)$, show that z is a decreasing function of w_c.

[1] M. Eswaran and A. Kotwal, "A Theory of Two-Tier Labor Markets in Agrarian Economics," *The American Economic Review*, 75, no. 1 (1985), 162–77.

73. Thermal Pollution In Shonle's discussion of thermal pollution,[2] the efficiency of a power plant is given by

$$E = 0.71\left(1 - \frac{T_c}{T_h}\right)$$

where T_h and T_c are the respective absolute temperatures of the hotter and colder reservoirs. Assume that T_c is a positive constant and that T_h is positive. Using calculus, show that as T_h increases, the efficiency increases.

74. Telephone Service In a discussion of the pricing of local telephone service, Renshaw[3] determines that total revenue r is given by

$$r = 2F + \left(1 - \frac{a}{b}\right)p - p^2 + \frac{a^2}{b}$$

where p is an indexed price per call, and a, b, and F are constants. Determine the value of p that maximizes revenue.

75. Storage and Shipping Costs In his model for storage and shipping costs of materials for a manufacturing process, Lancaster[4] derives the cost function

$$C(k) = 100\left(100 + 9k + \frac{144}{k}\right) \quad 1 \le k \le 100$$

where $C(k)$ is the total cost (in dollars) of storage and transportation for 100 days of operation if a load of k tons of material is moved every k days.

(a) Find $C(1)$.
(b) For what value of k does $C(k)$ have a minimum?
(c) What is the minimum value?

76. Physiology—The Bends When a deep-sea diver undergoes decompression or a pilot climbs to a high altitude, nitrogen may bubble out of the blood, causing what is commonly called *the bends*. Suppose the percentage P of people who suffer effects of the bends at an altitude of h thousand feet is given by[5]

$$P = \frac{100}{1 + 100{,}000e^{-0.36h}}$$

Is P an increasing function of h?

In Problems 77–80, from the graph of the function, find the coordinates of all relative extrema. Round your answers to two decimal places.

77. $y = 0.3x^2 + 2.3x + 5.1$ **78.** $y = 3x^4 - 4x^3 - 5x + 1$

79. $y = \dfrac{8.2x}{0.4x^2 + 3}$ **80.** $y = \dfrac{e^x(3 - x)}{7x^2 + 1}$

81. Graph the function

$$f(x) = [x(x - 2)(2x - 3)]^2$$

in the window $-1 \le x \le 3$, $-1 \le y \le 3$. Upon first glance, it may appear that this function has two relative minimum points and one relative maximum point. However, in reality, it has three relative minimum points and two relative maximum points. Determine the x-values of all these points. Round answers to two decimal places.

82. If $f(x) = 3x^3 - 7x^2 + 4x + 2$, display the graphs of f and f' on the same screen. Notice that $f'(x) = 0$ where relative extrema of f occur.

83. Let $f(x) = 6 + 4x - 3x^2 - x^3$. (a) Find $f'(x)$. (b) Graph $f'(x)$. (c) Observe where $f'(x)$ is positive and where it is negative. Give the intervals (rounded to two decimal places) where f is increasing and where f is decreasing. (d) Graph f and f' on the same screen, and verify your results to part (c).

84. If $f(x) = x^4 - x^2 - (x + 2)^2$, find $f'(x)$. Determine the critical values of f. Round your answers to two decimal places.

13.2 Absolute Extrema on a Closed Interval

Objective

To find extreme values on a closed interval.

If a function f is *continuous* on a *closed* interval $[a, b]$, it can be shown that of *all* the function values $f(x)$ for x in $[a, b]$, there must be an absolute maximum value and an absolute minimum value. These two values are called **extreme values** of f on that interval. This important property of continuous functions is called the *extreme-value theorem.*

Extreme-Value Theorem

If a function is continuous on a closed interval, then the function has *both* a maximum value *and* a minimum value on that interval.

For example, each function in Figure 13.25 is continuous on the closed interval [1, 3]. Geometrically, the extreme-value theorem assures us that over this interval each graph has a highest point and a lowest point.

In the extreme-value theorem, it is important that we are dealing with

1. a closed interval and
2. a function continuous on that interval

[2] J. I. Shonle, *Environmental Applications of General Physics* (Reading, MA: Addison-Wesley Publishing Company, Inc., 1975).

[3] E. Renshaw, "A Note on Equity and Efficiency in the Pricing of Local Telephone Services," *The American Economic Review*, 75, no. 3 (1985), 515–18.

[4] P. Lancaster, *Mathematics: Models of the Real World* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1976).

[5] Adapted from G. E. Folk, Jr., *Textbook of Environmental Physiology*, 2nd ed. (Philadelphia: Lea & Febiger, 1974).

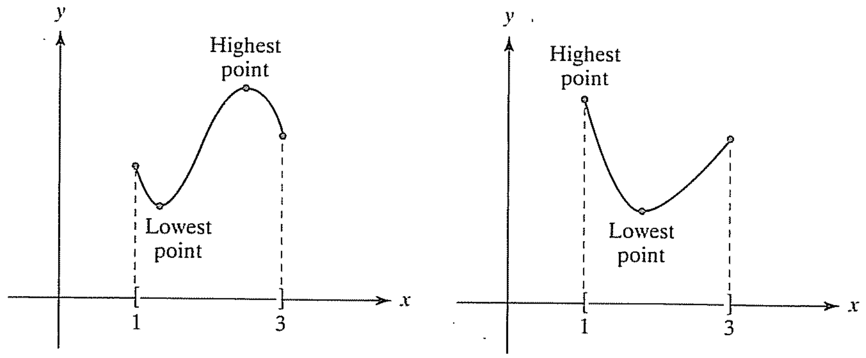


FIGURE 13.25 Illustrating the extreme-value theorem.

If either condition (1) or condition (2) is not met, then extreme values are not guaranteed. For example, Figure 13.26(a) shows the graph of the continuous function $f(x) = x^2$ on the *open* interval $(-1, 1)$. You can see that f has no maximum value on the interval (although f has a minimum value there). Now consider the function $f(x) = 1/x^2$ on the closed interval $[-1, 1]$. Here f is *not continuous* at 0. From the graph of f in Figure 13.26(b), you can see that f has no maximum value (although there is a minimum value).

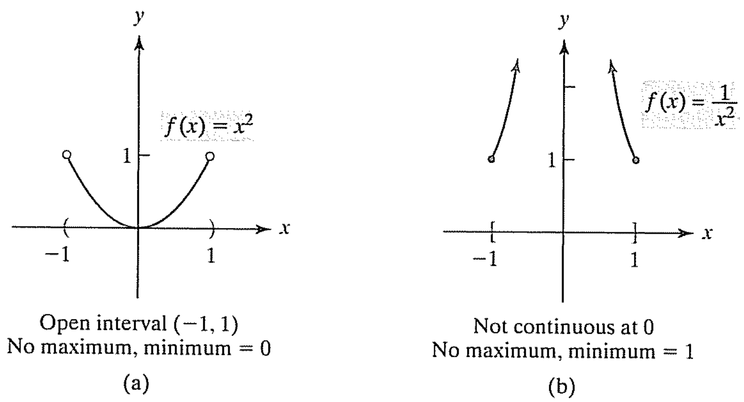


FIGURE 13.26 Extreme-value theorem does not apply.

