10 Limits and Continuity

The philosopher Zeno of Elea was fond of paradoxes about motion. His most famous one goes something like this: The warrior Achilles agrees to run a race against a tortoise. Achilles can run 10 meters per second and the tortoise only 1 meter per second, so the tortoise gets a 10-meter head start. Since Achilles is so much faster, he should still win. But by the time he has covered his first 10 meters and reached the place where the tortoise started, the tortoise has advanced 1 meter and is still ahead. And after Achilles has covered that 1 meter, the tortoise has advanced another 0.1 meter and is still ahead. And after Achilles has covered that 0.1 meter, the tortoise has advanced another 0.01 meter and is still ahead. And so on. Therefore, Achilles gets closer and closer to the tortoise but can never catch up.

Zeno's audience knew that the argument was fishy. The position of Achilles at time t after the race has begun is $(10\,\text{m/s})t$. The position of the tortoise at the same time t is $(1\,\text{m/s})t + 10\,\text{m}$. When these are equal, Achilles and the tortoise are side by side. To solve the resulting equation

$$(10\,\text{m/s})t = (1\,\text{m/s})t + 10\,\text{m}$$

for t is to find the time at which Achilles pulls even with the tortoise.

The solution is $t = 1\frac{1}{9}$ seconds, at which time Achilles will have run $\left(1\frac{1}{9}\,\text{s}\right)(10\,\text{m/s}) = 11\frac{1}{9}$ meters.

What puzzled Zeno and his listeners is how it could be that

$$10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots = 11\frac{1}{9}$$

where the left side represents an *infinite sum* and the right side is a finite result. The solution to this problem is the concept of a limit, which is the key topic of this chapter. The left side of the equation is the sum of an infinite geometric sequence. Using limit notation, summation notation, and the formula from Section 1.6 for the sum of a finite geometric sequence, we write

$$\lim_{k\to\infty} \sum_{n=0}^{k} 10^{1-n} = \lim_{k\to\infty} \frac{10\left(1 - \left(\frac{1}{10}\right)^{k+1}\right)}{1 - \frac{1}{10}} = \frac{100}{9} = 11\frac{1}{9}$$

and find the sum of this particular infinite geometric sequence. (In Section 1.6 we showed that, for an infinite sequence with first term a and common ratio r, the sum of the infinite sequence exists and is given by $\frac{a}{1-r}$, *provided that* $|r| < 1$.)

Objective

To study limits and their basic properties.

10.1 Limits

Perhaps you have been in a parking-lot situation in which you must "inch up" to the car in front, but yet you do not want to bump or touch it. This notion of getting closer and closer to something, but yet not touching it, is very important in mathematics and is involved in the concept of *limit*, which lies at the foundation of calculus. We will let a variable "inch up" to a particular value and examine the effect this process has on the values of a function.

For example, consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

Although this function is not defined at $x = 1$, we may be curious about the behavior of the function values as x gets very close to 1. Table 10.1 gives some values of x that are slightly less than 1 and some that are slightly greater, as well as their corresponding function values. Notice that as x takes on values closer and closer to 1, regardless of whether x approaches it *from the left* ($x < 1$) or *from the right* ($x > 1$), the corresponding values of $f(x)$ get closer and closer to one and only one number, namely 3. This is also clear from the graph of f in Figure 10.1. Notice there that even though the function is not defined at $x = 1$ (as indicated by the hollow dot), the function values get closer and closer to 3 as x gets closer and closer to 1. To express this, we say that the **limit** of $f(x)$ as x approaches 1 is 3 and write

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

We can make $f(x)$ as close as we like to 3, and keep it that close, by taking x sufficiently close to, but different from, 1. The limit exists at 1, even though 1 is not in the domain of f.

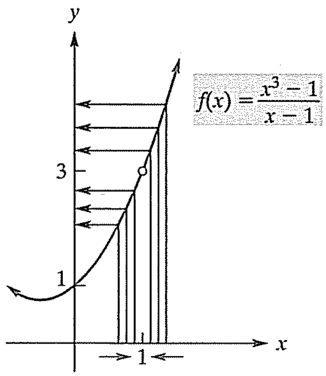


FIGURE 10.1 $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3.$

Table 10.1

$x < 1$		$x > 1$	
x	$f(x)$	x	$f(x)$
0.8	2.44	1.2	3.64
0.9	2.71	1.1	3.31
0.95	2.8525	1.05	3.1525
0.99	2.9701	1.01	3.0301
0.995	2.985025	1.005	3.015025
0.999	2.997001	1.001	3.003001

We can also consider the limit of a function as x approaches a number that is in the domain. Let us examine the limit of $f(x) = x + 3$ as x approaches 2:

$$\lim_{x \to 2} (x + 3)$$

Obviously, if x is close to 2 (but not equal to 2), then $x + 3$ is close to 5. This is also apparent from the table and graph in Figure 10.2. Thus,

$$\lim_{x \to 2} (x + 3) = 5$$

Given a function f and a number a, there *may* be two ways of associating a number to the pair (f, a). One such number is the *evaluation of f at a*, namely $f(a)$. It *exists* precisely when a is in the domain of f. For example, if $f(x) = \frac{x^3 - 1}{x - 1}$, our first example, then $f(1)$ does not *exist*. Another way of associating a number to the pair (f, a) is *the limit*

$x < 2$	
x	$f(x)$
1.5	4.5
1.9	4.9
1.95	4.95
1.99	4.99
1.999	4.999

$x > 2$	
x	$f(x)$
2.5	5.5
2.1	5.1
2.05	5.05
2.01	5.01
2.001	5.001

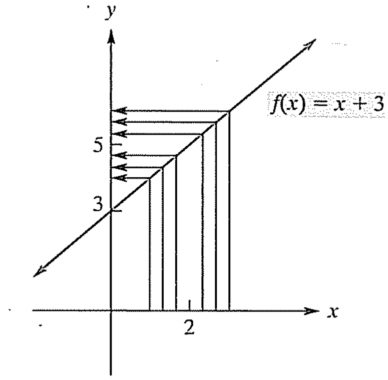


FIGURE 10.2 $\lim_{x \to 2} (x+3) = 5$.

of $f(x)$ *as* x *approaches* a, which is denoted $\lim_{x \to a} f(x)$. We have given two examples. Here is the general case.

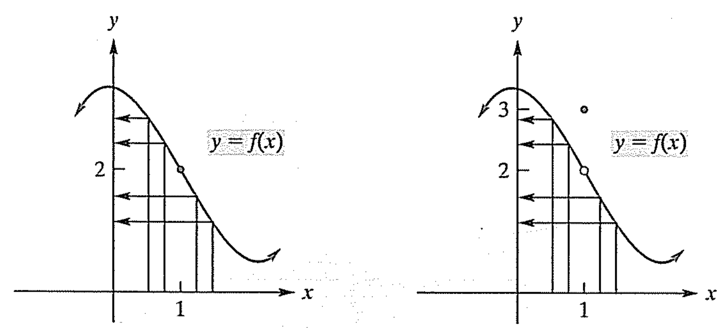
> **Definition**
>
> *The limit of* $f(x)$ *as* x *approaches* a is the number L, written
>
> $$\lim_{x \to a} f(x) = L$$
>
> provided that we can make the values $f(x)$ as close as we like to L, and keep them that close, by taking x sufficiently close to, but different from, a. If there is no such number, we say that the limit of $f(x)$ as x approaches a *does not exist.*

We emphasize that, when finding a limit, we are concerned not with what happens to $f(x)$ when x equals a, but only with what happens to $f(x)$ when x is close to a. In fact, even if $f(a)$ *exists,* the preceding definition explicitly rules out consideration of it. In our second example, $f(x) = x + 3$, we have $f(2) = 5$ and also $\lim_{x \to 2} (x + 3) = 5$, but it is quite possible to have a function f and a number a for which both $f(a)$ and $\lim_{x \to a} f(x)$ exist and are different. Moreover, a limit must be independent of the way in which x *approaches* a, meaning the way in which x gets close to a. That is, the limit must be the same whether x approaches a from the left or from the right (for $x < a$ or $x > a$, respectively).

EXAMPLE 1 Estimating a Limit from a Graph

a. Estimate $\lim_{x \to 1} f(x)$, where the graph of f is given in Figure 10.3(a).

Solution: If we look at the graph for values of x near 1, we see that $f(x)$ is near 2. Moreover, as x gets closer and closer to 1, $f(x)$ appears to get closer and closer to 2.



(a) (b)

FIGURE 10.3 Investigation of $\lim_{x \to 1} f(x)$.

Thus, we estimate that

$$\lim_{x\to 1} f(x) = 2$$

b. Estimate $\lim_{x\to 1} f(x)$, where the graph of f is given in Figure 10.3(b).

Solution: Although $f(1) = 3$, this fact has no bearing whatsoever on the limit of $f(x)$ as x approaches 1. We see that as x gets closer and closer to 1, $f(x)$ appears to get closer and closer to 2. Thus, we estimate that

$$\lim_{x\to 1} f(x) = 2$$

Now Work Problem 1 ◁

Up to now, all of the limits that we have considered did indeed exist. Next we look at some situations in which a limit does not exist.

APPLY IT ▷

1. The greatest integer function, denoted, $f(x) = \lfloor x \rfloor$, is used every day by cashiers making change for customers. This function tells the amount of paper money for each amount of change owed. (For example, if a customer is owed \$1.25 in change, he or she would get \$1 in paper money; thus, $\lfloor 1.25 \rfloor = 1$.) Formally, $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x. Graph f, sometimes called a step function, on a graphing calculator in the standard viewing rectangle. (It is in the numbers menu; it's called "integer part.") Explore this graph using TRACE. Determine whether $\lim_{x\to a} f(x)$ exists.

EXAMPLE 2 Limits That Do Not Exist

a. Estimate $\lim_{x\to -2} f(x)$ if it exists, where the graph of f is given in Figure 10.4.

Solution: As x approaches -2 from the left ($x < -2$), the values of $f(x)$ appear to get closer to 1. But as x approaches -2 from the right ($x > -2$), $f(x)$ appears to get closer to 3. Hence, as x approaches -2, the function values do not settle down to one and only one number. We conclude that

$$\lim_{x\to -2} f(x) \text{ does not exist}$$

Note that the limit does not exist even though the function is defined at $x = -2$.

b. Estimate $\lim_{x\to 0} \frac{1}{x^2}$ if it exists.

Solution: Let $f(x) = 1/x^2$. The table in Figure 10.5 gives values of $f(x)$ for some values of x near 0. As x gets closer and closer to 0, the values of $f(x)$ get larger and larger without bound. This is also clear from the graph. Since the values of $f(x)$ do not approach a *number* as x approaches 0,

$$\lim_{x\to 0} \frac{1}{x^2} \text{ does not exist}$$

Now Work Problem 3 ◁

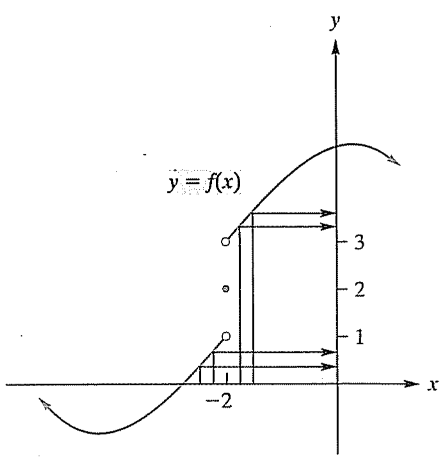


FIGURE 10.4 $\lim_{x\to -2} f(x)$ does not exist.

x	$f(x)$
± 1	1
± 0.5	4
± 0.1	100
± 0.01	10,000
± 0.001	1,000,000

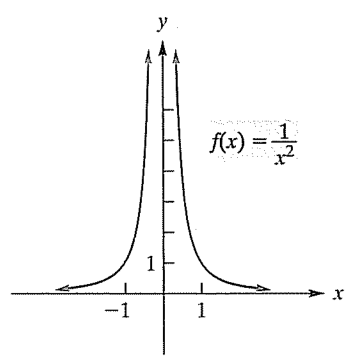


FIGURE 10.5 $\lim_{x\to 0} \frac{1}{x^2}$ does not exist.

TECHNOLOGY

Problem: Estimate $\lim_{x\to 2} f(x)$ if

$$f(x) = \frac{x^3 + 2.1x^2 - 10.2x + 4}{x^2 + 2.5x - 9}$$

Solution: One method of finding the limit is by constructing a table of function values $f(x)$ when x is close to 2. From Figure 10.6, we estimate the limit to be 1.57. Alternatively, we can estimate the limit from the graph of f. Figure 10.7 shows the graph of f in the standard window of $[-10, 10] \times [-10, 10]$. First we zoom in several times around $x = 2$ and obtain Figure 10.8. After tracing around $x = 2$, we estimate the limit to be 1.57.

X	Y1	
1.9	1.4688	
1.99	1.5592	
1.999	1.5682	
1.9999	1.5691	
2.01	1.5793	
2.001	1.5702	
2.0001	1.5693	

X=2.0001

FIGURE 10.6 $\lim_{x\to 2} f(x) \approx 1.57$.

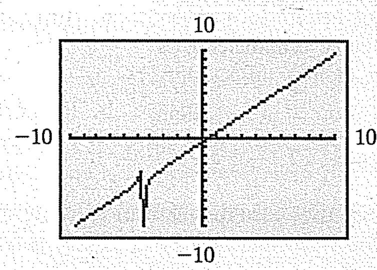


FIGURE 10.7 Graph of $f(x)$ in standard window.

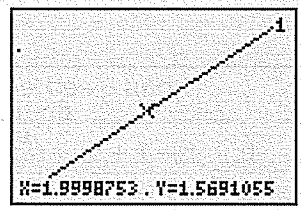


FIGURE 10.8 Zooming and tracing around $x = 2$ gives $\lim_{x\to 2} f(x) \approx 1.57$.

Properties of Limits

To determine limits, we do not always want to compute function values or sketch a graph. Alternatively, there are several properties of limits that we may be able to employ. The following properties may seem reasonable to you:

1. If $f(x) = c$ is a constant function, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} c = c$$

2. $\lim_{x\to a} x^n = a^n$, for any positive integer n

EXAMPLE 3 Applying Limit Properties 1 and 2

a. $\lim_{x\to 2} 7 = 7$; $\lim_{x\to -5} 7 = 7$

b. $\lim_{x\to 6} x^2 = 6^2 = 36$

c. $\lim_{t\to -2} t^4 = (-2)^4 = 16$

Now Work Problem 9 ◁

Some other properties of limits are as follows:

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

3. $$\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

That is, the limit of a sum or difference is the sum or difference, respectively, of the limits.

4. $$\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

That is, the limit of a product is the product of the limits.

5. $$\lim_{x\to a} [cf(x)] = c \cdot \lim_{x\to a} f(x), \text{ where } c \text{ is a constant}$$

That is, the limit of a constant times a function is the constant times the limit of the function.

APPLY IT ▷

2. The volume of helium in a spherical balloon (in cubic centimeters), as a function of the radius r in centimeters, is given by $V(r) = \frac{4}{3}\pi r^3$. Find $\lim_{r\to 1} V(r)$.

EXAMPLE 4 Applying Limit Properties

a. $\lim_{x\to 2}(x^2 + x) = \lim_{x\to 2} x^2 + \lim_{x\to 2} x$ Property 3

$= 2^2 + 2 = 6$ Property 2

b. Property 3 can be extended to the limit of a finite number of sums and differences. For example,

$$\lim_{q\to -1}(q^3 - q + 1) = \lim_{q\to -1} q^3 - \lim_{q\to -1} q + \lim_{q\to -1} 1$$
$$= (-1)^3 - (-1) + 1 = 1$$

c. $\lim_{x\to 2}[(x+1)(x-3)] = \lim_{x\to 2}(x+1) \cdot \lim_{x\to 2}(x-3)$ Property 4

$$= \left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right) \cdot \left(\lim_{x\to 2} x - \lim_{x\to 2} 3\right)$$
$$= (2+1)\cdot(2-3) = 3(-1) = -3$$

d. $\lim_{x\to -2} 3x^3 = 3 \cdot \lim_{x\to -2} x^3$ Property 5

$= 3(-2)^3 = -24$

Now Work Problem 11 ◁

APPLY IT ▷

3. The revenue function for a certain product is given by $R(x) = 500x - 6x^2$. Find $\lim_{x\to 8} R(x)$.

EXAMPLE 5 Limit of a Polynomial Function

Let $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ define a polynomial function. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)$$
$$= c_n \cdot \lim_{x\to a} x^n + c_{n-1} \cdot \lim_{x\to a} x^{n-1} + \cdots + c_1 \cdot \lim_{x\to a} x + \lim_{x\to a} c_0$$
$$= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = f(a)$$

Thus, we have the following property:

If f is a polynomial function, then

$$\lim_{x\to a} f(x) = f(a)$$

In other words, if f is a polynomial and a is any number, then both ways of associating a number to the pair (f, a), namely formation of the limit and evaluation, exist and are equal.

Now Work Problem 13 ◁

The result of Example 5 allows us to find many limits simply by evaluation. For example, we can find

$$\lim_{x\to -3}(x^3 + 4x^2 - 7)$$

by substituting -3 for x because $x^3 + 4x^2 - 7$ is a polynomial function:

$$\lim_{x\to -3}(x^3 + 4x^2 - 7) = (-3)^3 + 4(-3)^2 - 7 = 2$$

Similarly,

$$\lim_{h\to 3}(2(h-1)) = 2(3-1) = 4$$

We want to stress that we do not find limits simply by evaluating unless there is a rule that covers the situation. We were able to find the previous two limits by evaluation because we have a rule that applies to limits of polynomial functions. However, indiscriminate use of evaluation can lead to errors. To illustrate, in Example 1(b) we have $f(1) = 3$, which is not $\lim_{x\to 1} f(x)$; in Example 2(a), $f(-2) = 2$, which is not $\lim_{x\to -2} f(x)$.

The next two limit properties concern quotients and roots.

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

6. $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0$$

That is, the limit of a quotient is the quotient of limits, provided that the denominator does not have a limit of 0.

7. $$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \qquad \text{See footnote 1}$$

CAUTION!

Note that in Example 6(a) the numerator and denominator of the function are polynomials. In general, we can determine a limit of a rational function by evaluation, provided that the denominator is not 0 at a.

EXAMPLE 6 Applying Limit Properties 6 and 7

a. $\lim_{x\to 1} \dfrac{2x^2 + x - 3}{x^3 + 4} = \dfrac{\lim_{x\to 1}(2x^2 + x - 3)}{\lim_{x\to 1}(x^3 + 4)} = \dfrac{2 + 1 - 3}{1 + 4} = \dfrac{0}{5} = 0$

b. $\lim_{t\to 4} \sqrt{t^2 + 1} = \sqrt{\lim_{t\to 4}(t^2 + 1)} = \sqrt{17}$

c. $\lim_{x\to 3} \sqrt[3]{x^2 + 7} = \sqrt[3]{\lim_{x\to 3}(x^2 + 7)} = \sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = 2\sqrt[3]{2}$

Now Work Problem 15 ◁

Limits and Algebraic Manipulation

We now consider limits to which our limit properties do not apply and which cannot be determined by evaluation. A fundamental result is the following:

CAUTION!