In the previous section, our emphasis was on relative extrema. Now we will focus our attention on absolute extrema and make use of the extreme-value theorem where possible. If the domain of a function is a closed interval, to determine *absolute* extrema we must examine the function not only at critical values, but also at the endpoints. For example, Figure 13.27 shows the graph of the continuous function $y = f(x)$ over $[a, b]$. The extreme-value theorem guarantees absolute extrema over the interval. Clearly, the important points on the graph occur at $x = a, b, c$, and d, which correspond to endpoints

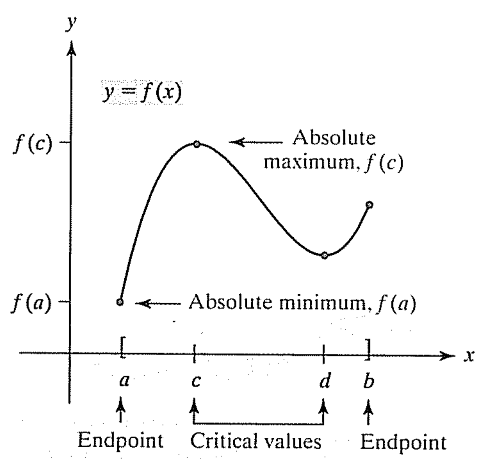


FIGURE 13.27 Absolute extrema.

or critical values. Notice that the absolute maximum occurs at the critical value c and the absolute minimum occurs at the endpoint a. These results suggest the following procedure:

Procedure to Find Absolute Extrema for a Function f That Is Continuous on $[a, b]$

Step 1. Find the critical values of f.

Step 2. Evaluate $f(x)$ at the endpoints a and b and at the critical values in (a, b).

Step 3. The maximum value of f is the greatest of the values found in step 2. The minimum value of f is the least of the values found in step 2.

EXAMPLE 1 Finding Extreme Values on a Closed Interval

Find absolute extrema for $f(x) = x^2 - 4x + 5$ over the closed interval $[1, 4]$.

Solution: Since f is continuous on $[1, 4]$, the foregoing procedure applies.

Step 1. To find the critical values of f, we first find f':

$$f'(x) = 2x - 4 = 2(x - 2)$$

This gives the critical value $x = 2$.

Step 2. Evaluating $f(x)$ at the endpoints 1 and 4 and at the critical value 2, we have

$$\left.\begin{aligned} f(1) &= 2 \\ f(4) &= 5 \end{aligned}\right\} \quad \text{values of } f \text{ at endpoints}$$

and

$$f(2) = 1 \quad \text{value of } f \text{ at critical value 2 in } (1, 4)$$

Step 3. From the function values in Step 2, we conclude that the maximum is $f(4) = 5$ and the minimum is $f(2) = 1$. (See Figure 13.28.)

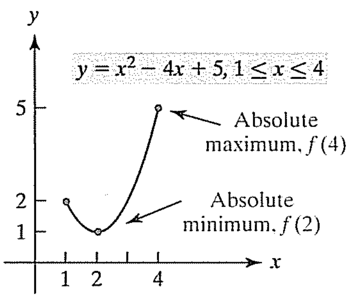


FIGURE 13.28 Extreme values for Example 1.

Now Work Problem 1 ◁

PROBLEMS 13.2

In Problems 1–14, find the absolute extrema of the given function on the given interval.

1. $f(x) = x^2 - 2x + 3$, $[0, 3]$
2. $f(x) = -2x^2 - 6x + 5$, $[-3, 2]$
3. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 1$, $[-1, 0]$
4. $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$, $[0, 1]$
5. $f(x) = x^3 - 5x^2 - 8x + 50$, $[0, 5]$
6. $f(x) = x^{2/3}$, $[-8, 8]$
7. $f(x) = -3x^5 + 5x^3$, $[-2, 0]$
8. $f(x) = \frac{7}{3}x^3 + 2x^2 - 3x + 1$, $[0, 3]$
9. $f(x) = 3x^4 - x^6$, $[-1, 2]$
10. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 2$, $[0, 4]$
11. $f(x) = x^4 - 9x^2 + 2$, $[-1, 3]$
12. $f(x) = \dfrac{x}{x^2 + 1}$, $[0, 2]$
13. $f(x) = (x - 1)^{2/3}$, $[-26, 28]$
14. $f(x) = 0.2x^3 - 3.6x^2 + 2x + 1$, $[-1, 2]$
15. Consider the function

$$f(x) = x^4 + 8x^3 + 21x^2 + 20x + 9$$

over the interval $[-4, 9]$.

(a) Determine the value(s) (rounded to two decimal places) of x at which f attains a minimum value.

(b) What is the minimum value (rounded to two decimal places) of f?

(c) Determine the value(s) of x at which f attains a maximum value.

(d) What is the maximum value of f?

Objective

To test a function for concavity and inflection points. Also, to sketch curves with the aid of the information obtained from the first and second derivatives.

13.3 Concavity

The first derivative provides much information for sketching curves. It is used to determine where a function is increasing, is decreasing, has relative maxima, and has relative minima. However, to be sure we know the true shape of a curve, we may need more

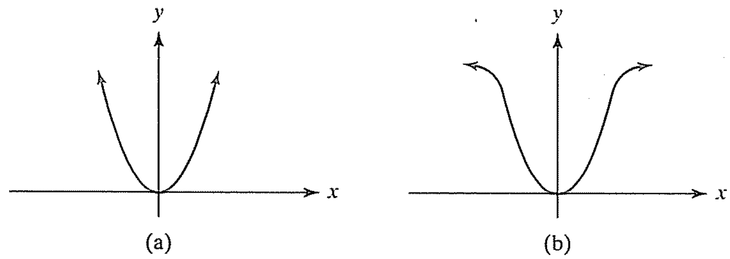


FIGURE 13.29 Two functions with $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

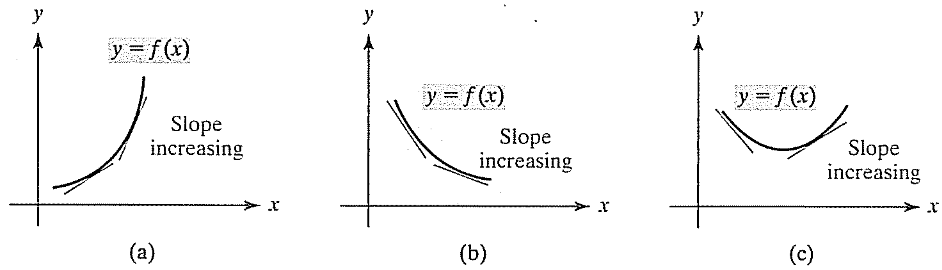


FIGURE 13.30 Each curve is concave up.

information. For example, consider the curve $y = f(x) = x^2$. Since $f'(x) = 2x, x = 0$ is a critical value. If $x < 0$, then $f'(x) < 0$, and f is decreasing; if $x > 0$, then $f'(x) > 0$, and f is increasing. Thus, there is a relative minimum when $x = 0$. In Figure 13.29, both curves meet the preceding conditions. But which one truly describes the curve $y = x^2$? This question will be settled easily by using the second derivative and the notion of *concavity*.

In Figure 13.30, note that each curve $y = f(x)$ "bends" (or opens) upward. This means that if tangent lines are drawn to each curve, the curves lie *above* them. Moreover, the slopes of the tangent lines *increase* in value as x increases: In part (a), the slopes go from small positive values to larger values; in part (b), they are negative and approaching zero (and thus increasing); in part (c), they pass from negative values to positive values. Since $f'(x)$ gives the slope at a point, an increasing slope means that f' must be an increasing function. To describe this property, each curve (or function f) in Figure 13.30 is said to be *concave up*.

In Figure 13.31, it can be seen that each curve lies *below* the tangent lines and the curves are bending downward. As x increases, the slopes of the tangent lines are *decreasing*. Thus, f' must be a decreasing function here, and we say that f is *concave down*.

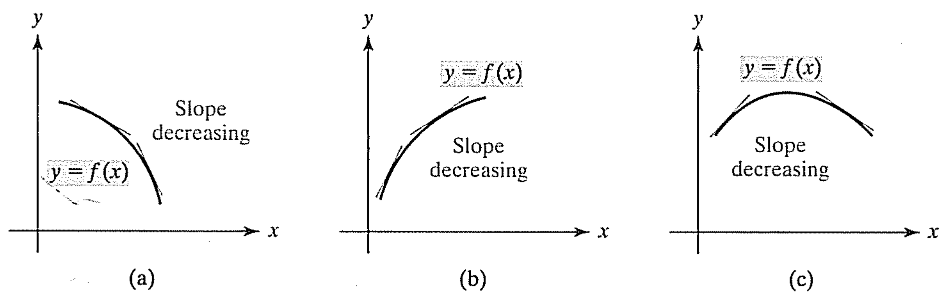


FIGURE 13.31 Each curve is concave down.

CAUTION!

Concavity relates to whether f', not f, is increasing or decreasing. In Figure 13.30(b), note that f is concave up and decreasing; however, in Figure 13.31(a), f is concave down and decreasing.

Definition

Let f be differentiable on the interval (a, b). Then f is said to be ***concave up*** [***concave down***] on (a, b) if f' is increasing [decreasing] on (a, b).

Remember: If f is concave up on an interval, then, geometrically, its graph is bending upward there. If f is concave down, then its graph is bending downward.

Since f' is increasing when its derivative $f''(x)$ is positive, and f' is decreasing when $f''(x)$ is negative, we can state the following rule:

Rule 1 Criteria for Concavity

Let f' be differentiable on the interval (a, b). If $f''(x) > 0$ for all x in (a, b), then f is concave up on (a, b). If $f''(x) < 0$ for all x in (a, b), then f is concave down on (a, b).

A function f is also said to be concave up at a point c if there exists an open interval around c on which f is concave up. In fact, for the functions that we will consider, if $f''(c) > 0$, then f is concave up at c. Similarly, f is concave down at c if $f''(c) < 0$.

EXAMPLE 1 Testing for Concavity

Determine where the given function is concave up and where it is concave down.

a. $y = f(x) = (x - 1)^3 + 1$.

Solution: To apply Rule 1, we must examine the signs of y''. Now, $y' = 3(x - 1)^2$, so

$$y'' = 6(x - 1)$$

Thus, f is concave up when $6(x - 1) > 0$; that is, when $x > 1$. And f is concave down when $6(x - 1) < 0$; that is, when $x < 1$. We now use a sign chart for f'' (together with an interpretation line for f) to organize our findings. (See Figure 13.32.)

	$-\infty$ to 1	1	1 to ∞
$x - 1$	$-$	0	$+$
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$		$\cup$

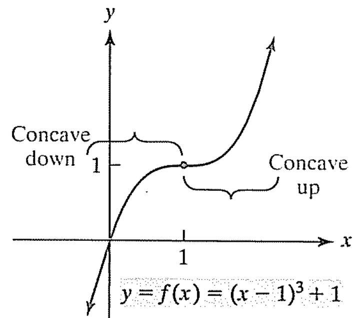


FIGURE 13.32 Sign chart for f'' and concavity for $f(x) = (x - 1)^3 + 1$.

b. $y = x^2$.

Solution: We have $y' = 2x$ and $y'' = 2$. Because y'' is always positive, the graph of $y = x^2$ must always be concave up, as in Figure 13.29(a). The graph cannot appear as in Figure 13.29(b), for that curve is sometimes concave down.

Now Work Problem 1 ◁

A point on a graph where concavity changes from concave down to concave up, or vice versa, such as (1, 1) in Figure 13.32, is called an *inflection point* or a *point of inflection*. Around such a point, the sign of $f''(x)$ goes from $-$ to $+$ or from $+$ to $-$. More precisely, we have the following definition:

The definition of an inflection point implies that a is in the domain of f.

Definition

A function f has an ***inflection point*** at a if and only if f is continuous at a and f changes concavity at a.

To test a function for concavity and inflection points, first find the values of x where $f''(x)$ is 0 or not defined. These values of x determine intervals. On each interval, determine whether $f''(x) > 0$ (f is concave up) or $f''(x) < 0$ (f is concave down). If concavity changes around one of these x-values and f is continuous there, then f has an inflection point at this x-value. The continuity requirement implies that the x-value must be in the domain of the function. In brief, a *candidate* for an inflection point must satisfy two conditions:

1. f'' must be 0 or fail to exist at that point.
2. f must be continuous at that point.

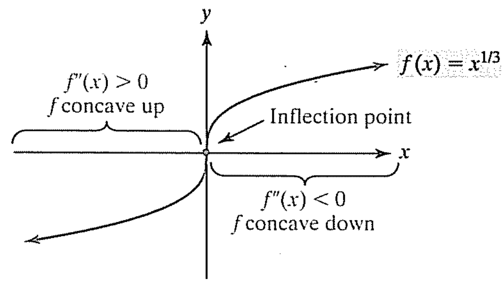


FIGURE 13.33 Inflection point for $f(x) = x^{1/3}$.

The candidate *will be* an inflection point if concavity changes around it. For example, if $f(x) = x^{1/3}$, then $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}}$$

Because f'' does not exist at 0, but f is continuous at 0, there is a candidate for an inflection point at 0. If $x > 0$, then $f''(x) < 0$, so f is concave down for $x > 0$; if $x < 0$, then $f''(x) > 0$, so f is concave up for $x < 0$. Because concavity changes at 0, there is an inflection point there. (See Figure 13.33.)

EXAMPLE 2 Concavity and Inflection Points

Test $y = 6x^4 - 8x^3 + 1$ for concavity and inflection points.

Solution: We have

$$y' = 24x^3 - 24x^2$$
$$y'' = 72x^2 - 48x = 24x(3x - 2)$$

	$-\infty$	0		2/3	∞
x	$-$	0	$+$		$+$
$3x - 2$	$-$		$-$	0	$+$
y''	$+$	0	$-$	0	$+$
y	$\cup$		$\cap$		$\cup$

FIGURE 13.34 Sign chart of $y'' = 24x(3x - 2)$ for $y = 6x^4 - 8x^3 + 1$.