The condition for equality of the limits does not preclude the possibility that $f(a) = g(a)$. The condition only concerns $x \neq a$.

If f and g are two functions for which $f(x) = g(x)$, for all $x \neq a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

(meaning that if either limit exists, then the other exists and they are equal).

The result follows directly from the definition of *limit* since the value of $\lim_{x\to a} f(x)$ depends only on those values $f(x)$ for x that are close to a. We repeat: The evaluation of f at a, $f(a)$, or lack of its existence, is irrelevant in the determination of $\lim_{x\to a} f(x)$ unless we have a specific rule that applies, such as in the case when f is a polynomial.

APPLY IT ▷

4. The rate of change of productivity p (in number of units produced per hour) increases with time on the job by the function

$$p(t) = \frac{50(t^2 + 4t)}{t^2 + 3t + 20}$$

Find $\lim_{t\to 2} p(t)$.

EXAMPLE 7 Finding a Limit

Find $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1}$.

Solution: As $x \to -1$, both numerator and denominator approach zero. Because the limit of the denominator is 0, we *cannot* use Property 6. However, since what happens to the quotient when x equals -1 is of no concern, we can assume that $x \neq -1$ and simplify the fraction:

$$\frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1} = x - 1 \quad \text{for } x \neq -1$$

---

[1] If n is even, we require that $\lim_{x\to a} f(x)$ be nonnegative.

This algebraic manipulation (factoring and cancellation) of the original function $\frac{x^2-1}{x+1}$ yields a new function $x-1$, which is the same as the original function for $x \neq -1$. Thus the fundamental result displayed in the box at the beginning of this subsection applies and we have

$$\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} (x-1) = -1-1 = -2$$

Notice that, although the original function is not defined at -1, it *does* have a limit as $x \to -1$.

Now Work Problem 21 ◁

When both $f(x)$ and $g(x)$ approach 0 as $x \to a$, then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is said to have the *form* $0/0$. Similarly, we speak of *form* $k/0$, for $k \neq 0$ if $f(x)$ approaches $k \neq 0$ as $x \to a$ but $g(x)$ approaches 0 as $x \to a$.

In Example 7, the method of finding a limit by evaluation does not work. Replacing x by -1 gives $0/0$, which has no meaning. When the meaningless form $0/0$ arises, algebraic manipulation (as in Example 7) may result in a function that agrees with the original function, except possibly at the limiting value. In Example 7 the new function, $x-1$, is a polynomial and its limit *can* be found by evaluation.

In the beginning of this section, we found

$$\lim_{x\to 1} \frac{x^3-1}{x-1}$$

by examining a table of function values of $f(x) = (x^3-1)/(x-1)$ and also by considering the graph of f. This limit has the form $0/0$. Now we will determine the limit by using the technique used in Example 7.

EXAMPLE 8 **Form 0/0**

Find $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

Solution: As $x \to 1$, both the numerator and denominator approach 0. Thus, we will try to express the quotient in a different form for $x \neq 1$. By factoring, we have

$$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{(x-1)} = x^2+x+1 \qquad \text{for } x \neq 1$$

(Alternatively, long division would give the same result.) Therefore,

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} (x^2+x+1) = 1^2+1+1 = 3$$

as we showed before.

Now Work Problem 23 ◁

CAUTION!

There is frequently confusion about which principle is being used in this example and in Example 7. It is this:

If $f(x) = g(x)$ for $x \neq a$,

then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

APPLY IT ▷

5. The length of a material increases as it is heated up according to the equation $l = 125 + 2x$. The rate at which the length is increasing is given by

$$\lim_{h\to 0} \frac{125+2(x+h)-(125+2x)}{h}$$

Calculate this limit.

EXAMPLE 9 **Form 0/0**

If $f(x) = x^2+1$, find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Solution:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2+1]-(x^2+1)}{h}$$

Here we treat x as a constant because h, not x, is changing. As $h \to 0$, both the numerator and denominator approach 0. Therefore, we will try to express the quotient in a different form, for $h \neq 0$. We have

$$\lim_{h\to 0} \frac{[(x+h)^2+1]-(x^2+1)}{h} = \lim_{h\to 0} \frac{[x^2+2xh+h^2+1]-x^2-1}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h}$$

The expression

$$\frac{f(x+h)-f(x)}{h}$$

is called a *difference quotient*. The limit of the difference quotient lies at the heart of differential calculus. We will encounter such limits in Chapter 11.

$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$
$$= \lim_{h\to 0} (2x+h)$$
$$= 2x$$

Note: It is the fourth equality above, $\lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h)$, that uses the fundamental result. When $\frac{h(2x+h)}{h}$ and $2x+h$ are considered as *functions of h*, they are seen to be equal, for all $h \neq 0$. It follows that their limits as h approaches 0 are equal.

Now Work Problem 35 ◁

A Special Limit

We conclude this section with a note concerning a most important limit, namely,

$$\lim_{x\to 0} (1+x)^{1/x}$$

Figure 10.9 shows the graph of $f(x) = (1+x)^{1/x}$. Although $f(0)$ does not exist, as $x \to 0$ it is clear that the limit of $(1+x)^{1/x}$ exists. It is approximately 2.71828 and is denoted by the letter e. This, you may recall, is the base of the system of natural logarithms. The limit

This limit will be used in Chapter 12.

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

can actually be considered the definition of e. It can be shown that this agrees with the definition of e that we gave in Section 4.1.

x	$(1+x)^{1/x}$
0.5	2.2500
0.1	2.5937
0.01	2.7048
0.001	2.7169

x	$(1+x)^{1/x}$
−0.5	4.0000
−0.1	2.8680
−0.01	2.7320
−0.001	2.7196

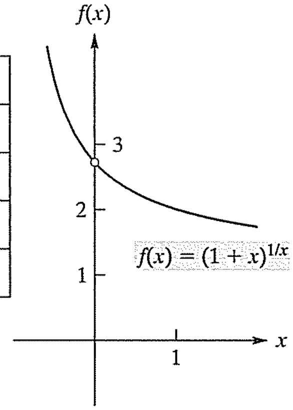


FIGURE 10.9 $\lim_{x\to 0}(1+x)^{1/x} = e$.

PROBLEMS 10.1

In Problems 1–4, use the graph of f to estimate each limit if it exists.

1. Graph of f appears in Figure 10.10.
(a) $\lim_{x\to 0} f(x)$ **(b)** $\lim_{x\to 1} f(x)$ **(c)** $\lim_{x\to 2} f(x)$

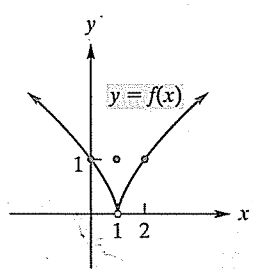


FIGURE 10.10

2. Graph of f appears in Figure 10.11.
(a) $\lim_{x\to -1} f(x)$ **(b)** $\lim_{x\to 0} f(x)$ **(c)** $\lim_{x\to 1} f(x)$

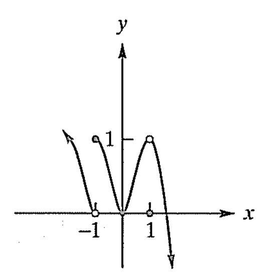


FIGURE 10.11

3. Graph of f appears in Figure 10.12.

(a) $\lim_{x\to -1} f(x)$ **(b)** $\lim_{x\to 1} f(x)$ **(c)** $\lim_{x\to 2} f(x)$

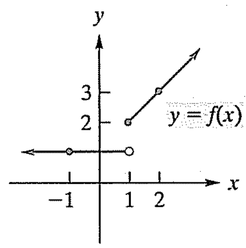


FIGURE 10.12

4. Graph of f appears in Figure 10.13.

(a) $\lim_{x\to -1} f(x)$ **(b)** $\lim_{x\to 0} f(x)$ **(c)** $\lim_{x\to 1} f(x)$

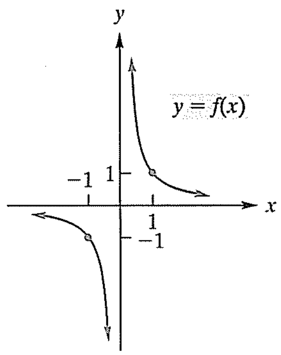


FIGURE 10.13

In Problems 5–8, use your calculator to complete the table, and use your results to estimate the given limit.

5. $\lim_{x\to -1} \dfrac{3x^2+2x-1}{x+1}$

x	-0.9	-0.99	-0.999	-1.001	-1.01	-1.1
$f(x)$						

6. $\lim_{x\to -3} \dfrac{x^2-9}{x+3}$

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$						

7. $\lim_{x\to 0} |x|^{|x|}$

x	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$f(x)$						

8. $\lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$

h	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

In Problems 9–34, find the limits.

9. $\lim_{x\to 2} 16$

10. $\lim_{x\to 3} 2x$

11. $\lim_{t\to -5} (t^2-5)$

12. $\lim_{t\to 1/2} (3t-5)$

13. $\lim_{x\to -2} (3x^3-4x^2+2x-3)$

14. $\lim_{r\to 9} \dfrac{4r-3}{11}$

15. $\lim_{t\to -3} \dfrac{t-2}{t+5}$

16. $\lim_{x\to -6} \dfrac{x^2+6}{x-6}$

17. $\lim_{t\to 0} \dfrac{t}{t^3-4t+3}$

18. $\lim_{z\to 0} \dfrac{z^2-5z-4}{z^2+1}$

19. $\lim_{p\to 4} \sqrt{p^2+p+5}$

20. $\lim_{y\to 15} \sqrt{y+3}$

21. $\lim_{x\to -2} \dfrac{x^2+2x}{x+2}$

22. $\lim_{x\to -1} \dfrac{x^2-1}{x^2-1}$

23. $\lim_{x\to 2} \dfrac{x^2-x-2}{x-2}$

24. $\lim_{t\to 0} \dfrac{t^3+3t^2}{t^3-4t^2}$

25. $\lim_{x\to 3} \dfrac{x^2-x-6}{x-3}$

26. $\lim_{t\to 2} \dfrac{t^2-4}{t-2}$

27. $\lim_{x\to -4} \dfrac{x+4}{x^2-16}$

28. $\lim_{x\to 0} \dfrac{x^2-2x}{x}$

29. $\lim_{x\to 4} \dfrac{x^2-9x+20}{x^2-3x-4}$

30. $\lim_{x\to -3} \dfrac{x^4-81}{x^2+8x+15}$

31. $\lim_{x\to 2} \dfrac{3x^2-x-10}{x^2+5x-14}$

32. $\lim_{x\to 3} \dfrac{x^2-2x-3}{x^2+2x-15}$

33. $\lim_{h\to 0} \dfrac{(2+h)^2-2^2}{h}$

34. $\lim_{x\to 0} \dfrac{(x+2)^2-4}{x}$

35. Find $\lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$ by treating x as a constant.

36. Find $\lim_{h\to 0} \dfrac{3(x+h)^2+7(x+h)-3x^2-7x}{h}$ by treating x as a constant.

In Problems 37–42, find $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

37. $f(x)=5+2x$

38. $f(x)=2x+3$

39. $f(x)=x^2-3$

40. $f(x)=x^2+x+1$

41. $f(x)=x^3-4x^2$

42. $f(x)=2-5x+x^2$

43. Find $\lim_{x\to 6} \dfrac{\sqrt{x-2}-2}{x-6}$ (*Hint:* First rationalize the numerator by multiplying both the numerator and denominator by $\sqrt{x-2}+2$.)

44. Find the constant c so that $\lim_{x\to 3} \dfrac{x^2+x+c}{x^2-5x+6}$ exists. For that value of c, determine the limit. (*Hint:* Find the value of c for which $x-3$ is a factor of the numerator.)

45. Power Plant The maximum theoretical efficiency of a power plant is given by

$$E=\frac{T_h-T_c}{T_h}$$

where T_h and T_c are the absolute temperatures of the hotter and colder reservoirs, respectively. Find (a) $\lim_{T_c\to 0} E$ and (b) $\lim_{T_c\to T_h} E$.

46. Satellite When a 3200-lb satellite revolves about the earth in a circular orbit of radius r ft, the total mechanical energy E of the earth–satellite system is given by

$$E=-\frac{7.0\times 10^{17}}{r}\text{ ft-lb}$$

Find the limit of E as $r\to 7.5\times 10^7$ ft.

In Problems 47–50, use a graphing calculator to graph the functions, and then estimate the limits. Round your answers to two decimal places.

47. $\lim_{x\to 3} \dfrac{x^4-2x^3+2x^2-2x-3}{x^2-9}$

48. $\lim_{x\to 0} x^x$

49. $\lim_{x\to 9} \dfrac{x-10\sqrt{x}+21}{3-\sqrt{x}}$

50. $\lim_{x\to 1} \dfrac{x^3+x^2-5x+3}{x^3+2x^2-7x+4}$

51. Water Purification The cost of purifying water is given by $C = \frac{50{,}000}{p} - 6500$, where p is the percent of impurities remaining after purification. Graph this function on your graphing calculator, and determine $\lim_{p \to 0} C$. Discuss what this means.

52. Profit Function The profit function for a certain business is given by $P(x) = 225x - 3.2x^2 - 700$. Graph this function on your graphing calculator, and use the evaluation function to determine $\lim_{x \to 40.2} P(x)$, using the rule about the limit of a polynomial function.

Objective

To study one-sided limits, infinite limits, and limits at infinity.

10.2 Limits (Continued)

One-Sided Limits

Figure 10.14 shows the graph of a function f. Notice that $f(x)$ is not defined when $x = 0$. As x approaches 0 *from the right*, $f(x)$ approaches 1. We write this as

$$\lim_{x \to 0^+} f(x) = 1$$

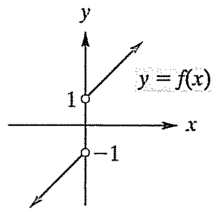


FIGURE 10.14 $\lim_{x \to 0} f(x)$ does not exist.

On the other hand, as x approaches 0 *from the left*, $f(x)$ approaches -1, and we write

$$\lim_{x \to 0^-} f(x) = -1$$

Limits like these are called **one-sided limits.** From the preceding section, we know that the limit of a function as $x \to a$ is independent of the way x approaches a. Thus, the limit will exist if and only if both one-sided limits exist and are equal. We therefore conclude that

$$\lim_{x \to 0} f(x) \text{ does not exist}$$

As another example of a one-sided limit, consider $f(x) = \sqrt{x-3}$ as x approaches 3. Since f is defined only when $x \geq 3$, we can speak of the limit of $f(x)$ as x approaches 3 from the right. If x is slightly greater than 3, then $x - 3$ is a positive number that is close to 0, so $\sqrt{x-3}$ is close to 0. We conclude that

$$\lim_{x \to 3^+} \sqrt{x-3} = 0$$

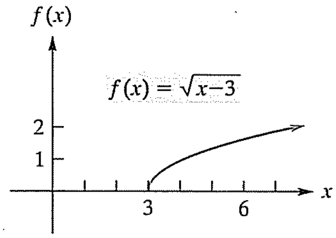


FIGURE 10.15 $\lim_{x \to 3^+} \sqrt{x-3} = 0$.

This limit is also evident from Figure 10.15.

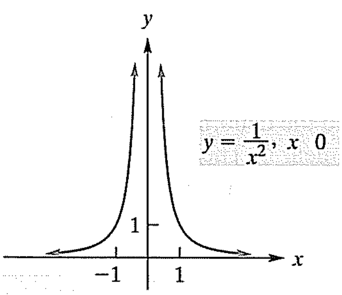
Infinite Limits

In the previous section, we considered limits of the form 0/0—that is, limits where both the numerator and denominator approach 0. Now we will examine limits where the denominator approaches 0, but the numerator approaches a number different from 0. For example, consider

$$\lim_{x \to 0} \frac{1}{x^2}$$

Here, as x approaches 0, the denominator approaches 0 and the numerator approaches 1. Let us investigate the behavior of $f(x) = 1/x^2$ when x is close to 0. The number x^2 is positive and also close to 0. Thus, dividing 1 by such a number results in a very large number. In fact, the closer x is to 0, the larger the value of $f(x)$. For example, see the table of values in Figure 10.16, which also shows the graph of f. Clearly, as $x \to 0$ both from the left and from the right, $f(x)$ increases without bound. Hence, no limit exists at 0. We say that as $x \to 0, f(x)$ becomes positively infinite, and symbolically we express this "infinite limit" by writing

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty = \infty$$



x	$f(x)$
±1	1
±0.5	4
±0.1	100
±0.01	10,000
±0.001	1,000,000

FIGURE 10.16 $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

If $\lim_{x \to a} f(x)$ does not exist, it may be for a reason other than that the values $f(x)$ become arbitrarily large as x gets close to a. For example, look again at the situation in Example 2(a) of Section 10.1. Here we have

$$\lim_{x \to -2} f(x) \text{ does not exist but } \lim_{x \to -2} f(x) \neq \infty$$

CAUTION!

The use of the "equality" sign in this situation does not mean that the limit exists. On the contrary, it is a way of saying specifically that there is no limit and *why* there is no limit.

Consider now the graph of $y = f(x) = 1/x$ for $x \neq 0$. (See Figure 10.17.) As x approaches 0 from the right, $1/x$ becomes positively infinite; as x approaches 0 from the left, $1/x$ becomes negatively infinite. Symbolically, these infinite limits are written

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

x	$f(x)$
0.01	100
0.001	1000
0.0001	10,000
−0.01	−100
−0.001	−1000
−0.0001	−10,000

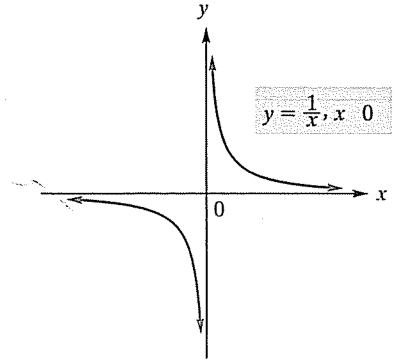


FIGURE 10.17 $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Either one of these facts implies that

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist}$$

EXAMPLE 1 Infinite Limits

Find the limit (if it exists).

a. $\lim_{x\to -1^+} \frac{2}{x+1}$

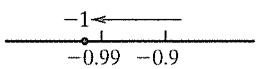


FIGURE 10.18 $x \to -1^+$.

Solution: As x approaches -1 from the right (think of values of x such as -0.9, -0.99, and so on, as shown in Figure 10.18), $x + 1$ approaches 0 but is always positive. Since we are dividing 2 by positive numbers approaching 0, the results, $2/(x+1)$, are positive numbers that are becoming arbitrarily large. Thus,

$$\lim_{x\to -1^+} \frac{2}{x+1} = \infty$$

and the limit does not exist. By a similar analysis, we can show that

$$\lim_{x\to -1^-} \frac{2}{x+1} = -\infty$$

b. $\lim_{x\to 2} \frac{x+2}{x^2-4}$

Solution: As $x \to 2$, the numerator approaches 4 and the denominator approaches 0. Hence, we are dividing numbers near 4 by numbers near 0. The results are numbers that become arbitrarily large in magnitude. At this stage, we can write

$$\lim_{x\to 2} \frac{x+2}{x^2-4} \text{ does not exist}$$

However, let us see if we can use the symbol ∞ or $-\infty$ to be more specific about "does not exist." Notice that

$$\lim_{x\to 2} \frac{x+2}{x^2-4} = \lim_{x\to 2} \frac{x+2}{(x+2)(x-2)} = \lim_{x\to 2} \frac{1}{x-2}$$

Since

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty \quad \text{and} \quad \lim_{x\to 2^-} \frac{1}{x-2} = -\infty$$

$\lim_{x\to 2} \frac{x+2}{x^2-4}$ is neither ∞ nor $-\infty$.

Now Work Problem 31 ◁

Example 1 considered limits of the form $k/0$, where $k \neq 0$. It is important to distinguish the form $k/0$ from the form $0/0$, which was discussed in Section 10.1. These two forms are handled quite differently.

EXAMPLE 2 Finding a Limit

Find $\lim_{t\to 2} \frac{t-2}{t^2-4}$.

Solution: As $t \to 2$, *both* numerator and denominator approach 0 (form $0/0$). Thus, we first simplify the fraction, for $t \neq 2$, as we did in Section 10.1, and then take the limit:

$$\lim_{t\to 2} \frac{t-2}{t^2-4} = \lim_{t\to 2} \frac{t-2}{(t+2)(t-2)} = \lim_{t\to 2} \frac{1}{t+2} = \frac{1}{4}$$

Now Work Problem 37 ◁

Limits at Infinity

Now let us examine the function

$$f(x) = \frac{1}{x}$$

as x becomes infinite, first in a positive sense and then in a negative sense. From Table 10.2, we can see that as x increases without bound through positive values, the values of $f(x)$ approach 0. Likewise, as x decreases without bound through negative values, the values of $f(x)$ also approach 0. These observations are also apparent from the graph in Figure 10.17. There, moving to the right along the curve through positive x-values, the corresponding y-values approach 0 through positive values. Similarly, moving to the left along the curve through negative x-values, the corresponding y-values approach 0 through negative values. Symbolically, we write

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0$$

Both of these limits are called *limits at infinity*.

We can obtain

$$\lim_{x\to\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x}$$

without the benefit of a graph or a table. Dividing 1 by a large positive number results in a small positive number, and as the divisors get arbitrarily large, the quotients get arbitrarily small. A similar argument can be made for the limit as $x \to -\infty$.

Table 10.2 Behavior of $f(x)$ as $x \to \pm\infty$

x	$f(x)$	x	$f(x)$
1000	0.001	−1000	−0.001
10,000	0.0001	−10,000	−0.0001
100,000	0.00001	−100,000	−0.00001
1,000,000	0.000001	−1,000,000	−0.000001

APPLY IT ▷

6. The demand function for a certain product is given by $p(x) = \frac{10,000}{(x+1)^2}$, where p is the price in dollars and x is the quantity sold. Graph this function on your graphing calculator in the window $[0, 10] \times [0, 10,000]$. Use the TRACE function to find $\lim_{x\to\infty} p(x)$. Determine what is happening to the graph and what this means about the demand function.

EXAMPLE 3 Limits at Infinity

Find the limit (if it exists).

a. $\lim_{x\to\infty} \frac{4}{(x-5)^3}$

Solution: As x becomes very large, so does $x - 5$. Since the cube of a large number is also large, $(x-5)^3 \to \infty$. Dividing 4 by very large numbers results in numbers near 0. Thus,

$$\lim_{x\to\infty} \frac{4}{(x-5)^3} = 0$$

b. $\lim_{x\to-\infty} \sqrt{4-x}$

Solution: As x gets negatively infinite, $4 - x$ becomes positively infinite. Because square roots of large numbers are large numbers, we conclude that

$$\lim_{x\to-\infty} \sqrt{4-x} = \infty$$

◁

In our next discussion we will need a certain limit, namely, $\lim_{x\to\infty} 1/x^p$, where $p > 0$. As x becomes very large, so does x^p. Dividing 1 by very large numbers results in numbers near 0. Thus, $\lim_{x\to\infty} 1/x^p = 0$. In general,

$$\lim_{x\to\infty} \frac{1}{x^p} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^p} = 0$$

for $p > 0$.[2] For example,

$$\lim_{x\to\infty} \frac{1}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1}{x^{1/3}} = 0$$

Let us now find the limit of the rational function

$$f(x) = \frac{4x^2+5}{2x^2+1}$$

as $x \to \infty$. (Recall from Section 2.2 that a rational function is a quotient of polynomials.) As x gets larger and larger, *both* the numerator and denominator of any rational function become infinite in absolute value. However, the form of the quotient can be changed, so that we can draw a conclusion as to whether or not it has a limit. To do this, we divide both the numerator and denominator by the greatest power of x that occurs in the denominator. Here it is x^2. This gives

$$\lim_{x\to\infty} \frac{4x^2+5}{2x^2+1} = \lim_{x\to\infty} \frac{\dfrac{4x^2+5}{x^2}}{\dfrac{2x^2+1}{x^2}} = \lim_{x\to\infty} \frac{\dfrac{4x^2}{x^2}+\dfrac{5}{x^2}}{\dfrac{2x^2}{x^2}+\dfrac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{4+\dfrac{5}{x^2}}{2+\dfrac{1}{x^2}} = \frac{\lim\limits_{x\to\infty} 4 + 5 \cdot \lim\limits_{x\to\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \dfrac{1}{x^2}}$$

Since $\lim_{x\to\infty} 1/x^p = 0$ for $p > 0$,

$$\lim_{x\to\infty} \frac{4x^2+5}{2x^2+1} = \frac{4+5(0)}{2+0} = \frac{4}{2} = 2$$

Similarly, the limit as $x \to -\infty$ is 2. These limits are clear from the graph of f in Figure 10.19.

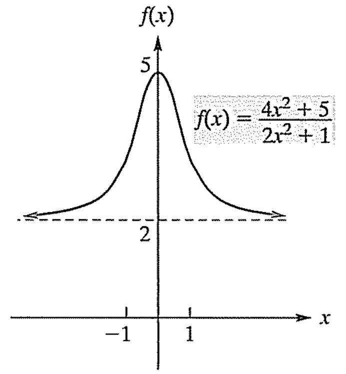


FIGURE 10.19 $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = 2$.

For the preceding function, there is an easier way to find $\lim_{x\to\infty} f(x)$. For *large* values of x, in the numerator the term involving the greatest power of x, namely, $4x^2$, dominates the sum $4x^2 + 5$, and the dominant term in the denominator, $2x^2 + 1$, is $2x^2$. Thus, as $x \to \infty$, $f(x)$ can be approximated by $(4x^2)/(2x^2)$. As a result, to determine the limit of $f(x)$, it suffices to determine the limit of $(4x^2)/(2x^2)$. That is,

$$\lim_{x\to\infty} \frac{4x^2+5}{2x^2+1} = \lim_{x\to\infty} \frac{4x^2}{2x^2} = \lim_{x\to\infty} 2 = 2$$

as we saw before. In general, we have the following rule:

Limits at Infinity for Rational Functions

If $f(x)$ is a *rational function* and $a_n x^n$ and $b_m x^m$ are the terms in the numerator and denominator, respectively, with the greatest powers of x, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{a_n x^n}{b_m x^m}$$

[2] For $\lim_{x\to-\infty} 1/x^p$, we assume that p is such that $1/x^p$ is defined for $x < 0$.

Let us apply this rule to the situation where the degree of the numerator is greater than the degree of the denominator. For example,

$$\lim_{x\to-\infty}\frac{x^4-3x}{5-2x}=\lim_{x\to-\infty}\frac{x^4}{-2x}=\lim_{x\to-\infty}\left(-\frac{1}{2}x^3\right)=\infty$$

(Note that in the next-to-last step, as x becomes very negative, so does x^3; moreover, $-\frac{1}{2}$ times a very negative number is very positive.) Similarly,

$$\lim_{x\to\infty}\frac{x^4-3x}{5-2x}=\lim_{x\to\infty}\left(-\frac{1}{2}x^3\right)=-\infty$$

From this illustration, we make the following conclusion:

> If the degree of the numerator of a *rational function* is greater than the degree of the denominator, then the function has no limit as $x\to\infty$ and no limit as $x\to-\infty$.

APPLY IT ▷

7. The yearly amount of sales y of a certain company (in thousands of dollars) is related to the amount the company spends on advertising, x (in thousands of dollars), according to the equation $y(x)=\frac{500x}{x+20}$. Graph this function on your graphing calculator in the window $[0, 1000]\times[0, 550]$. Use TRACE to explore $\lim_{x\to\infty} y(x)$, and determine what this means to the company.

EXAMPLE 4 Limits at Infinity for Rational Functions

Find the limit (if it exists).

a. $\lim_{x\to\infty}\frac{x^2-1}{7-2x+8x^2}$

Solution:

$$\lim_{x\to\infty}\frac{x^2-1}{7-2x+8x^2}=\lim_{x\to\infty}\frac{x^2}{8x^2}=\lim_{x\to\infty}\frac{1}{8}=\frac{1}{8}$$

b. $\lim_{x\to-\infty}\frac{x}{(3x-1)^2}$

Solution:

$$\lim_{x\to-\infty}\frac{x}{(3x-1)^2}=\lim_{x\to-\infty}\frac{x}{9x^2-6x+1}=\lim_{x\to-\infty}\frac{x}{9x^2}$$

$$=\lim_{x\to-\infty}\frac{1}{9x}=\frac{1}{9}\cdot\lim_{x\to-\infty}\frac{1}{x}=\frac{1}{9}(0)=0$$

c. $\lim_{x\to\infty}\frac{x^5-x^4}{x^4-x^3+2}$

Solution: Since the degree of the numerator is greater than that of the denominator, there is no limit. More precisely,

$$\lim_{x\to\infty}\frac{x^5-x^4}{x^4-x^3+2}=\lim_{x\to\infty}\frac{x^5}{x^4}=\lim_{x\to\infty}x=\infty$$

Now Work Problem 21 ◁

CAUTION!

The preceding technique applies only to limits of rational functions at *infinity*.

To find $\lim_{x\to0}\frac{x^2-1}{7-2x+8x^2}$, we cannot simply determine the limit of $\frac{x^2}{8x^2}$. That simplification applies only in case $x\to\infty$ or $x\to-\infty$. Instead, we have

$$\lim_{x\to0}\frac{x^2-1}{7-2x+8x^2}=\frac{\lim_{x\to0}x^2-1}{\lim_{x\to0}7-2x+8x^2}=\frac{0-1}{7-0+0}=-\frac{1}{7}$$

Let us now consider the limit of the polynomial function $f(x)=8x^2-2x$ as $x\to\infty$:

$$\lim_{x\to\infty}(8x^2-2x)$$

Because a polynomial is a rational function with denominator 1, we have

$$\lim_{x\to\infty}(8x^2-2x)=\lim_{x\to\infty}\frac{8x^2-2x}{1}=\lim_{x\to\infty}\frac{8x^2}{1}=\lim_{x\to\infty}8x^2$$

That is, the limit of $8x^2-2x$ as $x\to\infty$ is the same as the limit of the term involving the greatest power of x, namely, $8x^2$. As x becomes very large, so does $8x^2$. Thus,

$$\lim_{x\to\infty}(8x^2-2x)=\lim_{x\to\infty}8x^2=\infty$$

In general, we have the following:

> As $x \to \infty$ (or $x \to -\infty$), the limit of a *polynomial function* is the same as the limit of its term that involves the greatest power of x.

APPLY IT ▷

8. The cost C of producing x units of a certain product is given by $C(x) = 50{,}000 + 200x + 0.3x^2$. Use your graphing calculator to explore $\lim_{x\to\infty} C(x)$ and determine what this means.

EXAMPLE 5 Limits at Infinity for Polynomial Functions

a. $\lim_{x\to-\infty}(x^3 - x^2 + x - 2) = \lim_{x\to-\infty} x^3$. As x becomes very negative, so does x^3. Thus,

$$\lim_{x\to-\infty}(x^3 - x^2 + x - 2) = \lim_{x\to-\infty} x^3 = -\infty$$

b. $\lim_{x\to-\infty}(-2x^3 + 9x) = \lim_{x\to-\infty} -2x^3 = \infty$, because -2 times a very negative number is very positive.

Now Work Problem 9 ◁

Do not use dominant terms when a function is not rational.

The technique of focusing on dominant terms to find limits as $x \to \infty$ or $x \to -\infty$ is valid for *rational functions*, but it is not necessarily valid for other types of functions. For example, consider

$$\lim_{x\to\infty}\left(\sqrt{x^2 + x} - x\right) \tag{1}$$

Notice that $\sqrt{x^2 + x} - x$ is not a rational function. It is *incorrect* to infer that because x^2 dominates in $x^2 + x$, the limit in (1) is the same as

$$\lim_{x\to\infty}\left(\sqrt{x^2} - x\right) = \lim_{x\to\infty}(x - x) = \lim_{x\to\infty} 0 = 0$$

It can be shown (see Problem 62) that the limit in (1) is not 0, but is $\frac{1}{2}$.

The ideas discussed in this section will now be applied to a case-defined function.

APPLY IT ▷

9. A plumber charges \$100 for the first hour of work at your house and \$75 for every hour (or fraction thereof) afterward. The function for what an x-hour visit will cost you is

$$f(x) = \begin{cases} \$100 & \text{if } 0 < x \le 1 \\ \$175 & \text{if } 1 < x \le 2 \\ \$250 & \text{if } 2 < x \le 3 \\ \$325 & \text{if } 3 < x \le 4 \end{cases}$$

Find $\lim_{x\to1} f(x)$ and $\lim_{x\to2.5} f(x)$.

EXAMPLE 6 Limits for a Case-Defined Function

If $f(x) = \begin{cases} x^2 + 1 & \text{if } x \ge 1 \\ 3 & \text{if } x < 1 \end{cases}$, find the limit (if it exists).

a. $\lim_{x\to1^+} f(x)$

Solution: Here x gets close to 1 from the right. For $x > 1$, we have $f(x) = x^2 + 1$. Thus,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x^2 + 1)$$

If x is greater than 1, but close to 1, then $x^2 + 1$ is close to 2. Therefore,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x^2 + 1) = 2$$

b. $\lim_{x\to1^-} f(x)$

Solution: Here x gets close to 1 from the left. For $x < 1, f(x) = 3$. Hence,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} 3 = 3$$

c. $\lim_{x\to1} f(x)$

Solution: We want the limit as x approaches 1. However, the rule of the function depends on whether $x \ge 1$ or $x < 1$. Thus, we must consider one-sided limits. The limit as x approaches 1 will exist if and only if both one-sided limits exist and are the same. From parts (a) and (b),

$$\lim_{x\to1^+} f(x) \ne \lim_{x\to1^-} f(x) \qquad \text{since } 2 \ne 3$$

Therefore,

$$\lim_{x\to1} f(x) \qquad \text{does not exist}$$

d. $\lim_{x\to\infty} f(x)$

Solution: For very large values of x, we have $x \geq 1$, so $f(x) = x^2 + 1$. Thus,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^2 + 1) = \lim_{x\to\infty} x^2 = \infty$$

e. $\lim_{x\to-\infty} f(x)$

Solution: For very negative values of x, we have $x < 1$, so $f(x) = 3$. Hence,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 3 = 3$$

All the limits in parts (a) through (c) should be obvious from the graph of f in Figure 10.20.

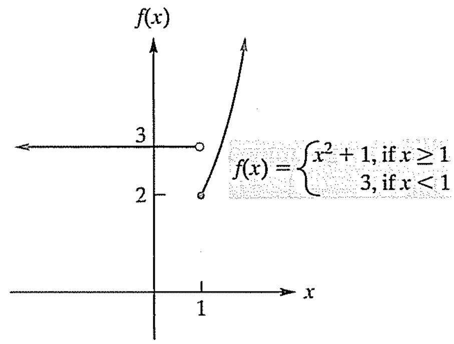


FIGURE 10.20 Graph of case-defined function.

Now Work Problem 57 ◁

PROBLEMS 10.2

1. For the function f given in Figure 10.21, find the following limits. If the limit does not exist, so state that, or use the symbol ∞ or $-\infty$ where appropriate.

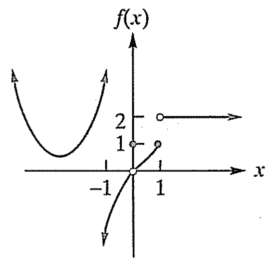


FIGURE 10.21

(a) $\lim_{x\to-\infty} f(x)$ **(b)** $\lim_{x\to-1^-} f(x)$ **(c)** $\lim_{x\to-1^+} f(x)$
(d) $\lim_{x\to-1} f(x)$ **(e)** $\lim_{x\to0^-} f(x)$ **(f)** $\lim_{x\to0^+} f(x)$
(g) $\lim_{x\to0} f(x)$ **(h)** $\lim_{x\to1^-} f(x)$ **(i)** $\lim_{x\to1^+} f(x)$
(j) $\lim_{x\to1} f(x)$ **(k)** $\lim_{x\to\infty} f(x)$

2. For the function f given in Figure 10.22, find the following limits. If the limit does not exist, so state that, or use the symbol ∞ or $-\infty$ where appropriate.

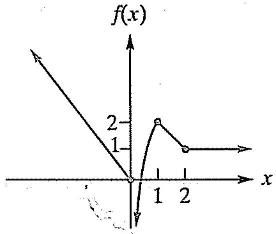


FIGURE 10.22

(a) $\lim_{x\to0^-} f(x)$ **(b)** $\lim_{x\to0^+} f(x)$ **(c)** $\lim_{x\to0} f(x)$
(d) $\lim_{x\to-\infty} f(x)$ **(e)** $\lim_{x\to1} f(x)$ **(f)** $\lim_{x\to\infty} f(x)$
(g) $\lim_{x\to2^+} f(x)$

In each of Problems 3–54, find the limit. If the limit does not exist, so state, or use the symbol ∞ or $-\infty$ where appropriate.