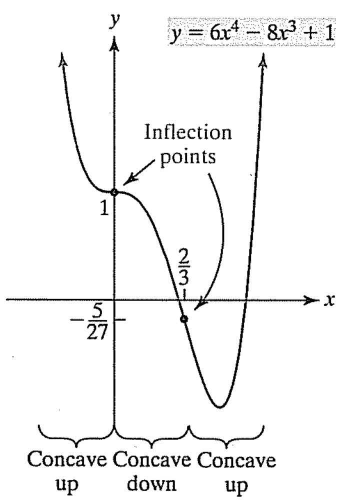


FIGURE 13.35 Graph of $y = 6x^4 - 8x^3 + 1$.

To find where $y'' = 0$, we set each factor in y'' equal to 0. This gives $x = 0, \frac{2}{3}$. We also note that y'' is never undefined. Thus, there are three intervals to consider, as recorded on the top of the sign chart in Figure 13.34. Since y is continuous at 0 and $\frac{2}{3}$, these points are candidates for inflection points. Having completed the sign chart, we see that concavity changes at 0 and at $\frac{2}{3}$. Thus these candidates are indeed inflection points. (See Figure 13.35.) In summary, the curve is concave up on $(-\infty, 0)$ and $(\frac{2}{3}, \infty)$ and is concave down on $(0, \frac{2}{3})$. Inflection points occur at 0 and at $\frac{2}{3}$. These points are $(0, y(0)) = (0, 1)$ and $(\frac{2}{3}, y(\frac{2}{3})) = (\frac{2}{3}, -\frac{5}{27})$.

Now Work Problem 13 ◁

As we did in the analysis of increasing and decreasing, so we must in concavity analysis consider also those points a that are not in the domain of f but that are near points in the domain of f. The next example will illustrate.

	$-\infty$	0	∞
$\frac{1}{x^3}$	$-$	$+$	
$f''(x)$	$-$	$+$	
$f(x)$	$\cap$	$\cup$	

FIGURE 13.36 Sign chart for $f''(x)$.

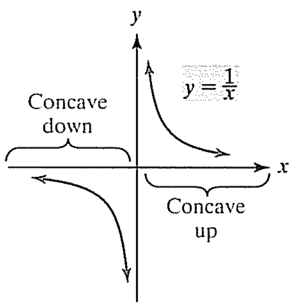


FIGURE 13.37 Graph of $y = \frac{1}{x}$.

CAUTION!

A candidate for an inflection point may not necessarily be an inflection point. For example, if $f(x) = x^4$, then $f''(x) = 12x^2$ and $f''(0) = 0$. But $f''(x) > 0$ both when $x < 0$ and when $x > 0$. Thus, concavity does not change, and there are no inflection points. (See Figure 13.38.)

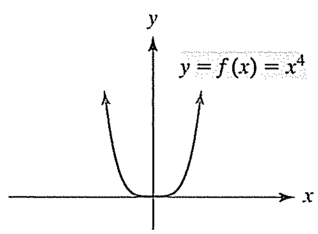


FIGURE 13.38 Graph of $f(x) = x^4$.

EXAMPLE 3 A Change in Concavity with No Inflection Point

Discuss concavity and find all inflection points for $f(x) = \frac{1}{x}$.

Solution: Since $f(x) = x^{-1}$ for $x \neq 0$,

$$f'(x) = -x^{-2} \quad \text{for } x \neq 0$$
$$f''(x) = 2x^{-3} = \frac{2}{x^3} \quad \text{for } x \neq 0$$

We see that $f''(x)$ is never 0 but it is not defined when $x = 0$. Since f is not continuous at 0, we conclude that 0 is not a candidate for an inflection point. Thus, the given function has no inflection point. However, 0 must be considered in an analysis of concavity. See the sign chart in Figure 13.36; note that we have a thick vertical line at 0 to indicate that 0 is not in the domain of f and cannot correspond to an inflection point. If $x > 0$, then $f''(x) > 0$; if $x < 0$, then $f''(x) < 0$. Hence, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. (See Figure 13.37.) Although concavity changes around $x = 0$, there is no inflection point there because f is not continuous at 0 (nor is it even defined there).

Now Work Problem 23 ◁

Curve Sketching

EXAMPLE 4 Curve Sketching

Sketch the graph of $y = 2x^3 - 9x^2 + 12x$.

Solution:

Intercepts If $x = 0$, then $y = 0$. Setting $y = 0$ gives $0 = x(2x^2 - 9x + 12)$. Clearly, $x = 0$ is a solution, and using the quadratic formula on $2x^2 - 9x + 12 = 0$ gives no real roots. Thus, the only intercept is $(0, 0)$. In fact, since $2x^2 - 9x + 12$ is a continuous function whose value at 0 is $2 \cdot 0^2 - 9 \cdot 0 + 12 = 12 > 0$, we conclude that $2x^2 - 9x + 12 > 0$ for all x, which gives the sign chart in Figure 13.39 for y.

Note that this chart tells us the graph of $y = 2x^3 - 9x^2 + 12x$ is confined to the third and first quadrants of the xy-plane.

Symmetry None.

Maxima and Minima We have

$$y' = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$$

The critical values are $x = 1, 2$, so these and the factors $x - 1$ and $x - 2$ determine the sign chart of y' (Figure 13.40).

From the sign chart for y' we see that there is a relative maximum at 1 and a relative minimum at 2. Note too that the bottom line of Figure 13.40, together with that of Figure 13.39, comes close to determining a precise graph of $y = 2x^3 - 9x^2 + 12x$.

	$-\infty$	0	∞
x	$-$	0	$+$
$2x^2 - 9x + 12$	$+$		$+$
y	$-$	0	$+$

FIGURE 13.39 Sign chart for y.

	$-\infty$	1		2	∞
$x - 1$	$-$	0	$+$		$+$
$x - 2$	$-$		$-$	0	$+$
y'	$+$	0	$-$	0	$+$
y	↗		↘		↗

FIGURE 13.40 Sign chart of $y' = 6(x - 1)(x - 2)$.

	$-\infty$	$3/2$	∞
$2x-3$	$-$	0	$+$
y''	$-$	0	$+$
y	$\cap$		$\cup$

FIGURE 13.41 Sign chart of y''.

Of course, it will help to know the relative maximum $y(1) = 5$, which occurs at 1, and the relative minimum $y(2) = 4$, which occurs at 2, so that in addition to the intercept $(0, 0)$ we will actually plot also $(1, 5)$ and $(2, 4)$.

Concavity

$$y'' = 12x - 18 = 6(2x - 3)$$

Setting $y'' = 0$ gives a possible inflection point at $x = \frac{3}{2}$ from which we construct the simple sign chart for y'' in Figure 13.41.

Since concavity changes at $x = \frac{3}{2}$, at which point f is certainly continuous, there is an inflection point at $\frac{3}{2}$.

Discussion We know the coordinates of three of the important points on the graph. The only other important point from our perspective is the inflection point, and since $y(3/2) = 2(3/2)^3 - 9(3/2)^2 + 12(3/2) = 9/2$ the inflection point is $(3/2, 9/2)$.

We plot the four points noted above and observe from all three sign charts jointly that the curve increases through the third quadrant and passes through $(0, 0)$, all the while concave down until a relative maximum is attained at $(1, 5)$. The curve then falls until it reaches a relative minimum at $(2, 4)$. However, along the way the concavity changes at $(3/2, 9/2)$ from concave down to concave up and remains so for the rest of the curve. After $(2, 4)$ the curve increases through the first quadrant. The curve is shown in Figure 13.42.

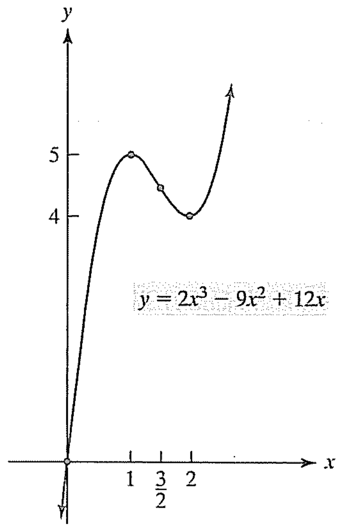


FIGURE 13.42 Graph of $y = 2x^3 - 9x^2 + 12x$.

Now Work Problem 39 ◁

TECHNOLOGY

Suppose that you need to find the inflection points for

$$f(x) = \frac{1}{20}x^5 - \frac{17}{16}x^4 + \frac{273}{32}x^3 - \frac{4225}{128}x^2 + \frac{750}{4}$$

The second derivative of f is given by

$$f''(x) = x^3 - \frac{51}{4}x^2 + \frac{819}{16}x - \frac{4225}{64}$$

Here the roots of $f'' = 0$ are not obvious. Thus, we will graph f'' using a graphing calculator. (See Figure 13.43.) We find that the roots of $f'' = 0$ are approximately 3.25 and 6.25. Around $x = 6.25$, $f''(x)$ goes from negative to positive values. Therefore, at $x = 6.25$, there is an inflection point. Around $x = 3.25$, $f''(x)$ does not change sign, so no inflection point exists at $x = 3.25$. Comparing our results with the graph of f in Figure 13.44, we see that everything checks out.

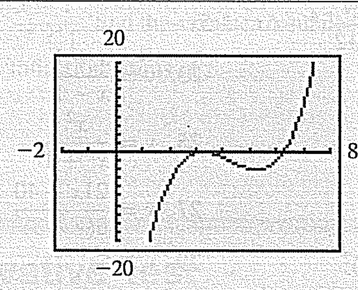


FIGURE 13.43 Graph of f''; roots of of $f'' = 0$ are approximately 3.25 and 6.25.

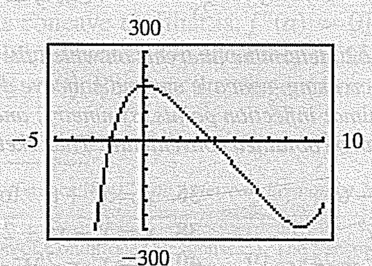


FIGURE 13.44 Graph of f; inflection point at $x = 6.25$, but not at $x = 3.25$.

PROBLEMS 13.3

In Problems 1–6, a function and its second derivative are given. Determine the concavity of f and find x-values where points of inflection occur.

1. $f(x) = x^4 - 3x^3 - 6x^2 + 6x + 1; f''(x) = 6(2x+1)(x-2)$

2. $f(x) = \frac{x^5}{20} + \frac{x^4}{4} - 2x^2; f''(x) = (x-1)(x+2)^2$

3. $f(x) = \frac{2+x-x^2}{x^2-2x+1}; f''(x) = \frac{2(7-x)}{(x-1)^4}$

4. $f(x) = \frac{x^2}{(x-1)^2}; f''(x) = \frac{2(2x+1)}{(x-1)^4}$

5. $f(x) = \frac{x^2+1}{x^2-2}; f''(x) = \frac{6(3x^2+2)}{(x^2-2)^3}$

6. $f(x) = x\sqrt{a^2-x^2}; f''(x) = \frac{x(2x^2-3a^2)}{(a^2-x^2)^{3/2}}$

In Problems 7–34, determine concavity and the x-values where points of inflection occur. Do not sketch the graphs.

7. $y = -2x^2 + 4x$

8. $y = -74x^2 + 19x - 37$

9. $y = 4x^3 + 12x^2 - 12x$

10. $y = x^3 - 6x^2 + 9x + 1$

11. $y = ax^3 + bx^2 + cx + d$

12. $y = x^4 - 8x^2 - 6$

13. $y = 2x^4 - 48x^2 + 7x + 3$

14. $y = -\frac{x^4}{4} + \frac{9x^2}{2} + 2x$

15. $y = 2x^{1/5}$

16. $y = \frac{a}{x^3}$

17. $y = \frac{x^4}{2} + \frac{19x^3}{6} - \frac{7x^2}{2} + x + 5$

18. $y = -\frac{5}{2}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x - \frac{2}{5}$

19. $y = \frac{1}{20}x^5 - \frac{1}{4}x^4 + \frac{1}{6}x^3 - \frac{1}{2}x - \frac{2}{3}$

20. $y = \frac{1}{10}x^5 - 3x^3 + 17x + 43$

21. $y = \frac{1}{30}x^6 - \frac{7}{12}x^4 + 6x^2 + 5x - 4$

22. $y = x^6 - 3x^4$

23. $y = \frac{x+1}{x-1}$

24. $y = 1 - \frac{1}{x^2}$

25. $y = \frac{x^2}{x^2+1}$

26. $y = \frac{ax^2}{x+b}$

27. $y = \frac{21x+40}{6(x+3)^2}$

28. $y = 3(x^2-2)^2$

29. $y = 5e^x$

30. $y = e^x - e^{-x}$

31. $y = axe^x$

32. $y = xe^{x^2}$

33. $y = \frac{\ln x}{2x}$

34. $y = \frac{x^2+1}{3e^x}$

In Problems 35–62, determine intervals on which the function is increasing, decreasing, concave up, and concave down; relative maxima and minima; inflection points; symmetry; and those intercepts that can be obtained conveniently. Then sketch the graph.

35. $y = x^2 - x - 6$

36. $y = x^2 + a$ for $a > 0$

37. $y = 5x - 2x^2$

38. $y = x - x^2 + 2$

39. $y = x^3 - 9x^2 + 24x - 19$

40. $y = x^3 - 25x^2$

41. $y = \frac{x^3}{3} - 5x$

42. $y = x^3 - 6x^2 + 9x$

43. $y = x^3 - 3x^2 + 3x - 3$

44. $y = 2x^3 + \frac{5}{2}x^2 + 2x$

45. $y = 4x^3 - 3x^4$

46. $y = -x^3 + 8x^2 - 5x + 3$

47. $y = -2 + 12x - x^3$

48. $y = (3+2x)^3$

49. $y = 2x^3 - 6x^2 + 6x - 2$

50. $y = \frac{x^5}{100} - \frac{x^4}{20}$

51. $y = 16x - x^5$

52. $y = x^2(x-1)^2$

53. $y = 3x^4 - 4x^3 + 1$

54. $y = 3x^5 - 5x^3$

55. $y = 4x^2 - x^4$

56. $y = x^2e^x$

57. $y = x^{1/3}(x-8)$

58. $y = (x-1)^2(x+2)^2$

59. $y = 4x^{1/3} + x^{4/3}$

60. $y = (x+1)\sqrt{x+4}$

61. $y = 2x^{2/3} - x$

62. $y = 5x^{2/3} - x^{5/3}$

63. Sketch the graph of a continuous function f such that $f(2) = 4, f'(2) = 0, f'(x) < 0$ if $x < 2$, and $f''(x) > 0$ if $x > 2$.