3. $\lim_{x\to3^+} (x-2)$ **4.** $\lim_{x\to-1^+} (1-x^2)$ **5.** $\lim_{x\to-\infty} 5x$

6. $\lim_{x\to-\infty} -6$ **7.** $\lim_{x\to0^-} \frac{6x}{x^4}$ **8.** $\lim_{x\to2} \frac{7}{x-1}$

9. $\lim_{x\to-\infty} x^2$ **10.** $\lim_{t\to\infty} (t-1)^3$ **11.** $\lim_{h\to1^+} \sqrt{h-1}$

12. $\lim_{h\to5^-} \sqrt{5-h}$ **13.** $\lim_{x\to-2^-} \frac{-3}{x+2}$ **14.** $\lim_{x\to0^-} 2^{1/2}$

15. $\lim_{x\to1^+} (4\sqrt{x-1})$ **16.** $\lim_{x\to2^-} (x\sqrt{4-x^2})$ **17.** $\lim_{x\to\infty} \sqrt{x+10}$

18. $\lim_{x\to-\infty} -\sqrt{1-10x}$ **19.** $\lim_{x\to\infty} \frac{3}{\sqrt{x}}$

20. $\lim_{x\to\infty} \frac{-6}{5x\sqrt[3]{x}}$ **21.** $\lim_{x\to\infty} \frac{x-5}{2x+1}$ **22.** $\lim_{x\to\infty} \frac{2x-4}{3-2x}$

23. $\lim_{x\to-\infty} \frac{x^2-1}{x^3+4x-3}$ **24.** $\lim_{r\to\infty} \frac{r^3}{r^2+1}$

25. $\lim_{t\to\infty} \frac{3t^3+2t^2+9t-1}{5t^2-5}$ **26.** $\lim_{x\to\infty} \frac{4x^2}{3x^3-x^2+2}$

27. $\lim_{x\to\infty} \frac{7}{2x+1}$ **28.** $\lim_{x\to-\infty} \frac{2}{(4x-1)^3}$

29. $\lim_{x\to\infty} \frac{3-4x-2x^3}{5x^3-8x+1}$ **30.** $\lim_{x\to-\infty} \frac{3-2x-2x^3}{7-5x^3+2x^2}$

31. $\lim_{x\to 3^+} \frac{x+3}{x^2-9}$ **32.** $\lim_{x\to -3^-} \frac{3x}{9-x^2}$ **33.** $\lim_{w\to\infty} \frac{2w^2-3w+4}{5w^2+7w-1}$

34. $\lim_{x\to\infty} \frac{4-3x^3}{x^3-1}$ **35.** $\lim_{x\to\infty} \frac{6-4x^2+x^3}{4+5x-7x^2}$

36. $\lim_{x\to -\infty} \frac{2x-x^2}{x^2+19x-64}$ **37.** $\lim_{x\to -3^-} \frac{5x^2+14x-3}{x^2+3x}$

38. $\lim_{t\to 3} \frac{t^2-4t+3}{t^2-2t-3}$ **39.** $\lim_{x\to 1} \frac{x^2-3x+1}{x^2+1}$

40. $\lim_{x\to -1} \frac{3x^3-x^2}{2x+1}$ **41.** $\lim_{x\to 2^-} \left(2-\frac{1}{x-2}\right)$

42. $\lim_{x\to -\infty} -\frac{x^5+2x^3-1}{x^5-4x^2}$ **43.** $\lim_{x\to -7^-} \frac{x^2+1}{\sqrt{x^2-49}}$

44. $\lim_{x\to -2^+} \frac{x}{\sqrt{16-x^4}}$ **45.** $\lim_{x\to 0^+} \frac{5}{x+x^2}$

46. $\lim_{x\to -\infty} \left(x^2+\frac{1}{x}\right)$ **47.** $\lim_{x\to 1} x(x-1)^{-1}$ **48.** $\lim_{x\to 1/2} \frac{1}{2x-1}$

49. $\lim_{x\to 1^+} \left(\frac{-5}{1-x}\right)$ **50.** $\lim_{x\to 3} \left(-\frac{7}{x-3}\right)$ **51.** $\lim_{x\to 1} |x-1|$

52. $\lim_{x\to 0} \left|\frac{1}{x}\right|$ **53.** $\lim_{x\to -\infty} \frac{x+1}{x}$

54. $\lim_{x\to\infty} \left(\frac{3}{x}-\frac{2x^2}{x^2+1}\right)$

In Problems 55–58, find the indicated limits. If the limit does not exist, so state, or use the symbol ∞ or $-\infty$ where appropriate.

55. $f(x) = \begin{cases} 2 & \text{if } x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$

(a) $\lim_{x\to 2^+} f(x)$ **(b)** $\lim_{x\to 2^-} f(x)$ **(c)** $\lim_{x\to 2} f(x)$
(d) $\lim_{x\to\infty} f(x)$ **(e)** $\lim_{x\to -\infty} f(x)$

56. $f(x) = \begin{cases} 2-x & \text{if } x \le 3 \\ -1+3x-x^2 & \text{if } x > 3 \end{cases}$

(a) $\lim_{x\to 3^+} f(x)$ **(b)** $\lim_{x\to 3^-} f(x)$ **(c)** $\lim_{x\to 3} f(x)$
(d) $\lim_{x\to\infty} f(x)$ **(e)** $\lim_{x\to -\infty} f(x)$

57. $g(x) = \begin{cases} x & \text{if } x < 0 \\ -x & \text{if } x > 0 \end{cases}$

(a) $\lim_{x\to 0^+} g(x)$ **(b)** $\lim_{x\to 0^-} g(x)$ **(c)** $\lim_{x\to 0} g(x)$
(d) $\lim_{x\to\infty} g(x)$ **(e)** $\lim_{x\to -\infty} g(x)$

58. $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -x & \text{if } x > 0 \end{cases}$

(a) $\lim_{x\to 0^+} g(x)$ **(b)** $\lim_{x\to 0^-} g(x)$ **(c)** $\lim_{x\to 0} g(x)$
(d) $\lim_{x\to\infty} g(x)$ **(e)** $\lim_{x\to -\infty} g(x)$

59. Average Cost If c is the total cost in dollars to produce q units of a product, then the average cost per unit for an output of q units is given by $\bar{c} = c/q$. Thus, if the total cost equation is $c = 5000 + 6q$, then

$$\bar{c} = \frac{5000}{q} + 6$$

For example, the total cost of an output of 5 units is \$5030, and the average cost per unit at this level of production is \$1006. By finding $\lim_{q\to\infty} \bar{c}$, show that the average cost approaches a level of stability if the producer continually increases output. What is the limiting value of the average cost? Sketch the graph of the average-cost function.

60. Average Cost Repeat Problem 59, given that the fixed cost is \$12,000 and the variable cost is given by the function $c_v = 7q$.

61. Population The population of a certain small city t years from now is predicted to be

$$N = 40{,}000 - \frac{5000}{t+3}$$

Find the population in the long run; that is, find $\lim_{t\to\infty} N$.

62. Show that

$$\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \frac{1}{2}$$

(*Hint:* Rationalize the numerator by multiplying the expression $\sqrt{x^2+x} - x$ by

$$\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$$

Then express the denominator in a form such that x is a factor.)

63. Host–Parasite Relationship For a particular host–parasite relationship, it was determined that when the host density (number of hosts per unit of area) is x, the number of hosts parasitized over a period of time is

$$y = \frac{900x}{10+45x}$$

If the host density were to increase without bound, what value would y approach?

64. If $f(x) = \begin{cases} \sqrt{2-x} & \text{if } x < 2 \\ x^3 + k(x+1) & \text{if } x \ge 2 \end{cases}$, determine the value of the constant k for which $\lim_{x\to 2} f(x)$ exists.

In Problems 65 and 66, use a calculator to evaluate the given function when $x = 1, 0.5, 0.2, 0.1, 0.01, 0.001$, and 0.0001. From your results, speculate about $\lim_{x\to 0^+} f(x)$.

65. $f(x) = x^{2x}$ **66.** $f(x) = e^{1/x}$

67. Graph $f(x) = \sqrt{4x^2-1}$. Use the graph to estimate $\lim_{x\to 1/2^+} f(x)$.

68. Graph $f(x) = \frac{\sqrt{x^2-9}}{x+3}$. Use the graph to estimate $\lim_{x\to -3^-} f(x)$ if it exists. Use the symbol ∞ or $-\infty$ if appropriate.

69. Graph $f(x) = \begin{cases} 2x^2+3 & \text{if } x < 2 \\ 2x+5 & \text{if } x \ge 2 \end{cases}$. Use the graph to estimate each of the following limits if it exists:
(a) $\lim_{x\to 2^-} f(x)$ **(b)** $\lim_{x\to 2^+} f(x)$ **(c)** $\lim_{x\to 2} f(x)$

Objective

To study continuity and to find points of discontinuity for a function.

10.3 Continuity

Many functions have the property that there is no "break" in their graphs. For example, compare the functions

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} x & \text{if } x \ne 1 \\ 2 & \text{if } x = 1 \end{cases}$$

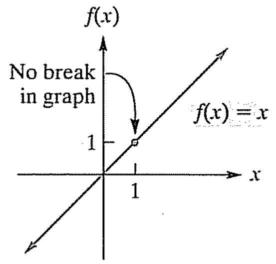


FIGURE 10.23 Continuous at 1.

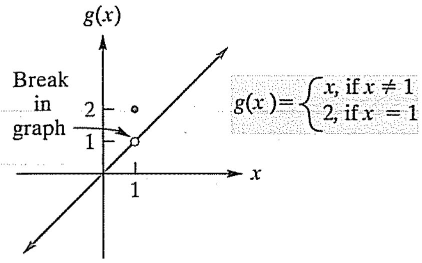


FIGURE 10.24 Discontinuous at 1.

whose graphs appear in Figures 10.23 and 10.24, respectively. The graph of f is unbroken, but the graph of g has a break at $x = 1$. Stated another way, if you were to trace both graphs with a pencil, you would have to lift the pencil off the graph of g when $x = 1$, but you would not have to lift it off the graph of f. These situations can be expressed by limits. As x approaches 1, compare the limit of each function with the value of the function at $x = 1$:

$$\lim_{x\to 1} f(x) = 1 = f(1)$$

whereas

$$\lim_{x\to 1} g(x) = 1 \neq 2 = g(1)$$

In Section 10.1 we stressed that given a function f and a number a, there are two important ways to associate a number to the pair (f, a). One is simple evaluation, $f(a)$, which *exists* precisely if a is in the domain of f. The other is $\lim_{x\to a} f(x)$, whose existence and determination can be more challenging. For the functions f and g above, the limit of f as $x \to 1$ is the same as $f(1)$, but the limit of g as $x \to 1$ is *not* the same as $g(1)$. For these reasons, we say that f is *continuous* at 1 and g is *discontinuous* at 1.

Definition

A function f is **continuous** at a if and only if the following three conditions are met:

1. $f(a)$ exists
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

If f is not continuous at a, then f is said to be **discontinuous** at a, and a is called a **point of discontinuity** of f.

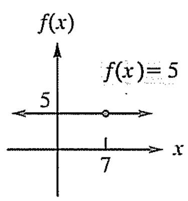


FIGURE 10.25 f is continuous at 7.

EXAMPLE 1 Applying the Definition of Continuity

a. Show that $f(x) = 5$ is continuous at 7.

Solution: We must verify that the preceding three conditions are met. First, $f(7) = 5$, so f is defined at $x = 7$. Second,

$$\lim_{x\to 7} f(x) = \lim_{x\to 7} 5 = 5$$

Thus, f has a limit as $x \to 7$. Third,

$$\lim_{x\to 7} f(x) = 5 = f(7)$$

Therefore, f is continuous at 7. (See Figure 10.25.)

b. Show that $g(x) = x^2 - 3$ is continuous at -4.

Solution: The function g is defined at $x = -4 : g(-4) = 13$. Also,

$$\lim_{x\to -4} g(x) = \lim_{x\to -4} (x^2 - 3) = 13 = g(-4)$$

Therefore, g is continuous at -4. (See Figure 10.26.)

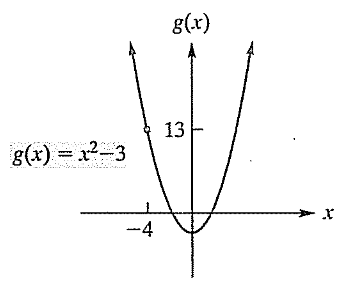


FIGURE 10.26 g is continuous at -4.

Now Work Problem 1 ◁

We say that a function is *continuous on an interval* if it is continuous at each point there. In this situation, the graph of the function is connected over the interval. For example, $f(x) = x^2$ is continuous on the interval [2, 5]. In fact, in Example 5 of Section 10.1, we showed that, for *any* polynomial function f, for any number a, $\lim_{x\to a} f(x) = f(a)$. This means that

A polynomial function is continuous at every point.

It follows that such a function is continuous on every interval. We say that a function is **continuous on its domain** if it is continuous at each point in its domain. If the domain of such a function is the set of all real numbers, we may simply say that the function is continuous.

EXAMPLE 2 Continuity of Polynomial Functions

The functions $f(x) = 7$ and $g(x) = x^2 - 9x + 3$ are polynomial functions. Therefore, they are continuous on their domains. For example, they are continuous at 3.

Now Work Problem 13 ◁

When is a function discontinuous? We can say that a function f defined on an open interval containing a is discontinuous at a if

1. f has no limit as $x \to a$

or

2. as $x \to a$, f has a limit that is different from $f(a)$

If f is not defined at a, we will say also, in that case, that f is discontinuous at a. In Figure 10.27, we can find points of discontinuity by inspection.

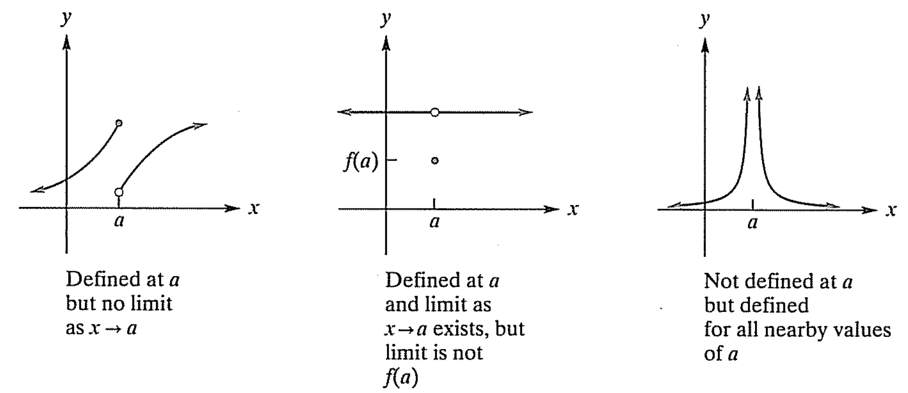


FIGURE 10.27 Discontinuities at a.

EXAMPLE 3 Discontinuities

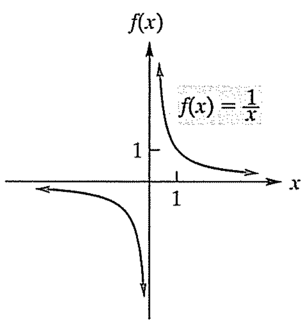


FIGURE 10.28 Infinite discontinuity at 0.

a. Let $f(x) = 1/x$. (See Figure 10.28.) Note that f is not defined at $x = 0$, but it is defined for all other x nearby. Thus, f is discontinuous at 0. Moreover, $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$. A function is said to have an **infinite discontinuity** at a when at least one of the one-sided limits is either ∞ or $-\infty$ as $x \to a$. Hence, f has an *infinite discontinuity* at $x = 0$.

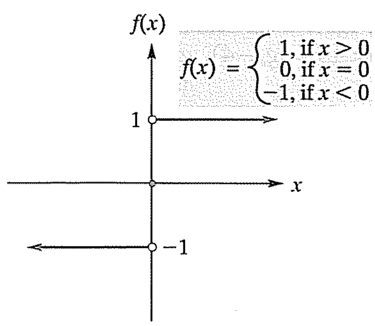


FIGURE 10.29 Discontinuous case-defined function.

b. Let $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.

(See Figure 10.29.) Although f is defined at $x = 0$, $\lim_{x \to 0} f(x)$ does not exist. Thus, f is discontinuous at 0.

Now Work Problem 29 ◁

The following property indicates where the discontinuities of a rational function occur:

Discontinuities of a Rational Function

A rational function is discontinuous at points where the denominator is 0 and is continuous otherwise. Thus, a rational function is continuous on its domain.

EXAMPLE 4 Locating Discontinuities in Rational Functions

The rational function $f(x) = \frac{x+1}{x+1}$ is continuous on its domain but it is not defined at -1. It is discontinuous at -1. The graph of f is a horizontal straight line with a "hole" in it at -1.

For each of the following functions, find all points of discontinuity.

a. $f(x) = \dfrac{x^2 - 3}{x^2 + 2x - 8}$

Solution: This rational function has denominator

$$x^2 + 2x - 8 = (x + 4)(x - 2)$$

which is 0 when $x = -4$ or $x = 2$. Thus, f is discontinuous only at -4 and 2.

b. $h(x) = \dfrac{x + 4}{x^2 + 4}$

Solution: For this rational function, the denominator is never 0. (It is always positive.) Therefore, h has no discontinuity.

Now Work Problem 19 ◁

EXAMPLE 5 Locating Discontinuities in Case-Defined Functions

For each of the following functions, find all points of discontinuity.

a. $f(x) = \begin{cases} x + 6 & \text{if } x \geq 3 \\ x^2 & \text{if } x < 3 \end{cases}$

Solution: The cases defining the function are given by polynomials, which are continuous, so the only possible place for a discontinuity is at $x = 3$, where the separation of cases occurs. We know that $f(3) = 3 + 6 = 9$. So because

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x + 6) = 9$$

and

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 = 9$$

we can conclude that $\lim_{x \to 3} f(x) = 9 = f(3)$ and the function has no points of discontinuity. We can reach the same conclusion by inspecting the graph of f in Figure 10.30.

b. $f(x) = \begin{cases} x + 2 & \text{if } x > 2 \\ x^2 & \text{if } x < 2 \end{cases}$

Solution: Since f is not defined at $x = 2$, it is discontinuous at 2. Note, however, that

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4 = \lim_{x \to 2^+} x + 2 = \lim_{x \to 2^+} f(x)$$

shows that $\lim_{x \to 2} f(x)$ exists. (See Figure 10.31.)

Now Work Problem 31 ◁

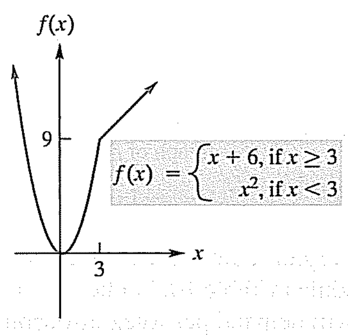


FIGURE 10.30 Continuous case-defined function.