64. Sketch the graph of a continuous function f such that $f(4) = 4, f'(4) = 0, f''(x) < 0$ for $x < 4$, and $f''(x) > 0$ for $x > 4$.

65. Sketch the graph of a continuous function f such that $f(1) = 1, f'(1) = 0$, and $f''(x) < 0$ for all x.

66. Sketch the graph of a continuous function f such that $f(1) = 1$, both $f'(x) < 0$ and $f''(x) < 0$ for $x < 1$, and both $f(x) > 0$ and $f''(x) < 0$ for $x > 1$.

67. Demand Equation Show that the graph of the demand equation $p = \frac{100}{q+2}$ is decreasing and concave up for $q > 0$.

68. Average Cost For the cost function

$$c = q^2 + 2q + 1$$

show that the graph of the average-cost function $\bar{c}$ is always concave up for $q > 0$.

69. Species of Plants The number of species of plants on a plot may depend on the size of the plot. For example, in Figure 13.45, we see that on 1-m² plots there are three species (A, B, and C on the left plot, A, B, and D on the right plot), and on a 2-m² plot there are four species (A, B, C, and D).

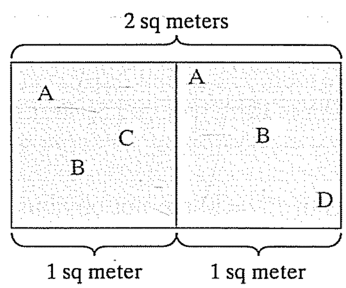


FIGURE 13.45

In a study of rooted plants in a certain geographic region,[6] it was determined that the average number of species, S, occurring on

[6]Adapted from R. W. Poole, *An Introduction to Quantitative Ecology* (New York; McGraw-Hill Book Company, 1974).

plots of size A (in square meters) is given by

$$S = f(A) = 12\sqrt[4]{A} \quad 0 \le A \le 625$$

Sketch the graph of f. (*Note:* Your graph should be rising and concave down. Thus, the number of species is increasing with respect to area, but at a decreasing rate.)

70. Inferior Good In a discussion of an inferior good, Persky[7] considers a function of the form

$$g(x) = e^{(U_0/A)}e^{-x^2/(2A)}$$

where x is a quantity of a good, U_0 is a constant that represents utility, and A is a positive constant. Persky claims that the graph of g is concave down for $x < \sqrt{A}$ and concave up for $x > \sqrt{A}$. Verify this.

71. Psychology In a psychological experiment involving conditioned response,[8] subjects listened to four tones, denoted 0, 1, 2, and 3. Initially, the subjects were conditioned to tone 0 by receiving a shock whenever this tone was heard. Later, when each of the four tones (stimuli) were heard without shocks, the subjects' responses were recorded by means of a tracking device that measures galvanic skin reaction. The average response to each stimulus (without shock) was determined, and the results were plotted on a coordinate plane where the x- and y-axes represent the stimuli (0, 1, 2, 3) and the average galvanic responses, respectively. It was determined that the points fit a curve that is approximated by the graph of

$$y = 12.5 + 5.8(0.42)^x$$

Show that this function is decreasing and concave up.

72. Entomology In a study of the effects of food deprivation on hunger,[9] an insect was fed until its appetite was completely satisfied. Then it was deprived of food for t hours (the deprivation period). At the end of this period, the insect was re-fed until its appetite was again completely satisfied. The weight H (in grams) of the food that was consumed at this time was statistically found to be a function of t, where

$$H = 1.00[1 - e^{-(0.0464t+0.0670)}]$$

Here H is a measure of hunger. Show that H is increasing with respect to t and is concave down.

73. Insect Dispersal In an experiment on the dispersal of a particular insect,[10] a large number of insects are placed at a release point in an open field. Surrounding this point are traps that are placed in a concentric circular arrangement at a distance of 1 m, 2 m, 3 m, and so on from the release point. Twenty-four hours after the insects are released, the number of insects in each trap is counted. It is determined that at a distance of r meters from the release point, the average number of insects contained in a trap is

$$n = f(r) = 0.1\ln(r) + \frac{7}{r} - 0.8 \quad 1 \le r \le 10$$

(a) Show that the graph of f is always falling and concave up. (b) Sketch the graph of f. (c) When $r = 5$, at what rate is the average number of insects in a trap decreasing with respect to distance?

74. Graph $y = -0.35x^3 + 4.1x^2 + 8.3x - 7.4$, and from the graph determine the number of (a) relative maximum points, (b) relative minimum points, and (c) inflection points.

75. Graph $y = x^5(x - 2.3)$, and from the graph determine the number of inflection points. Now, prove that for any $a \ne 0$, the curve $y = x^5(x - a)$ has two points of inflection.

76. Graph $y = xe^{-x}$ and determine the number of inflection points, first using a graphing calculator and then using the techniques of this chapter. If a demand equation has the form $q = q(p) = Qe^{-Rp}$ for constants Q and R, relate the graph of the resulting revenue function to that of the function graphed above, by taking $Q = 1 = R$.

77. Graph the curve $y = x^3 - 2x^2 + x + 3$, and also graph the tangent line to the curve at $x = 2$. Around $x = 2$, does the curve lie above or below the tangent line? From your observation determine the concavity at $x = 2$.

78. If $f(x) = 2x^3 + 3x^2 - 6x + 1$, find $f'(x)$ and $f''(x)$. Note that where f' has a relative minimum, f changes its direction of bending. Why?

79. If $f(x) = x^6 + 3x^5 - 4x^4 + 2x^2 + 1$, find the x-values (rounded to two decimal places) of the inflection points of f.

80. If $f(x) = \dfrac{x+1}{x^2+1}$, find the x-values (rounded to two decimal places) of the inflection points of f.

Objective

To locate relative extrema by applying the second-derivative test.

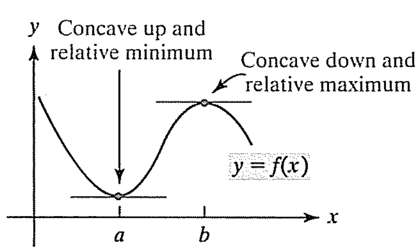


FIGURE 13.46 Relating concavity to relative extrema.

13.4 The Second-Derivative Test

The second derivative can be used to test certain critical values for relative extrema. Observe in Figure 13.46 that at a, there is a horizontal tangent; that is, $f'(a) = 0$. Furthermore, around a, the function is concave up [that is, $f''(a) > 0$]. This leads us to conclude that there is a relative minimum at a. On the other hand, around b, the function is concave down [that is, $f''(b) < 0$]. Because the tangent line is horizontal at

[7] A. L. Persky, "An Inferior Good and a Novel Indifference Map," *The American Economist* XXIX, no. 1 (1985), 67–69.

[8] Adapted from C. I. Hovland, "The Generalization of Conditioned Responses: I. The Sensory Generalization of Conditioned Responses with Varying Frequencies of Tone," *Journal of General Psychology,* 17 (1937), 125–48.

[9] C. S. Holling, "The Functional Response of Invertebrate Predators to Prey Density," *Memoirs of the Entomological Society of Canada,* no. 48 (1966).

[10] Adapted from Poole, op. cit.

b, we conclude that a relative maximum exists there. This technique of examining the second derivative at points where the first derivative is 0 is called the *second-derivative test* for relative extrema.

Second-Derivative Test for Relative Extrema

Suppose $f'(a) = 0$.

If $f''(a) < 0$, then f has a relative maximum at a.

If $f''(a) > 0$, then f has a relative minimum at a.

We want to emphasize that *the second-derivative test does* not *apply when* $f''(a) = 0$. If both $f'(a) = 0$ and $f''(a) = 0$, then there may be a relative maximum, a relative minimum, or neither, at a. In such cases, the first-derivative test should be used to analyze what is happening at a. [Also, the second-derivative test does not apply when $f'(a)$ does not exist.]

EXAMPLE 1 Second-Derivative Test

Test the following for relative maxima and minima. Use the second-derivative test, if possible.

a. $y = 18x - \frac{2}{3}x^3$.

Solution:

$$y' = 18 - 2x^2 = 2(9 - x^2) = 2(3 + x)(3 - x)$$

$$y'' = -4x \qquad \text{taking } \frac{d}{dx} \text{ of } 18 - 2x^2$$

Solving $y' = 0$ gives the critical values $x = \pm 3$.

$$\text{If } x = 3, \quad \text{then } y'' = -4(3) = -12 < 0.$$

There is a relative maximum when $x = 3$.

$$\text{If } x = -3, \quad \text{then } y'' = -4(-3) = 12 > 0.$$

There is a relative minimum when $x = -3$. (Refer to Figure 13.4.)

b. $y = 6x^4 - 8x^3 + 1$.

Solution:

$$y' = 24x^3 - 24x^2 = 24x^2(x - 1)$$

$$y'' = 72x^2 - 48x$$

Solving $y' = 0$ gives the critical values $x = 0, 1$. We see that

$$\text{if } x = 0, \quad \text{then } y'' = 0$$

and

$$\text{if } x = 1, \quad \text{then } y'' > 0$$

By the second-derivative test, there is a relative minimum when $x = 1$. We cannot apply the test when $x = 0$ because $y'' = 0$ there. To analyze what is happening at 0, we turn to the first-derivative test:

$$\text{If } x < 0, \quad \text{then } y' < 0.$$

$$\text{If } 0 < x < 1, \quad \text{then } y' < 0.$$

Thus, no maximum or minimum exists when $x = 0$. (Refer to Figure 13.35.)

Now Work Problem 5 ◁

CAUTION!

Although the second-derivative test can be very useful, do not depend entirely on it. Not only may the test fail to apply, but also it may be awkward to find the second derivative.

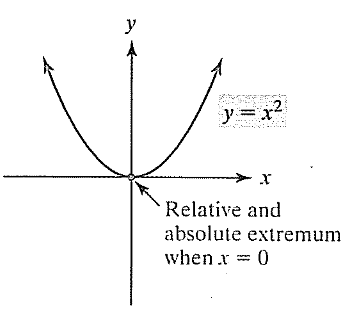


FIGURE 13.47 Exactly one relative extremum implies an absolute extremum.

If a continuous function has *exactly one* relative extremum on an interval, it can be shown that the relative extremum must also be an *absolute* extremum on the interval. To illustrate, in Figure 13.47 the function $y = x^2$ has a relative minimum when $x = 0$, and there are no other relative extrema. Since $y = x^2$ is continuous, this relative minimum is also an absolute minimum for the function.

EXAMPLE 2 Absolute Extrema

If $y = f(x) = x^3 - 3x^2 - 9x + 5$, determine when absolute extrema occur on the interval $(0, \infty)$.

Solution: We have

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3)$$
$$= 3(x + 1)(x - 3)$$

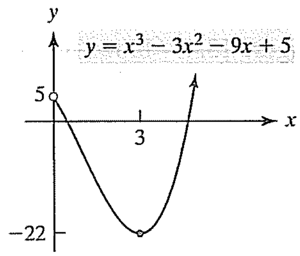


FIGURE 13.48 On $(0, \infty)$, there is an absolute minimum at 3.

The only critical value on the interval $(0, \infty)$ is 3. Applying the second-derivative test at this point gives

$$f''(x) = 6x - 6$$
$$f''(3) = 6(3) - 6 = 12 > 0$$

Thus, there is a relative minimum at 3. Since this is the only relative extremum on $(0, \infty)$ and f is continuous there, we conclude by our previous discussion that there is an *absolute* minimum value at 3; this value is $f(3) = -22$. (See Figure 13.48.)

Now Work Problem 3 ◁

PROBLEMS 13.4

In Problems 1–14, test for relative maxima and minima. Use the second-derivative test, if possible. In Problems 1–4, state whether the relative extrema are also absolute extrema.

1. $y = x^2 - 5x + 6$

2. $y = 3x^2 + 12x + 14$

3. $y = -4x^2 + 2x - 8$

4. $y = 3x^2 - 5x + 6$

5. $y = \frac{1}{3}x^3 + 2x^2 - 5x + 1$

6. $y = x^3 - 12x + 1$

7. $y = 2x^3 - 3x^2 - 36x + 17$

8. $y = x^4 - 2x^2 + 4$

9. $y = 7 - 2x^4$

10. $y = -2x^7$

11. $y = 81x^5 - 5x$

12. $y = 15x^3 + x^2 - 15x + 2$

13. $y = (x^2 + 7x + 10)^2$

14. $y = -x^3 + 3x^2 + 9x - 2$

Objective

To determine horizontal and vertical asymptotes for a curve and to sketch the graphs of functions having asymptotes.

13.5 Asymptotes

Vertical Asymptotes

In this section, we conclude our discussion of curve-sketching techniques by investigating functions having *asymptotes*. An asymptote is a line that a curve approaches arbitrarily closely. For example, in each part of Figure 13.49, the dashed line $x = a$ is an asymptote. But to be precise about it, we need to make use of infinite limits.

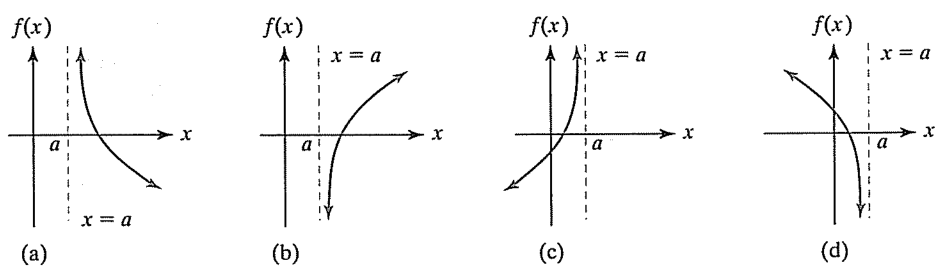


FIGURE 13.49 Vertical asymptotes $x = a$.