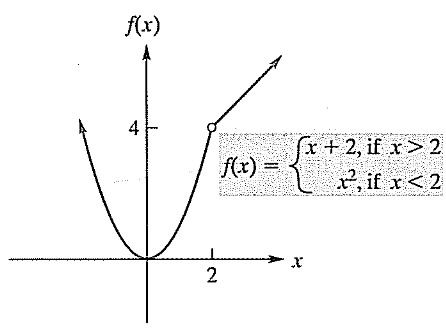


FIGURE 10.31 Discontinuous at 2.

EXAMPLE 6 Post-Office Function

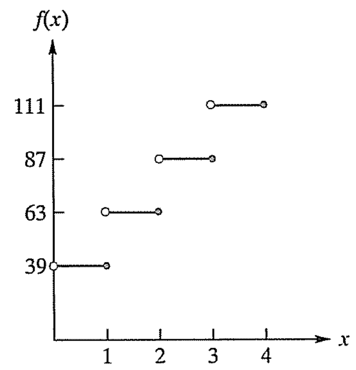


FIGURE 10.32 Post-office function.

The post-office function

$$c = f(x) = \begin{cases} 39 & \text{if } 0 < x \le 1 \\ 63 & \text{if } 1 < x \le 2 \\ 87 & \text{if } 2 < x \le 3 \\ 111 & \text{if } 3 < x \le 4 \end{cases}$$

gives the cost c (in cents) of mailing, first class, an item of weight x (ounces), for $0 < x \le 4$, in July 2006. It is clear from its graph in Figure 10.32 that f has discontinuities at 1, 2, and 3 and is constant for values of x between successive discontinuities. Such a function is called a *step function* because of the appearance of its graph.

Now Work Problem 35 ◁

There is another way to express continuity besides that given in the definition. If we take the statement

$$\lim_{x \to a} f(x) = f(a)$$

and replace x by $a + h$, then as $x \to a$, we have $h \to 0$; and as $h \to 0$ we have $x \to a$. It follows that $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$, provided the limits exist (Figure 10.33). Thus, the statement

This method of expressing continuity at a is used frequently in mathematical proofs.

$$\lim_{h \to 0} f(a + h) = f(a)$$

assuming both sides exist, also defines continuity at a.

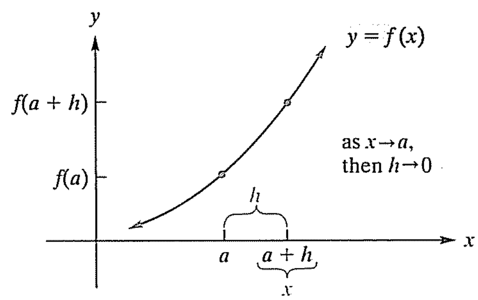


FIGURE 10.33 Diagram for continuity at a.

TECHNOLOGY

By observing the graph of a function, we may be able to determine where a discontinuity occurs. However, we can be fooled. For example, the function

$$f(x) = \frac{x - 1}{x^2 - 1}$$

is discontinuous at ± 1, but the discontinuity at 1 is not obvious from the graph of f in Figure 10.34. On the other hand, the discontinuity at -1 is obvious. Note that f is defined neither at -1 nor at 1.

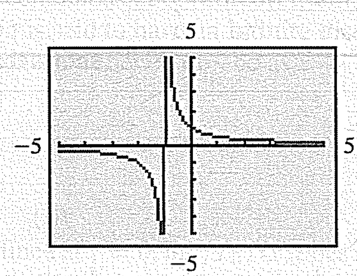


FIGURE 10.34 Discontinuity at 1 is not apparent from graph of $f(x) = \dfrac{x - 1}{x^2 - 1}$.

Often, it is helpful to describe a situation by a continuous function. For example, the demand schedule in Table 10.3 indicates the number of units of a particular product that consumers will demand per week at various prices. This information can be given graphically, as in Figure 10.35(a), by plotting each quantity–price pair as a point.

Table 10.3 Demand Schedule

Price/Unit, p	Quantity/Week, q
\$20	0
10	5
5	15
4	20
2	45
1	95

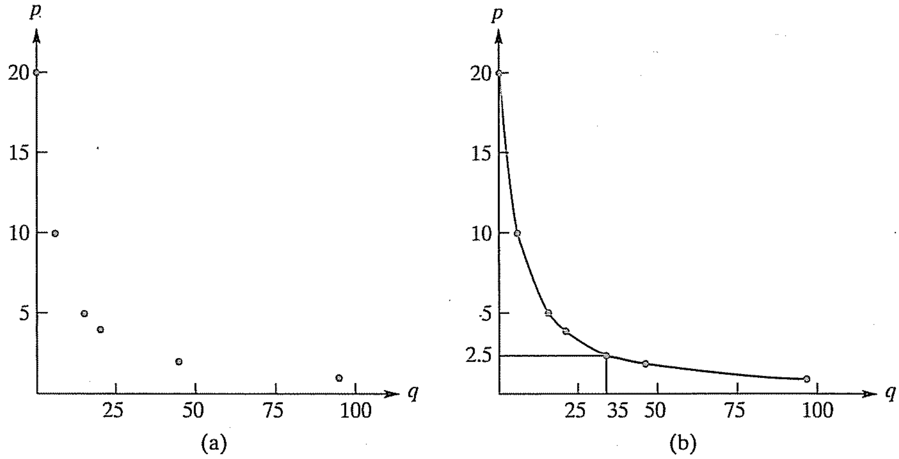


FIGURE 10.35 Viewing data via a continuous function.

Clearly, the graph does not represent a continuous function. Furthermore, it gives us no information as to the price at which, say, 35 units would be demanded. However, if we connect the points in Figure 10.35(a) by a smooth curve [see Figure 10.35(b)], we get a so-called demand curve. From it, we could guess that at about \$2.50 per unit, 35 units would be demanded.

Frequently, it is possible and useful to describe a graph, as in Figure 10.35(b), by means of an equation that defines a continuous function f. Such a function not only gives us a demand equation, $p = f(q)$, which allows us to anticipate corresponding prices and quantities demanded, but also permits a convenient mathematical analysis of the nature and basic properties of demand. Of course, some care must be used in working with equations such as $p = f(q)$. Mathematically, f may be defined when $q = \sqrt{37}$, but from a practical standpoint, a demand of $\sqrt{37}$ units could be meaningless to our particular situation. For example, if a unit is an egg, then a demand of $\sqrt{37}$ eggs make no sense.

We remark that functions of the form $f(x) = x^a$, for fixed a, are continuous on their domains. In particular, (square) root functions are continuous. Also, exponential functions and logarithmic functions are continuous on their domains. Thus, exponential functions have no discontinuities while a logarithmic function has only a discontinuity at 0 (which is an infinite discontinuity). Many more examples of continuous functions are provided by the observation that if f and g are continuous on their domains, then the composite function $f \circ g$, given by $f \circ g(x) = f(g(x))$ is continuous on its domain. For example, the function

$$f(x) = \sqrt{\ln\left(\frac{x^2+1}{x-1}\right)}$$

is continuous on its domain. Determining the domain of such a function may, of course, be fairly involved.

PROBLEMS 10.3

In Problems 1–6, use the definition of continuity to show that the given function is continuous at the indicated point.

1. $f(x) = x^3 - 5x; x = 2$

2. $f(x) = \dfrac{x-3}{5x}; x = -3$

3. $g(x) = \sqrt{2-3x}; x = 0$

4. $f(x) = \dfrac{x}{8}; x = 2$

5. $h(x) = \dfrac{x+3}{x-3}; x = -3$

6. $f(x) = \sqrt[3]{x}; x = -1$

In Problems 7–12, determine whether the function is continuous at the given points.

7. $f(x) = \dfrac{x+4}{x-2}; -2, 0$

8. $f(x) = \dfrac{x^2-4x+4}{6}; 2, -2$

9. $g(x) = \dfrac{x-3}{x^2-9}; 3, -3$

10. $h(x) = \dfrac{3}{x^2+9}; 3, -3$

11. $f(x) = \begin{cases} x+2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}; 2, 0$

12. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}; 0, -1$

In Problems 13–16, give a reason why the function is continuous on its domain.

13. $f(x) = 2x^2 - 3$

14. $f(x) = \dfrac{2+3x-x^2}{5}$

15. $f(x) = \ln(\sqrt[3]{x})$

16. $f(x) = x(1 - x)$

In Problems 17–34, find all points of discontinuity.

17. $f(x) = 3x^2 - 3$

18. $h(x) = x - 2$

19. $f(x) = \dfrac{3}{x+4}$

20. $f(x) = \dfrac{x^2 + 5x - 2}{x^2 - 9}$

21. $g(x) = \dfrac{(2x^2 - 3)^3}{15}$

22. $f(x) = -1$

23. $f(x) = \dfrac{x^2 + 6x + 9}{x^2 + 2x - 15}$

24. $g(x) = \dfrac{x - 3}{x^2 + x}$

25. $h(x) = \dfrac{x - 3}{x^3 - 9x}$

26. $f(x) = \dfrac{2x - 3}{3 - 2x}$

27. $p(x) = \dfrac{x}{x^2 + 1}$

28. $f(x) = \dfrac{x^4}{x^4 - 1}$

29. $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

30. $f(x) = \begin{cases} 3x + 5 & \text{if } x \geq -2 \\ 2 & \text{if } x < -2 \end{cases}$

31. $f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$

32. $f(x) = \begin{cases} x - 3 & \text{if } x > 2 \\ 3 - 2x & \text{if } x < 2 \end{cases}$

33. $f(x) = \begin{cases} x^2 + 1 & \text{if } x > 2 \\ 8x & \text{if } x < 2 \end{cases}$

34. $f(x) = \begin{cases} \dfrac{16}{x^2} & \text{if } x \geq 2 \\ 3x - 2 & \text{if } x < 2 \end{cases}$

35. Telephone Rates Suppose the long-distance rate for a telephone call from Hazleton, Pennsylvania to Los Angeles, California, is \$0.08 for the first minute or fraction thereof and \$0.04 for each additional minute or fraction thereof. If $y = f(t)$ is a function that indicates the total charge y for a call of t minutes duration, sketch the graph of f for $0 < t \leq 3\frac{1}{2}$. Use your graph to determine the values of t, where $0 < t \leq 3\frac{1}{2}$, at which discontinuities occur.

36. The *greatest integer function*, $f(x) = \lfloor x \rfloor$, is defined to be the greatest integer less than or equal to x, where x is any real number. For example, $\lfloor 3 \rfloor = 3$, $\lfloor 1.999 \rfloor = 1$, $\lfloor \frac{1}{4} \rfloor = 0$, and $\lfloor -4.5 \rfloor = -5$. Sketch the graph of this function for $-3.5 \leq x \leq 3.5$. Use your sketch to determine the values of x at which discontinuities occur.

37. Inventory Sketch the graph of

$$y = f(x) = \begin{cases} -100x + 600 & \text{if } 0 \leq x < 5 \\ -100x + 1100 & \text{if } 5 \leq x < 10 \\ -100x + 1600 & \text{if } 10 \leq x < 15 \end{cases}$$

A function such as this might describe the inventory y of a company at time x. Is f continuous at 2? At 5? At 10?

38. Graph $g(x) = e^{-1/x^2}$. Because g is not defined at $x = 0$, g is discontinuous at 0. Based on the graph of g, is

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

continuous at 0?

Objective

To develop techniques for solving nonlinear inequalities.

10.4 Continuity Applied to Inequalities

In Section 1.2, we solved linear inequalities. We now turn our attention to showing how the notion of continuity can be applied to solving a nonlinear inequality such as $x^2 + 3x - 4 < 0$. The ability to do this will be important in our study of calculus.

Recall (from Section 2.5) that the x-intercepts of the graph of a function g are precisely the roots of the equation $g(x) = 0$. Hence, from the graph of $y = g(x)$ in Figure 10.36, we conclude that r_1, r_2, and r_3 are roots of $g(x) = 0$ and any other roots will give rise to x-intercepts (beyond what is actually shown of the graph). Assume that in fact all the roots of $g(x) = 0$, and hence all the x-intercepts, are shown. Note further from Figure 10.36 that the three roots determine four open intervals on the x-axis:

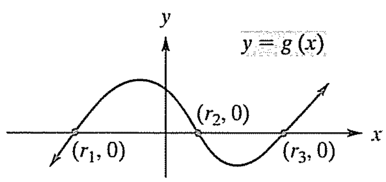


FIGURE 10.36 r_1, r_2, and r_3 are roots of $g(x) = 0$.

$$(-\infty, r_1) \quad (r_1, r_2) \quad (r_2, r_3) \quad (r_3, \infty)$$

To solve $x^2 + 3x - 4 > 0$, we let

$$f(x) = x^2 + 3x - 4 = (x + 4)(x - 1)$$

Because f is a polynomial function, it is continuous. The roots of $f(x) = 0$ are -4 and 1; hence, the graph of f has x-intercepts $(-4, 0)$ and $(1, 0)$. (See Figure 10.37.) The roots determine three intervals on the x-axis:

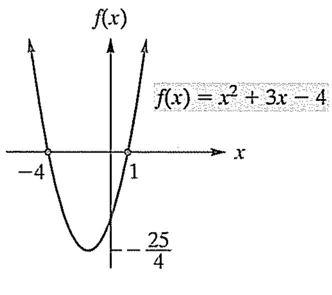


FIGURE 10.37 -4 and 1 are roots of $f(x) = 0$.

$$(-\infty, -4) \quad (-4, 1) \quad (1, \infty)$$

Consider the interval $(-\infty, -4)$. Since f is continuous on this interval, we claim that either $f(x) > 0$ or $f(x) < 0$ *throughout* the interval. If this were not the case, then $f(x)$ would indeed change sign on the interval. By the continuity of f, there would be a point where the graph intersects the x-axis—for example, at $(x_0, 0)$. (See Figure 10.38.) But then x_0 would be a root of $f(x) = 0$. However, this cannot be, because there is no root less than -4. Hence, $f(x)$ must be strictly positive or strictly negative on $(-\infty, -4)$. A similar argument can be made for each of the other intervals.

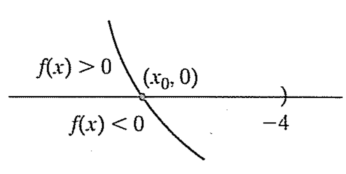


FIGURE 10.38 Change of sign for a continuous function.

To determine the sign of $f(x)$ on any one of the three intervals, it suffices to determine its sign at *any* point in the interval. For instance, -5 is in $(-\infty, -4)$ and

$$f(-5) = 6 > 0 \qquad \text{Thus, } f(x) > 0 \text{ on } (-\infty, -4)$$

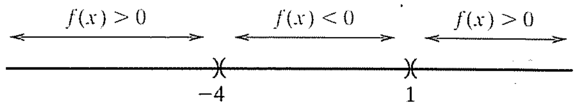


FIGURE 10.39 Simple sign chart for $x^2 + 3x - 4$.

Similarly, 0 is in $(-4, 1)$, and

$$f(0) = -4 < 0 \qquad \text{Thus, } f(x) < 0 \text{ on } (-4, 1)$$

Finally, 3 is in $(1, \infty)$, and

$$f(3) = 14 > 0 \qquad \text{Thus, } f(x) > 0 \text{ on } (1, \infty)$$

(See the "sign chart" in Figure 10.39.) Therefore,

$$x^2 + 3x - 4 > 0 \quad \text{on} \quad (-\infty, -4) \text{ and } (1, \infty)$$

so we have solved the inequality. These results are obvious from the graph in Figure 10.37. The graph lies above the x-axis, meaning that $f(x) > 0$, on $(-\infty, -4)$ and on $(1, \infty)$.

In more complicated examples it will be useful to exploit the multiplicative nature of signs. We noted that $f(x) = x^2 + 3x - 4 = (x + 4)(x - 1)$. Each of $x + 4$ and $x - 1$ has a sign chart that is simpler than that of $x^2 + 3x - 4$. Consider the "sign chart" in Figure 10.40. As before, we placed the roots of $f(x) = 0$ in ascending order, from left to right, so as to subdivide $(-\infty, \infty)$ into three open intervals. This forms the top line of the box. Directly below the top line we determined the signs of $x + 4$ on the three subintervals. We know that for the linear function $x + 4$ there is exactly one root of the equation $x + 4 = 0$, namely -4. We placed a 0 at -4 in the row labeled $x + 4$. By the argument illustrated in Figure 10.38, it follows that the sign of the function $x + 4$ is constant on $(-\infty, -4)$ and on $(-4, \infty)$ and two evaluations of $x + 4$ settle the distribution of signs for $x + 4$. From $(-5) + 4 = -1 < 0$, we have $x + 4$ *negative* on $(-\infty, -4)$, so we entered a $-$ sign in the $(-\infty, -4)$ space of the $x + 4$ row. From $(0) + 4 = 4 > 0$, we have $x + 4$ *positive* on $(-4, \infty)$. Since $(-4, \infty)$ has been further subdivided at 1, we entered a $+$ sign in each of the $(-4, 1)$ and $(1, \infty)$ spaces of the $x + 4$ row. In a similar way we constructed the row labeled $x - 1$.

	$-\infty$	-4		1		∞
$x + 4$	$-$	0	$+$		$+$	
$x - 1$	$-$		$-$	0	$+$	
$f(x)$	$+$	0	$-$	0	$+$	

FIGURE 10.40 Sign chart for $x^2 + 3x - 4$.

Now the bottom row is obtained by taking, for each component, the product of the entries above. Thus we have $(x + 4)(x - 1) = f(x)$, $(-)(-) = +$, $0(\text{any number}) = 0$, $(+)(-) = -$, $(\text{any number})0 = 0$, and $(+)(+) = +$. Sign charts of this kind are useful whenever a continuous function can be expressed as a product of several simpler, continuous functions, each of which has a simple sign chart. In Chapter 13 we will rely heavily on such sign charts.

EXAMPLE 1 Solving a Quadratic Inequality

Solve $x^2 - 3x - 10 > 0$.

Solution: If $f(x) = x^2 - 3x - 10$, then f is a polynomial (quadratic) function and thus is continuous everywhere. To find the real roots of $f(x) = 0$, we have

$$x^2 - 3x - 10 = 0$$
$$(x + 2)(x - 5) = 0$$
$$x = -2, 5$$

	$-\infty$	-2		5	∞
$x+2$	$-$	0	$+$		$+$
$x-5$	$-$		$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$+$

FIGURE 10.41 Sign chart for $x^2-3x-10$.

The roots -2 and 5 determine three intervals:

$$(-\infty,-2) \quad (-2,5) \quad (5,\infty)$$

In the manner of the last example, we construct the sign chart in Figure 10.41. We see that $x^2-3x-10>0$ on $(-\infty,-2)\cup(5,\infty)$.

Now Work Problem 1 ◁

APPLY IT ▷

10. An open box is formed by cutting a square piece out of each corner of an 8-inch-by-10-inch piece of metal. If each side of the cut-out squares is x inches long, the volume of the box is given by $V(x)=x(8-2x)(10-2x)$. This problem makes sense only when this volume is positive. Find the values of x for which the volume is positive.