In Figure 13.49(a), notice that as $x \to a^+, f(x)$ becomes positively infinite:

$$\lim_{x \to a^+} f(x) = \infty$$

In Figure 13.49(b), as $x \to a^+, f(x)$ becomes negatively infinite:

$$\lim_{x \to a^+} f(x) = -\infty$$

In Figures 13.49(c) and (d), we have

$$\lim_{x \to a^-} f(x) = \infty \quad \text{and} \quad \lim_{x \to a^-} \text{f(x)} = -\infty$$

respectively.

Loosely speaking, we can say that each graph in Figure 13.49 "blows up" around the dashed vertical line $x = a$, in the sense that a one-sided limit of $f(x)$ at a is either ∞ or $-\infty$. The line $x = a$ is called a *vertical asymptote* for the graph. A vertical asymptote is not part of the graph but is a useful aid in sketching it because part of the graph approaches the asymptote. Because of the explosion around $x = a$, the function is *not* continuous at a.

Definition

The line $x = a$ is a ***vertical asymptote*** for the graph of the function f if and only if at least one of the following is true:

$$\lim_{x \to a^+} f(x) = \pm\infty$$

or

$$\lim_{x \to a^-} f(x) = \pm\infty$$

To determine vertical asymptotes, we must find values of x around which $f(x)$ increases or decreases without bound. For a rational function (a quotient of two polynomials) *expressed in lowest terms*, these x-values are precisely those for which the denominator is zero but the numerator is not zero. For example, consider the rational function

$$f(x) = \frac{3x - 5}{x - 2}$$

When x is 2, the denominator is 0, but the numerator is not. If x is slightly larger than 2, then $x - 2$ is both close to 0 and positive, and $3x - 5$ is close to 1. Thus, $(3x - 5)/(x - 2)$ is very large, so

$$\lim_{x \to 2^+} \frac{3x - 5}{x - 2} = \infty$$

This limit is sufficient to conclude that the line $x = 2$ is a vertical asymptote. Because we are ultimately interested in the behavior of a function around a vertical asymptote, it is worthwhile to examine what happens to this function as x approaches 2 from the left. If x is slightly less than 2, then $x - 2$ is very close to 0 but negative, and $3x - 5$ is close to 1. Hence, $(3x - 5)/(x - 2)$ is "very negative," so

$$\lim_{x \to 2^-} \frac{3x - 5}{x - 2} = -\infty$$

We conclude that the function increases without bound as $x \to 2^+$ and decreases without bound as $x \to 2^-$. The graph appears in Figure 13.50.

CAUTION!

To see that the proviso about *lowest terms* is necessary, observe that $f(x) = \frac{3x - 5}{x - 2} = \frac{(3x - 5)(x - 2)}{(x - 2)^2}$ so that $x = 2$ is a vertical asymptote of $\frac{(3x - 5)(x - 2)}{(x - 2)^2}$, and here 2 makes both the denominator *and* the numerator 0.

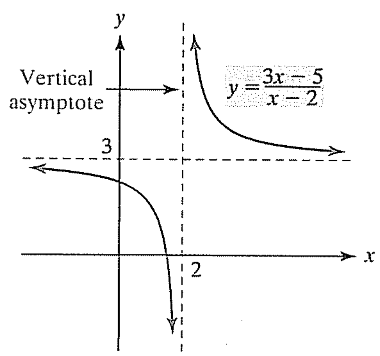


FIGURE 13.50 Graph of $y = \frac{3x - 5}{x - 2}$.

In summary, we have a rule for vertical asymptotes.

Vertical-Asymptote Rule for Rational Functions

Suppose that

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions and the quotient is in lowest terms. The line $x = a$ is a vertical asymptote for the graph of f if and only if $Q(a) = 0$ and $P(a) \neq 0$.

[It might be thought here that "lowest terms" rules out the possibility of a value a making *both* denominator *and* numerator 0, but consider the rational function $\frac{(3x-5)(x-2)}{(x-2)}$. Here we cannot divide numerator and denominator by $x-2$, to obtain the polynomial $3x-5$, because the domain of the latter is not equal to the domain of the former.]

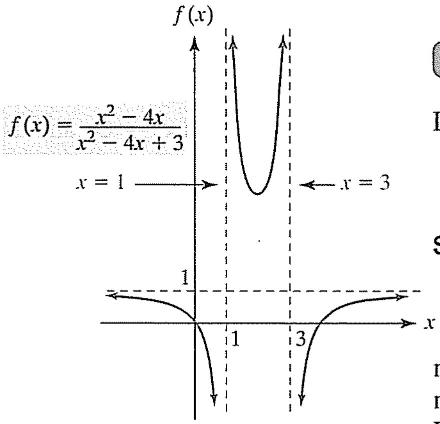


FIGURE 13.51 Graph of $f(x) = \frac{x^2-4x}{x^2-4x+3}$.

Although the vertical-asymptote rule guarantees that the lines $x = 3$ and $x = 1$ are vertical asymptotes, it does not indicate the precise nature of the "blow-up" around these lines. A precise analysis requires the use of one-sided limits.

EXAMPLE 1 **Finding Vertical Asymptotes**

Determine vertical asymptotes for the graph of

$$f(x) = \frac{x^2-4x}{x^2-4x+3}$$

Solution: Since f is a rational function, the vertical-asymptote rule applies. Writing

$$f(x) = \frac{x(x-4)}{(x-3)(x-1)} \quad \text{factoring}$$

makes it clear that the denominator is 0 when x is 3 or 1. Neither of these values makes the numerator 0. Thus, the lines $x = 3$ and $x = 1$ are vertical asymptotes. (See Figure 13.51.)

Now Work Problem 1 ◁

Horizontal and Oblique Asymptotes

A curve $y = f(x)$ may have other kinds of asymptote. In Figure 13.52(a), as x increases without bound ($x \to \infty$), the graph approaches the horizontal line $y = b$. That is,

$$\lim_{x\to\infty} f(x) = b$$

In Figure 13.52(b), as x becomes negatively infinite, the graph approaches the horizontal line $y = b$. That is,

$$\lim_{x\to-\infty} f(x) = b$$

In each case, the dashed line $y = b$ is called a *horizontal asymptote* for the graph. It is a horizontal line around which the graph "settles" either as $x \to \infty$ or as $x \to -\infty$.

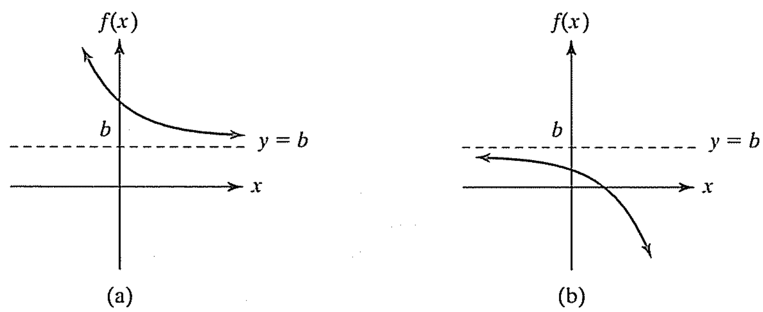


FIGURE 13.52 Horizontal asymptotes $y = b$.