EXAMPLE 2 Solving a Polynomial Inequality

Solve $x(x-1)(x+4)\le 0$.

Solution: If $f(x)=x(x-1)(x+4)$, then f is a polynomial function and hence continuous everywhere. The roots of $f(x)=0$ are (in ascending order) -4, 0, and 1 and lead to the sign chart in Figure 10.42.

	$-\infty$	-4		0		1	∞
x	$-$		$-$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

FIGURE 10.42 Sign chart for $x(x-1)(x+4)$.

From the sign chart, noting the endpoints required, $x(x-1)(x+4)\le 0$ on $(-\infty,-4]\cup[0,1]$.

Now Work Problem 11 ◁

The sign charts we have described are certainly not limited to solving polynomial inequalities. The reader will have noticed that we used thicker vertical lines at the endpoints, $-\infty$ and ∞, of the chart. These symbols do not denote real numbers, let alone points in the domain of a function. We extend the thick vertical line convention to single out isolated real numbers that are not in the domain of the function in question. The next example will illustrate.

EXAMPLE 3 Solving a Rational Function Inequality

Solve $\dfrac{x^2-6x+5}{x}\ge 0$.

Solution: Let

$$f(x)=\frac{x^2-6x+5}{x}=\frac{(x-1)(x-5)}{x}$$

For a rational function $f=g/h$, we solve the inequality by considering the intervals determined by both the roots of $g(x)=0$ and the roots of $h(x)=0$. Observe that the roots of $g(x)=0$ are the roots of $f(x)=0$ because the only way for a fraction to be 0 is for its numerator to be 0. On the other hand, the roots of $h(x)=0$ are precisely

	$-\infty$	0		1		5	∞
$x-1$	$-$		$-$	0	$+$		$+$
$x-5$	$-$		$-$		$-$	0	$+$
$1/x$	$-$	$\times$	$+$		$+$		$+$
$f(x)$	$-$	$\times$	$+$	0	$-$	0	$+$

FIGURE 10.43 Sign chart for $\frac{(x-1)(x-5)}{x}$.

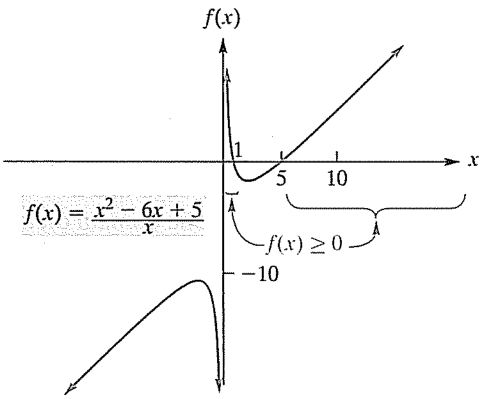


FIGURE 10.44 Graph of $f(x) = \frac{x^2-6x+5}{x}$.

the points at which f is not defined and these are also precisely the points at which f is discontinuous. The sign of f may change at a root and it may change at a discontinuity. Here the roots of the numerator are 1 and 5 and the root of the denominator is 0. In ascending order these give us 0, 1, and 5, which determine the open intervals

$$(-\infty, 0) \quad (0, 1) \quad (1, 5) \quad (5, \infty)$$

These, together with the observation that $1/x$ is a *factor* of f, lead to the sign chart in Figure 10.43.

Here the first two rows of the sign chart are constructed as before. In the third row we have placed a $\times$ sign at 0 to indicate that the factor $1/x$ is not defined at 0. The bottom row, as before, is constructed by taking the products of the entries above. Observe that a product is not defined at any point at which any of its factors is not defined. Hence we also have a $\times$ entry at 0 in the bottom row.

From the bottom row of the sign chart we can read that the solution of $\frac{(x-1)(x-5)}{x} \geq 0$ is $(0, 1] \cup [5, \infty]$. Observe that 1 and 5 are in the solution and 0 is not.

In Figure 10.44 we have graphed $f(x) = \frac{x^2-6x+5}{x}$, and we can confirm visually that the solution of the inequality $f(x) \geq 0$ is precisely the set of all real numbers at which the graph lies on or above the x-axis.

Now Work Problem 17 ◁

A sign chart is not always necessary, as the following example shows.

EXAMPLE 4 Solving Nonlinear Inequalities

a. Solve $x^2 + 1 > 0$.

Solution: The equation $x^2 + 1 = 0$ has no real roots. Thus, the continuous function $f(x) = x^2 + 1$ has no x-intercepts. It follows that either $f(x)$ is always positive or $f(x)$ is always negative. But x^2 is always positive or zero, so $x^2 + 1$ is always positive. Hence, the solution of $x^2 + 1 > 0$ is $(-\infty, \infty)$.

b. Solve $x^2 + 1 < 0$.

Solution: From part (a), $x^2 + 1$ is always positive, so $x^2 + 1 < 0$ has no solution, meaning that the set of solutions is $\emptyset$, the empty set.

Now Work Problem 7 ◁

We conclude with a nonrational example. The importance of the function introduced will become clear in later chapters.

EXAMPLE 5 Solving a Nonrational Function Inequality

Solve $x \ln x - x \geq 0$.

Solution: Let $f(x) = x \ln x - x = x(\ln x - 1)$, which, being a product of continuous functions, is continuous. From the *factored* form for f we see that the roots of $f(x) = 0$ are 0 and the roots of $\ln x - 1 = 0$. The latter is equivalent to $\ln x = 1$, which is equivalent to $e^{\ln x} = e^1$, since the exponential function is one-to-one. However, the last equality says that $x = e$. The domain of f is $(0, \infty)$ because $\ln x$ is only defined for $x > 0$. The domain dictates the top line of our sign chart in Figure 10.45.

The first row of Figure 10.45 is straightforward. For the second row, we placed a 0 at e, the only root of $\ln x - 1 = 0$. By continuity of $\ln x - 1$, the sign of $\ln x - 1$ on $(0, e)$ and on (e, ∞) can be determined by suitable evaluations. For the first we evaluate at 1 in $(0, e)$ and get $\ln 1 - 1 = 0 - 1 = -1 < 0$. For the second we evaluate at e^2 in (e, ∞) and get $\ln e^2 - 1 = 2 - 1 = 1 > 0$. The bottom row is, as usual, determined by multiplying the others. From the bottom row of Figure 10.45 the solution of $x \ln x - x \geq 0$ is evidently $[e, \infty)$.

	0		e		∞
x		+		+	
$\ln x - 1$		−	0	+	
$f(x)$		−	0	+	

FIGURE 10.45 Sign chart for $x \ln x - x$.

Now Work Problem 35 ◁

PROBLEMS 10.4

In Problems 1–26, solve the inequalities by the technique discussed in this section.

1. $x^2 - 3x - 4 > 0$

2. $x^2 - 8x + 15 > 0$

3. $x^2 - 3x - 10 \leq 0$

4. $15 - 2x - x^2 \geq 0$

5. $2x^2 + 11x + 14 < 0$

6. $x^2 - 4 < 0$

7. $x^2 + 4 < 0$

8. $2x^2 - x - 2 \leq 0$

9. $(x + 1)(x - 2)(x + 7) \leq 0$

10. $(x + 5)(x + 2)(x - 7) \leq 0$

11. $-x(x - 5)(x + 4) > 0$

12. $(x + 2)^2 > 0$

13. $x^3 + 4x \geq 0$

14. $(x + 3)^2(x^2 - 4) < 0$

15. $x^3 + 8x^2 + 15x \leq 0$

16. $x^3 + 6x^2 + 9x < 0$

17. $\dfrac{x}{x^2 - 9} < 0$

18. $\dfrac{x^2 - 1}{x} < 0$

19. $\dfrac{3}{x + 1} \geq 0$

20. $\dfrac{3}{x^2 - 5x + 6} > 0$

21. $\dfrac{x^2 - x - 6}{x^2 + 4x - 5} \geq 0$

22. $\dfrac{x^2 + 4x - 5}{x^2 + 3x + 2} \leq 0$

23. $\dfrac{3}{x^2 + 6x + 5} \leq 0$

24. $\dfrac{3x + 2}{(x - 1)^2} \leq 0$

25. $x^2 + 2x \geq 2$

26. $x^4 - 16 \geq 0$

27. Revenue Suppose that consumers will purchase q units of a product when the price of *each* unit is $28 - 0.2q$ dollars. How many units must be sold for the sales revenue to be at least \$750?

28. Forest Management A lumber company owns a forest that is of rectangular shape, 1 mi × 2 mi. The company wants to cut a uniform strip of trees along the outer edges of the forest. At most, how wide can the strip be if the company wants at least $1\frac{5}{16}$ mi^2 of forest to remain?

29. Container Design A container manufacturer wishes to make an open box by cutting a 3-in.-by-3-in. square from each corner of a square sheet of aluminum and then turning up the sides. The box is to contain at least 192 cubic inches. Find the dimensions of the smallest square sheet of aluminum that can be used.

30. Workshop Participation Imperial Education Services (I.E.S.) is offering a workshop in data processing to key personnel at Zeta Corporation. The price per person is \$50, and Zeta Corporation guarantees that at least 50 people will attend. Suppose I.E.S. offers to reduce the charge for *everybody* by \$0.50 for each person over the 50 who attends. How should I.E.S. limit the size of the group so that the total revenue it receives will never be less than that received for 50 persons?

31. Graph $f(x) = x^3 + 7x^2 - 5x + 4$. Use the graph to determine the solution of

$$x^3 + 7x^2 - 5x + 4 \leq 0$$

32. Graph $f(x) = \dfrac{3x^2 - 0.5x + 2}{6.2 - 4.1x}$. Use the graph to determine the solution of

$$\frac{3x^2 - 0.5x + 2}{6.2 - 4.1x} > 0$$

A novel way of solving a nonlinear inequality like $f(x) > 0$ is by examining the graph of $g(x) = f(x)/|f(x)|$, whose range consists only of 1 and −1:

$$g(x) = \frac{f(x)}{|f(x)|} = \begin{cases} 1 & \text{if } f(x) > 0 \\ -1 & \text{if } f(x) < 0 \end{cases}$$

The solution of $f(x) > 0$ consists of all intervals for which $g(x) = 1$. Using this technique, solve the inequalities in Problems 33 and 34.

33. $6x^2 - x - 2 > 0$

34. $\dfrac{x^2 + x - 1}{x^2 + x - 6} < 0$

35. Graph $x \ln x - x$. Does the function appear to be continuous? Does the graph support the conclusions of Example 5? At what value does the function appear to have a minimum value?

36. Graph e^{-x^2}. Does the function appear to be continuous? Can the conclusion be confirmed by invoking facts about continuous functions? At what value does the function appear to have a maximum value?

Chapter 10 Review

Important Terms and Symbols

		Examples
Section 10.1	**Limits**	
	$\lim_{x \to a} f(x) = L$	Ex. 8, p. 466
Section 10.2	**Limits (Continued)**	
	$\lim_{x \to a^-} f(x) = L$ $\quad \lim_{x \to a^+} f(x) = L$ $\quad \lim_{x \to a} f(x) = \infty$	Ex. 1, p. 470
	$\lim_{x \to \infty} f(x) = L$ $\quad \lim_{x \to -\infty} f(x) = L$	Ex. 3, p. 471
Section 10.3	**Continuity**	
	continuous at a discontinuous at a	Ex. 3, p. 478
	continuous on an interval continuous on its domain	Ex. 4, p. 479
Section 10.4	**Continuity Applied to Inequalities**	
	sign chart	Ex. 1, p. 483

Summary

The notion of limit is fundamental for calculus. To say that $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ can be made as close to the number L as we like by taking x sufficiently close to, but different from, a. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and c is a constant, then

1. $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x^n = a^n$
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} [cf(x)] = c \cdot \lim_{x \to a} f(x)$
6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$,
7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$
8. If f is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.

Property 8 implies that the limit of a polynomial function as $x \to a$ can be found by simply evaluating the polynomial at a. However, with other functions, f, evaluation at a may lead to the meaningless form 0/0. In such cases, algebraic manipulation such as factoring and canceling may yield a function g that agrees with f, for $x \neq a$, and for which the limit can be determined.

If $f(x)$ approaches L as x approaches a from the right, then we write $\lim_{x \to a^+} f(x) = L$. If $f(x)$ approaches L as x approaches a from the left, we write $\lim_{x \to a^-} f(x) = L$. These limits are called one-sided limits.

The infinity symbol ∞, which does not represent a number, is used in describing limits. The statement

$$\lim_{x \to \infty} f(x) = L$$

means that as x increases without bound, the values of $f(x)$ approach the number L. A similar statement applies for the situation when $x \to -\infty$, which means that x is decreasing without bound. In general, if $p > 0$, then

$$\lim_{x \to \infty} \frac{1}{x^p} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^p} = 0$$

If $f(x)$ increases without bound as $x \to a$, then we write $\lim_{x \to a} f(x) = \infty$. Similarly, if $f(x)$ decreases without bound, we have $\lim_{x \to a} f(x) = -\infty$. To say that the limit of a function is ∞ (or $-\infty$) does not mean that the limit exists. Rather, it is a way of saying that the limit does not exist and tells *why* there is no limit.

There is a rule for evaluating the limit of a rational function (quotient of polynomials) as $x \to \infty$ or $-\infty$. If $f(x)$ is a rational function and $a_n x^n$ and $b_m x^m$ are the terms in the numerator and denominator, respectively, with the greatest powers of x, then

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a_n x^n}{b_m x^m}$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{a_n x^n}{b_m x^m}$$

In particular, as $x \to \infty$ or $-\infty$, the limit of a polynomial is the same as the limit of the term that involves the greatest power of x. This means that, for a nonconstant polynomial, the limit as $x \to \infty$ or $-\infty$ is either ∞ or $-\infty$.

A function f is continuous at a if and only if

1. $f(a)$ exists
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Geometrically this means that the graph of f has no break at $x = a$. If a function is not continuous at a, then the function is said to be discontinuous at a. Polynomial functions and rational functions are continuous on their domains. Thus polynomial functions have no discontinuities and a rational function is discontinuous only at points where its denominator is zero.

To solve the inequality $f(x) > 0$ (or $f(x) < 0$), we first find the real roots of $f(x) = 0$ and the values of x for which f is discontinuous. These values determine intervals, and on each interval, $f(x)$ is either always positive or always negative. To find the sign on any one of these intervals, it suffices to find the sign of $f(x)$ at any point there. After the signs are determined for all intervals and assembled on a sign chart, it is easy to give the solution of $f(x) > 0$ (or $f(x) < 0$).

Review Problems

In Problems 1–28, find the limits if they exist. If the limit does not exist, so state, or use the symbol ∞ or $-\infty$ where appropriate.

1. $\lim_{x\to -1} (2x^2 + 6x - 1)$

2. $\lim_{x\to 0} \frac{2x^2 - 3x + 1}{2x^2 - 2}$

3. $\lim_{x\to 4} \frac{x^2 - 16}{x^2 - 4x}$

4. $\lim_{x\to -4} \frac{2x + 3}{x^2 - 4}$

5. $\lim_{h\to 0} (x + h)$

6. $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 3x + 2}$

7. $\lim_{x\to -4} \frac{x^3 + 4x^2}{x^2 + 2x - 8}$

8. $\lim_{x\to 2} \frac{x^2 - 7x + 10}{x^2 + x - 6}$

9. $\lim_{x\to \infty} \frac{2}{x + 1}$

10. $\lim_{x\to \infty} \frac{x^2 + 1}{2x^2}$

11. $\lim_{x\to \infty} \frac{2x + 5}{7x - 4}$

12. $\lim_{x\to -\infty} \frac{1}{x^4}$

13. $\lim_{t\to 4} \frac{3t - 4}{t - 4}$

14. $\lim_{x\to -\infty} \frac{x^6}{x^5}$

15. $\lim_{x\to -\infty} \frac{x + 3}{1 - x}$

16. $\lim_{x\to 4} \sqrt[3]{64}$

17. $\lim_{x\to \infty} \frac{x^2 - 1}{(3x + 2)^2}$

18. $\lim_{x\to 5} \frac{x^2 - 2x - 15}{x - 5}$

19. $\lim_{x\to 3^-} \frac{x + 3}{x^2 - 9}$

20. $\lim_{x\to 2} \frac{2 - x}{x - 2}$

21. $\lim_{x\to \infty} \sqrt{3x}$

22. $\lim_{y\to 5^+} \sqrt{y - 5}$

23. $\lim_{x\to \infty} \frac{x^{100} + (1/x^4)}{e - x^{96}}$

24. $\lim_{x\to -\infty} \frac{ex^2 - x^4}{31x - 2x^3}$

25. $\lim_{x\to 1} f(x)$ if $f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 1 \\ x & \text{if } x > 1 \end{cases}$

26. $\lim_{x\to 3} f(x)$ if $f(x) = \begin{cases} x + 5 & \text{if } x < 3 \\ 6 & \text{if } x \ge 3 \end{cases}$

27. $\lim_{x\to 4^+} \frac{\sqrt{x^2 - 16}}{4 - x}$ (*Hint:* For $x > 4$, $\sqrt{x^2 - 16} = \sqrt{x - 4}\sqrt{x + 4}$.)

28. $\lim_{x\to 3^+} \frac{x^2 + x - 12}{\sqrt{x - 3}}$ (*Hint:* For $x > 3$, $\frac{x - 3}{\sqrt{x - 3}} = \sqrt{x - 3}$.)

29. If $f(x) = 8x - 2$, find $\lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$.

30. If $f(x) = 2x^2 - 3$, find $\lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$.

31. Host–Parasite Relationship For a particular host–parasite relationship, it was determined that when the host density (number of hosts per unit of area) is x, then the number of hosts parasitized over a certain period of time is

$$y = 23\left(1 - \frac{1}{1 + 2x}\right)$$

If the host density were to increase without bound, what value would y approach?

32. Predator–Prey Relationship For a particular predator–prey relationship, it was determined that the number y of prey consumed by an individual predator over a period of time was a function of the prey density x (the number of prey per unit of area). Suppose

$$y = f(x) = \frac{10x}{1 + 0.1x}$$

If the prey density were to increase without bound, what value would y approach?

33. Using the definition of *continuity*, show that the function $f(x) = x + 3$ is continuous at $x = 2$.

34. Using the definition of *continuity*, show that the function $f(x) = \frac{x - 5}{x^2 + 2}$ is continuous at $x = 5$.

35. State whether $f(x) = x^2/5$ is continuous at each real number. Give a reason for your answer.

36. State whether $f(x) = x^2 - 2$ is continuous everywhere. Give a reason for your answer.

In Problems 37–44, find the points of discontinuity (if any) for each function.

37. $f(x) = \frac{x^2}{x + 3}$

38. $f(x) = \frac{0}{x^2}$

39. $f(x) = \frac{x - 1}{2x^2 + 3}$

40. $f(x) = (2 - 3x)^3$

41. $f(x) = \frac{4 - x^2}{x^2 + 3x - 4}$

42. $f(x) = \frac{2x + 6}{x^3 + x}$

43. $f(x) = \begin{cases} 2x + 3 & \text{if } x > 2 \\ 3x + 5 & \text{if } x \le 2 \end{cases}$

44. $f(x) = \begin{cases} 1/x & \text{if } x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$

In Problems 45–52, solve the given inequalities.

45. $x^2 + 4x - 12 > 0$

46. $3x^2 - 3x - 6 \leq 0$

47. $x^5 \leq 7x^4$

48. $x^3 + 9x^2 + 14x < 0$

49. $\dfrac{x+5}{x^2-1} < 0$

50. $\dfrac{x(x+5)(x+8)}{3} < 0$

51. $\dfrac{x^2+3x}{x^2+2x-8} \geq 0$

52. $\dfrac{x^2-9}{x^2-16} \leq 0$

53. Graph $f(x) = \dfrac{x^3+3x^2-19x+18}{x^3-2x^2+x-2}$. Use the graph to estimate $\lim_{x\to 2} f(x)$.

54. Graph $f(x) = \dfrac{\sqrt{x+3}-2}{x-1}$. From the graph, estimate $\lim_{x\to 1} f(x)$.

55. Graph $f(x) = x \ln x$. From the graph, estimate the one-sided limit $\lim_{x\to 0^+} f(x)$.

56. Graph $f(x) = \dfrac{e^x - 1}{(e^x+1)(e^{2x}-e^x)}$. Use the graph to estimate $\lim_{x\to 0} f(x)$.

57. Graph $f(x) = x^3 - x^2 + x - 6$. Use the graph to determine the solution of

$$x^3 - x^2 + x - 6 \geq 0$$

58. Graph $f(x) = \dfrac{x^5-4}{x^3+1}$. Use the graph to determine the solution of

$$\frac{x^5-4}{x^3+1} \leq 0$$

EXPLORE & EXTEND National Debt

The size of the U.S. national debt is of great concern to many people and is frequently a topic in the news. The magnitude of the debt affects the confidence in the U.S. economy of both domestic and foreign investors, corporate officials, and political leaders. There are those who believe that to reduce the debt there must be cuts in government spending, which could affect government programs, or there must be an increase in revenues, possibly through tax increases.

Suppose that it is possible for the debt to be reduced continuously at an annual fixed rate. This is similar to compounding interest continuously, as studied in Chapter 5, except that instead of adding interest to an amount at each instant of time, you would be subtracting from the debt at each instant. Let us see how you could model this situation.

Suppose the debt D_0 at time $t = 0$ is reduced at an annual rate r. Furthermore, assume that there are k time periods of equal length in a year. At the end of the first period, the original debt is reduced by $D_0\left(\frac{r}{k}\right)$, so the new debt is

$$D_0 - D_0\left(\frac{r}{k}\right) = D_0\left(1 - \frac{r}{k}\right)$$

At the end of the second period, this debt is reduced by $D_0\left(1 - \frac{r}{k}\right)\frac{r}{k}$, so the new debt is

$$\begin{aligned} D_0\left(1-\frac{r}{k}\right) - D_0\left(1-\frac{r}{k}\right)\frac{r}{k} &\\ = D_0\left(1-\frac{r}{k}\right)\left(1-\frac{r}{k}\right) &\\ = D_0\left(1-\frac{r}{k}\right)^2 & \end{aligned}$$

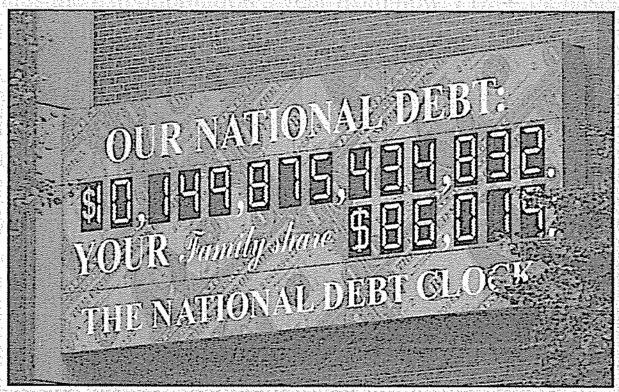
The pattern continues. At the end of the third period the debt is $D_0\left(1-\frac{r}{k}\right)^3$, and so on. At the end of t years the number of periods is kt and the debt is $D_0\left(1-\frac{r}{k}\right)^{kt}$. If the debt is to be reduced at each instant of time, then $k \to \infty$.

Thus you want to find

$$\lim_{k\to\infty} D_0\left(1-\frac{r}{k}\right)^{kt}$$

which can be rewritten as

$$D_0\left[\lim_{k\to\infty}\left(1-\frac{r}{k}\right)^{-k/r}\right]^{-rt}$$



If you let $x = -r/k$, then the condition $k \to \infty$ implies that $x \to 0$. Hence the limit inside the brackets has the form $\lim_{x\to 0}(1+x)^{1/x}$, which, as we pointed out in Section 10.1, is e. Therefore, if the debt D_0 at time $t = 0$ is reduced continuously at an annual rate r, then t years later the debt D is given by

$$D = D_0 e^{-rt}$$

For example, assume the U.S. national debt of \$11,195 billion (rounded to the nearest billion) in the middle of April 2009 and a continuous reduction rate of 3% annually. Then the debt t years from now is given by

$$D = 11{,}195e^{-0.03t}$$

where D is in billions of dollars. This means that in 10 years, the debt will be $11{,}195e^{-0.3} \approx \$8293$ billion. Figure 10.46 shows the graph of $D = 11{,}195e^{-rt}$ for various rates r. Of course, the greater the value of r, the faster the debt reduction. Notice that for $r = 0.03$, the debt at the end of 30 years is still considerable (approximately \$4552 billion).

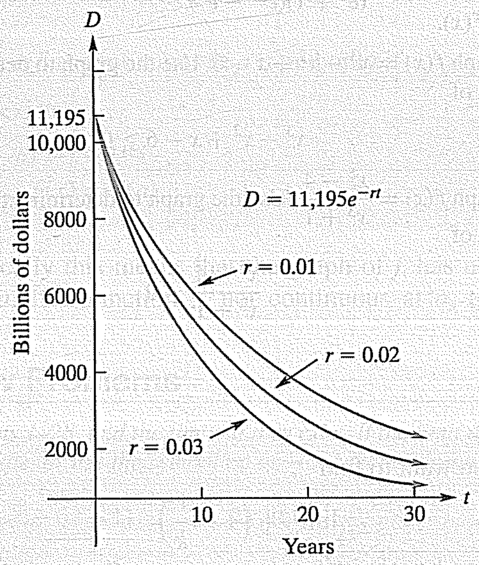


FIGURE 10.46 Budget debt reduced continuously.

It is interesting to note that decaying radioactive elements also follow the model of continuous debt reduction, $D = D_0e^{-rt}$.

To find out where the U.S. national debt currently stands, visit one of the national debt clocks on the Internet. You can find them by looking for "national debt clock" using any search engine.

Problems

In the following problems, assume a current national debt of \$11,195 billion.

1. If the debt were reduced to \$10,000 billion a year from now, what annual rate of continuous debt reduction would be involved? Give your answer to the nearest percent.

2. For a continuous debt reduction at an annual rate of 3%, determine the number of years from now required for the debt to be reduced by one-half. Give your answer to the nearest year.

3. What assumptions underlie a model of debt reduction that uses an exponential function?

11 Differentiation

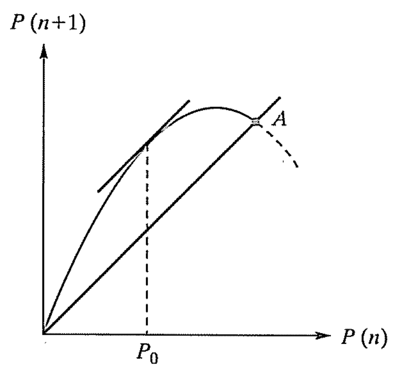
Government regulations generally limit the number of fish taken from a given fishing ground by commercial fishing boats in a season. This prevents overfishing, which depletes the fish population and leaves, in the long run, fewer fish to catch.

From a strictly commercial perspective, the ideal regulations would maximize the number of fish available for the year-to-year fish harvest. The key to finding those ideal regulations is a mathematical function called the reproduction curve. For a given fish habitat, this function estimates the fish population a year from now, $P(n+1)$, based on the population now, $P(n)$, assuming no external interventions such as fishing or influx of predators.

The figure to the bottom left shows a typical reproduction curve. Also graphed is the line $P(n+1) = P(n)$, the line along which the populations $P(n+1)$ and $P(n)$ would be equal. Notice the intersection of the curve and the straight line at point A. This is where, because of habitat crowding, the population has reached its maximum sustainable size. A population that is this size one year will be the same size the next year.

For any point on the horizontal axis, the distance between the reproduction curve and the line $P(n+1) = P(n)$ represents the sustainable harvest: the number of fish that could be caught, after the spawn have grown to maturity, so that in the end the population is back at the same size it was a year ago.

Commercially speaking, the optimal population size is the one where the distance between the reproduction curve and the line $P(n+1) = P(n)$ is the greatest. This condition is met where the slopes of the reproduction curve and the line $P(n+1) = P(n)$ are equal. [The slope of $P(n+1) = P(n)$ is, of course, 1.] Thus, for a maximum fish harvest year after year, regulations should aim to keep the fish population fairly close to P_0.



A central idea here is that of the slope of a curve at a given point. That idea is the cornerstone concept of this chapter.

Now we begin our study of calculus. The ideas involved in calculus are completely different from those of algebra and geometry. The power and importance of these ideas and their applications will become clear later in the book. In this chapter we introduce the *derivative* of a function and the important rules for finding derivatives. We also show how the derivative is used to analyze the rate of change of a quantity, such as the rate at which the position of a body is changing.

Objective

To develop the idea of a tangent line to a curve, to define the slope of a curve, and to define a derivative and give it a geometric interpretation. To compute derivatives by using the limit definition.

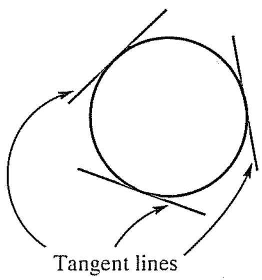


FIGURE 11.1 Tangent lines to a circle.

11.1 The Derivative

The main problem of differential calculus deals with finding the slope of the *tangent line* at a point on a curve. In high school geometry a tangent line, or *tangent*, to a circle is often defined as a line that meets the circle at exactly one point (Figure 11.1). However, this idea of a tangent is not very useful for other kinds of curves. For example, in Figure 11.2(a), the lines L_1 and L_2 intersect the curve at exactly one point P. Although we would not think of L_2 as the tangent at this point, it seems natural that L_1 is. In Figure 11.2(b) we intuitively would consider L_3 to be the tangent at point P, even though L_3 intersects the curve at other points.

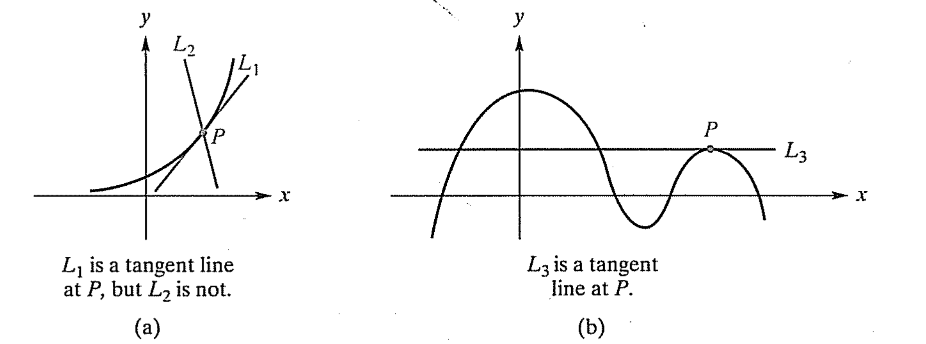


FIGURE 11.2 Tangent line at a point.

From these examples, we see that the idea of a tangent as simply a line that intersects a curve at only one point is inadequate. To obtain a suitable definition of tangent line, we use the limit concept and the geometric notion of a *secant line*. A **secant line** is a line that intersects a curve at two or more points.

Look at the graph of the function $y = f(x)$ in Figure 11.3. We wish to define the tangent line at point P. If Q is a different point on the curve, the line PQ is a secant line. If Q moves along the curve and approaches P from the right (see Figure 11.4), typical secant lines are PQ', PQ'', and so on. As Q approaches P from the left, typical secant lines are PQ_1, PQ_2, and so on. *In both cases, the secant lines approach the same limiting position.* This common limiting position of the secant lines is defined to be the **tangent line** to the curve at P. This definition seems reasonable and applies to curves in general, not just circles.

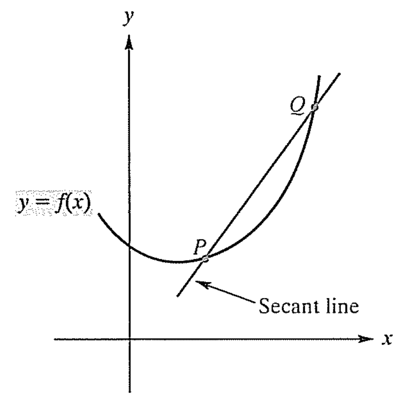


FIGURE 11.3 Secant line PQ.

A curve does not necessarily have a tangent line at each of its points. For example, the curve $y = |x|$ does not have a tangent at (0,0). As can be seen in Figure 11.5, a secant line through (0,0) and a nearby point to its right on the curve must always be the line

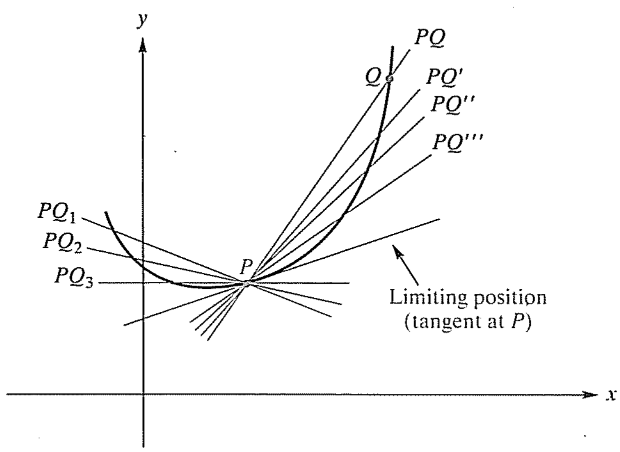


FIGURE 11.4 The tangent line is a limiting position of secant lines.

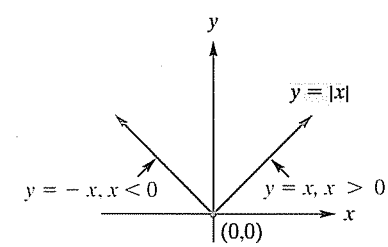


FIGURE 11.5 No tangent line to graph of $y = |x|$ at (0,0).

$y = x$. Thus the limiting position of such secant lines is also the line $y = x$. However, a secant line through (0,0) and a nearby point to its left on the curve must always be the line $y = -x$. Hence, the limiting position of such secant lines is also the line $y = -x$. Since there is no common limiting position, there is no tangent line at (0,0).

Now that we have a suitable definition of a tangent to a curve at a point, we can define the *slope of a curve* at a point.

Definition

The **slope of a curve** at a point P is the slope, if it exists, of the tangent line at P.

Since the tangent at P is a limiting position of secant lines PQ, we consider the slope of the tangent to be the limiting value of the slopes of the secant lines as Q approaches P. For example, let us consider the curve $f(x) = x^2$ and the slopes of some secant lines PQ, where $P = (1, 1)$. For the point $Q = (2.5, 6.25)$, the slope of PQ (see Figure 11.6) is

$$m_{PQ} = \frac{\text{rise}}{\text{run}} = \frac{6.25 - 1}{2.5 - 1} = 3.5$$

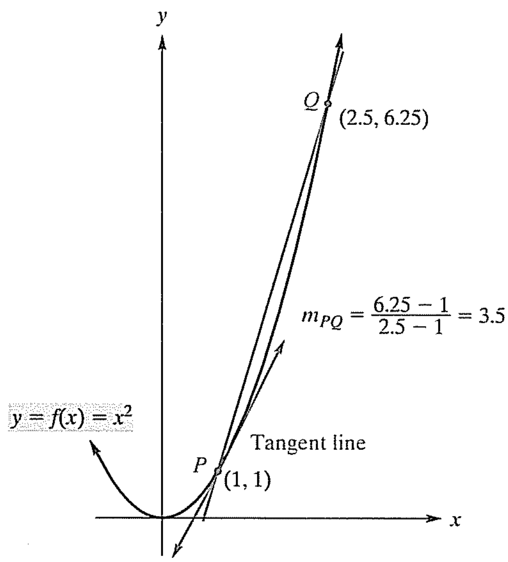


FIGURE 11.6 Secant line to $f(x) = x^2$ through (1, 1) and (2.5, 6.25).

Table 11.1 includes other points Q on the curve, as well as the corresponding slopes of PQ. Notice that as Q approaches P, the slopes of the secant lines seem to approach 2. Thus, we expect the slope of the indicated tangent line at (1, 1) to be 2. This will be confirmed later, in Example 1. But first, we wish to generalize our procedure.

Table 11.1 Slopes of Secant Lines to the Curve $f(x) = x^2$ at $P = (1, 1)$

Q	Slope of PQ
(2.5, 6.25)	$(6.25 - 1)/(2.5 - 1) = 3.5$
(2, 4)	$(4 - 1)/(2 - 1) = 3$
(1.5, 2.25)	$(2.25 - 1)/(1.5 - 1) = 2.5$
(1.25, 1.5625)	$(1.5625 - 1)/(1.25 - 1) = 2.25$
(1.1, 1.21)	$(1.21 - 1)/(1.1 - 1) = 2.1$
(1.01, 1.0201)	$(1.0201 - 1)/(1.01 - 1) = 2.01$

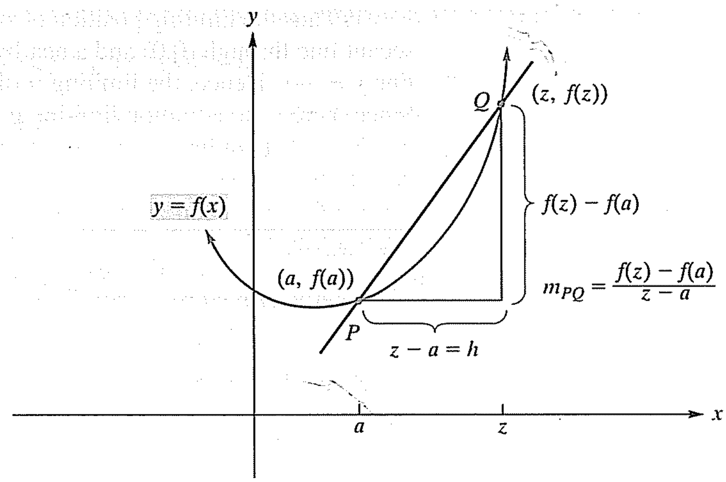


FIGURE 11.7 Secant line through P and Q.

For the curve $y = f(x)$ in Figure 11.7, we will find an expression for the slope at the point $P = (a, f(a))$. If $Q = (z, f(z))$, the slope of the secant line PQ is

$$m_{PQ} = \frac{f(z) - f(a)}{z - a}$$

If the difference $z - a$ is called h, then we can write z as $a + h$. Here we must have $h \neq 0$, for if $h = 0$, then $z = a$, and no secant line exists. Accordingly,

$$m_{PQ} = \frac{f(z) - f(a)}{z - a} = \frac{f(a + h) - f(a)}{h}$$

Which of these two forms for m_{PQ} is most convenient depends on the nature of the function f. As Q moves along the curve toward P, z approaches a. This means that h approaches zero. The limiting value of the slopes of the secant lines—which is the slope of the tangent line at $(a, f(a))$—is

$$m_{\tan} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad (1)$$

Again, which of these two forms is most convenient—which limit is easiest to determine—depends on the nature of the function f. In Example 1, we will use this limit to confirm our previous expectation that the slope of the tangent line to the curve $f(x) = x^2$ at (1, 1) is 2.

EXAMPLE 1 Finding the Slope of a Tangent Line

Find the slope of the tangent line to the curve $y = f(x) = x^2$ at the point (1, 1).

Solution: The slope is the limit in Equation (1) with $f(x) = x^2$ and $a = 1$:

$$\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{(1 + h)^2 - (1)^2}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2 + h)}{h} = \lim_{h \to 0} (2 + h) = 2$$

Therefore, the tangent line to $y = x^2$ at (1, 1) has slope 2. (Refer to Figure 11.6.)

Now Work Problem 1 ◁

We can generalize Equation (1) so that it applies to any point $(x,f(x))$ on a curve. Replacing a by x gives a function, called the *derivative* of f, whose input is x and whose output is the slope of the tangent line to the curve at $(x,f(x))$, provided that the tangent line *exists* and *has* a slope. (If the tangent line exists but is *vertical*, then it has no slope.) We thus have the following definition, which forms the basis of differential calculus:

Definition

The **derivative** of a function f is the function denoted f' (read "f prime") and defined by

$$f'(x) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \tag{2}$$

provided that this limit exists. If $f'(a)$ can be found [while perhaps not all $f'(x)$ can be found] f is said to be ***differentiable*** at a, and $f'(a)$ is called the derivative of f at a or the derivative of f with respect to x at a. The process of finding the derivative is called ***differentiation.***

In the definition of the derivative, the expression

$$\frac{f(z)-f(x)}{z-x} = \frac{f(x+h)-f(x)}{h}$$

where $z = x+h$, is called a **difference quotient.** Thus $f'(x)$ is the limit of a difference quotient.

EXAMPLE 2 Using the Definition to Find the Derivative

If $f(x) = x^2$, find the derivative of f.

Calculating a derivative via the definition requires precision. Typically, the difference quotient requires considerable manipulation before the limit step is taken. This requires that each written step be preceded by "$\lim_{h\to 0}$" to acknowledge that the limit step is still pending. Observe that after the limit step is taken, h will no longer be present.

Solution: Applying the definition of a derivative gives

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x \end{aligned}$$

Observe that, in taking the limit, we treated x as a constant, because it was h, not x, that was changing. Also, note that $f'(x) = 2x$ defines a function of x, which we can interpret as giving the slope of the tangent line to the graph of f at $(x,f(x))$. For example, if $x = 1$, then the slope is $f'(1) = 2 \cdot 1 = 2$, which confirms the result in Example 1.

Now Work Problem 3 ◁

Besides the notation $f'(x)$, other common ways to denote the derivative of $y = f(x)$ at x are

$\dfrac{dy}{dx}$	pronounced "dee y, dee x" or "dee y by dee x"
$\dfrac{d}{dx}(f(x))$	"dee $f(x)$, dee x" or "dee by dee x of $f(x)$"
y'	"y prime"
D_xy	"dee x of y"
$D_x(f(x))$	"dee x of $f(x)$"

CAUTION!

The notation $\dfrac{dy}{dx}$, which is called *Leibniz notation*, should **not** be thought of as a fraction, although it looks like one. It is a single symbol for a derivative. We have not yet attached any meaning to individual symbols such as dy and dx.

Because the derivative gives the slope of the tangent line, $f'(a)$ is the slope of the line tangent to the graph of $y = f(x)$ at $(a,f(a))$.

Two other notations for the derivative of f at a are

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{and} \quad y'(a)$$

EXAMPLE 3 Finding an Equation of a Tangent Line

If $f(x) = 2x^2 + 2x + 3$, find an equation of the tangent line to the graph of f at $(1, 7)$.

Solution:

Strategy We will first determine the slope of the tangent line by computing the derivative and evaluating it at $x = 1$. Using this result and the point $(1, 7)$ in a point-slope form gives an equation of the tangent line.

We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(2(x+h)^2 + 2(x+h) + 3) - (2x^2 + 2x + 3)}{h} \\ &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + 2x + 2h + 3 - 2x^2 - 2x - 3}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 + 2h}{h} = \lim_{h\to 0}(4x + 2h + 2) \end{aligned}$$

In Example 3 it is *not* correct to say that, since the derivative is $4x + 2$, the tangent line at $(1, 7)$ is $y - 7 = (4x + 2)(x - 1)$. (This is not even the equation of a line.) The derivative must be **evaluated** at the point of tangency to determine the slope of the tangent line.

So

$$f'(x) = 4x + 2$$

and

$$f'(1) = 4(1) + 2 = 6$$

Thus, the tangent line to the graph at (1, 7) has slope 6. A point-slope form of this tangent is

$$y - 7 = 6(x - 1)$$

which in slope-intercept form is

$$y = 6x + 1$$

Now Work Problem 25 ◁

EXAMPLE 4 Finding the Slope of a Curve at a Point

Find the slope of the curve $y = 2x + 3$ at the point where $x = 6$.

Solution: The slope of the curve is the slope of the tangent line. Letting $y = f(x) = 2x + 3$, we have

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(2(x+h) + 3) - (2x + 3)}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2 \end{aligned}$$

Since $dy/dx = 2$, the slope when $x = 6$, or in fact at any point, is 2. Note that the curve is a straight line and thus has the same slope at each point.

Now Work Problem 19 ◁

EXAMPLE 5 A Function with a Vertical Tangent Line

Find $\dfrac{d}{dx}(\sqrt{x})$.

Solution: Letting $f(x) = \sqrt{x}$, we have

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Rationalizing numerators or denominators of fractions is often helpful in calculating limits.

As $h \to 0$, both the numerator and denominator approach zero. This can be avoided by rationalizing the *numerator*:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

Therefore,

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

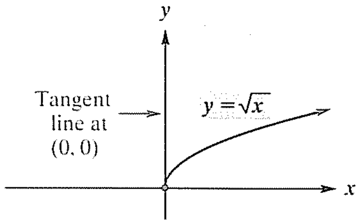


FIGURE 11.8 Vertical tangent line at (0, 0).

Note that the original function, $\sqrt{x}$, is defined for $x \geq 0$, but its derivative, $1/(2\sqrt{x})$, is defined only when $x > 0$. The reason for this is clear from the graph of $y = \sqrt{x}$ in Figure 11.8. When $x = 0$, the tangent is a vertical line, so its slope is not defined.

Now Work Problem 17 ◁

In Example 5 we saw that the function $y = \sqrt{x}$ is not differentiable when $x = 0$, because the tangent line is vertical at that point. It is worthwhile to mention that $y = |x|$ also is not differentiable when $x = 0$, but for a different reason: There is *no* tangent line at all at that point. (Refer to Figure 11.5.) Both examples show that the domain of f' may be strictly contained in the domain of f.

Variables other than x and y are often more natural in applied problems. Time denoted by t, quantity by q, and price by p are obvious examples. Example 6 illustrates.

To indicate a derivative, Leibniz notation is often useful because it makes it convenient to emphasize the independent and dependent variables involved. For example, if the variable p is a function of the variable q, we speak of the derivative of p with respect to q, written dp/dq.

APPLY IT ▷

1. If a ball is thrown upward at a speed of 40 ft/s from a height of 6 feet, its height H in feet after t seconds is given by $H = 6 + 40t - 16t^2$. Find $\frac{dH}{dt}$.

EXAMPLE 6 Finding the Derivative of p with Respect to q

If $p = f(q) = \frac{1}{2q}$, find $\frac{dp}{dq}$.

Solution: We will do this problem first using the $h \to 0$ limit (the only one we have used so far) and then using $r \to q$ to illustrate the other variant of the limit.

$$\frac{dp}{dq} = \frac{d}{dq}\left(\frac{1}{2q}\right) = \lim_{h\to 0}\frac{f(q+h)-f(q)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{2(q+h)}-\frac{1}{2q}}{h} = \lim_{h\to 0}\frac{\frac{q-(q+h)}{2q(q+h)}}{h}$$

$$= \lim_{h\to 0}\frac{q-(q+h)}{h(2q(q+h))} = \lim_{h\to 0}\frac{-h}{h(2q(q+h))}$$

$$= \lim_{h\to 0}\frac{-1}{2q(q+h)} = -\frac{1}{2q^2}$$

We also have

$$\frac{dp}{dq} = \lim_{r\to q}\frac{f(r)-f(q)}{r-q}$$

$$= \lim_{r\to q}\frac{\frac{1}{2r}-\frac{1}{2q}}{r-q} = \lim_{r\to q}\frac{\frac{q-r}{2rq}}{r-q}$$

$$= \lim_{r\to q}\frac{-1}{2rq} = \frac{-1}{2q^2}$$

We leave it you to decide which form leads to the simpler limit calculation in this case.

Note that when $q = 0$ the function is not defined, so the derivative is also not even defined when $q = 0$.

Now Work Problem 15 ◁

Keep in mind that the derivative of $y = f(x)$ at x is nothing more than a limit, namely

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

equivalently

$$\lim_{z\to x} \frac{f(z)-f(x)}{z-x}$$

whose use we have just illustrated. Although we can interpret the derivative as a function that gives the slope of the tangent line to the curve $y = f(x)$ at the point $(x, f(x))$, this interpretation is simply a geometric convenience that assists our understanding. The preceding limit may exist, aside from any geometric considerations at all. As we will see later, there are other useful interpretations of the derivative.

In Section 11.4, we will make technical use of the following relationship between differentiability and continuity. However, it is of fundamental importance and needs to be understood from the outset.

If f is differentiable at a, then f is continuous at a.

To establish this result, we will assume that f is differentiable at a. Then $f'(a)$ exists, and

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

Consider the numerator $f(a+h) - f(a)$ as $h \to 0$. We have

$$\begin{aligned}\lim_{h\to 0}(f(a+h)-f(a)) &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\cdot h\right)\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\lim_{h\to 0} h\\ &= f'(a)\cdot 0 = 0\end{aligned}$$

Thus, $\lim_{h\to 0}(f(a+h)-f(a)) = 0$. This means that $f(a+h)-f(a)$ approaches 0 as $h \to 0$. Consequently,

$$\lim_{h\to 0} f(a+h) = f(a)$$

As stated in Section 10.3, this condition means that f is continuous at a. The foregoing, then, proves that f is continuous at a when f is differentiable there. More simply, we say that **differentiability at a point implies continuity at that point.**

If a function is not continuous at a point, then it cannot have a derivative there. For example, the function in Figure 11.9 is discontinuous at a. The curve has no tangent at that point, so the function is not differentiable there.

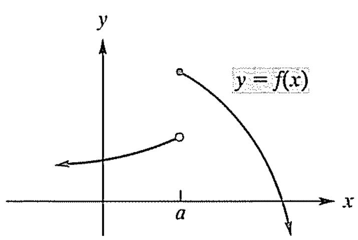


FIGURE 11.9 f is not continuous at a, so f is not differentiable at a.

EXAMPLE 7 Continuity and Differentiability

a. Let $f(x) = x^2$. The derivative, $2x$, is defined for all values of x, so $f(x) = x^2$ must be continuous for all values of x.

b. The function $f(p) = \dfrac{1}{2p}$ is not continuous at $p = 0$ because f is not defined there. Thus, the derivative does not exist at $p = 0$.

◁

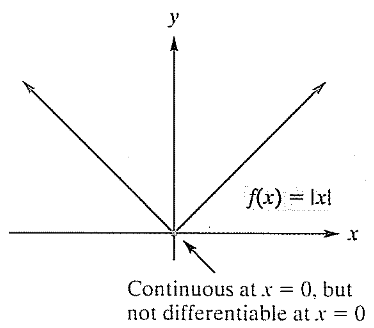


FIGURE 11.10 Continuity does not imply differentiability.

The converse of the statement that differentiability implies continuity is *false*. That is, continuity does not imply differentiability. In Example 8, we give a function that is continuous at a point, but not differentiable there.

EXAMPLE 8 Continuity Does Not Imply Differentiability

The function $y = f(x) = |x|$ is continuous at $x = 0$. (See Figure 11.10.) As we mentioned earlier, there is no tangent line at $x = 0$. Thus, the derivative does not exist there. This shows that continuity does *not* imply differentiability.

◁

Finally, we remark that while differentiability of f at a implies continuity of f at a, the derivative function, f', is not necessarily continuous at a. Unfortunately, the classic example is constructed from a function not considered in this book.

PROBLEMS 11.1

In Problems 1 and 2, a function f and a point P on its graph are given.

(a) *Find the slope of the secant line PQ for each point $Q = (x, f(x))$ whose x-value is given in the table. Round your answers to four decimal places.*

(b) *Use your results from part (a) to estimate the slope of the tangent line at P.*

1. $f(x) = x^3 + 3, P = (-2, -5)$

x-value of Q	−3	−2.5	−2.2	−2.1	−2.01	−2.001
m_{PQ}						

2. $f(x) = e^x, P = (0, 1)$

x-value of Q	1	0.5	0.2	0.1	0.01	0.001
m_{PQ}						

In Problems 3–18, use the definition of the derivative to find each of the following.

3. $f'(x)$ if $f(x) = x$

4. $f'(x)$ if $f(x) = 4x - 1$

5. $\dfrac{dy}{dx}$ if $y = 3x + 5$

6. $\dfrac{dy}{dx}$ if $y = -5x$

7. $\dfrac{d}{dx}(3 - 2x)$

8. $\dfrac{d}{dx}\left(1 - \dfrac{x}{2}\right)$

9. $f'(x)$ if $f(x) = 3$

10. $f'(x)$ if $f(x) = 7.01$

11. $\dfrac{d}{dx}(x^2 + 4x - 8)$

12. y' if $y = x^2 + 3x + 2$

13. $\dfrac{dp}{dq}$ if $p = 3q^2 + 2q + 1$

14. $\dfrac{d}{dx}(x^2 - x - 3)$

15. y' if $y = \dfrac{6}{x}$

16. $\dfrac{dC}{dq}$ if $C = 7 + 2q - 3q^2$

17. $f'(x)$ if $f(x) = \sqrt{2x}$

18. $H'(x)$ if $H(x) = \dfrac{3}{x - 2}$

19. Find the slope of the curve $y = x^2 + 4$ at the point $(-2, 8)$.

20. Find the slope of the curve $y = 1 - x^2$ at the point $(1, 0)$.

21. Find the slope of the curve $y = 4x^2 - 5$ when $x = 0$.

22. Find the slope of the curve $y = \sqrt{2x}$ when $x = 18$.

In Problems 23–28, find an equation of the tangent line to the curve at the given point.

23. $y = x + 4; (3, 7)$

24. $y = 3x^2 - 4; (1, -1)$

25. $y = x^2 + 2x + 3; (1, 6)$

26. $y = (x - 7)^2; (6, 1)$

27. $y = \dfrac{4}{x + 1}; (3, 1)$

28. $y = \dfrac{5}{1 - 3x}; (2, -1)$

29. Banking Equations may involve derivatives of functions. In an article on interest rate deregulation, Christofi and Agapos[1] solve the equation

$$r = \left(\frac{\eta}{1 + \eta}\right)\left(r_L - \frac{dC}{dD}\right)$$

for η (the Greek letter "eta"). Here r is the deposit rate paid by commercial banks, r_L is the rate earned by commercial banks, C is the administrative cost of transforming deposits into return-earning assets, D is the savings deposits level, and η is the deposit elasticity with respect to the deposit rate. Find η.

In Problems 30 and 31, use the numerical derivative feature of your graphing calculator to estimate the derivatives of the functions at the indicated values. Round your answers to three decimal places.

30. $f(x) = \sqrt{2x^2 + 3x}; x = 1, x = 2$

31. $f(x) = e^x(4x - 7); x = 0, x = 1.5$

In Problems 32 and 33, use the "limit of a difference quotient" approach to estimate $f'(x)$ at the indicated values of x. Round your answers to three decimal places.

32. $f(x) = x \ln x - x; x = 1, x = 10$

33. $f(x) = \dfrac{x^2 + 4x + 2}{x^3 - 3}; x = 2, x = -4$

[1] A. Christofi and A. Agapos, "Interest Rate Deregulation: An Empirical Justification," *Review of Business and Economic Research*, XX, no. 1 (1984), 39–49.

34. Find an equation of the tangent line to the curve $f(x) = x^2 + x$ at the point $(-2, 2)$. Graph both the curve and the tangent line. Notice that the tangent line is a good approximation to the curve near the point of tangency.

35. The derivative of $f(x) = x^3 - x + 2$ is $f'(x) = 3x^2 - 1$. Graph both the function f and its derivative f'. Observe that there are two points on the graph of f where the tangent line is horizontal. For the x-values of these points, what are the corresponding values of $f'(x)$? Why are these results expected? Observe the intervals where $f'(x)$ is positive. Notice that tangent lines to the graph of f have positive slopes over these intervals. Observe the interval where $f'(x)$ is negative. Notice that tangent lines to the graph of f have negative slopes over this interval.

In Problems 36 and 37, verify the identity $(z - x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) = z^n - x^n$ *for the indicated values of n and calculate the derivative using the* $z \to x$ *form of the definition of the derivative in Equation (2).*

36. $n = 4, n = 3, n = 2;\quad f'(x)$ if $f(x) = 2x^4 + x^3 - 3x^2$

37. $n = 5, n = 3;\quad f'(x)$ if $f(x) = 4x^5 - 3x^3$

11.2 Rules for Differentiation

Objective

To develop the basic rules for differentiating constant functions and power functions and the combining rules for differentiating a constant multiple of a function and a sum of two functions.

Differentiating a function by direct use of the definition of derivative can be tedious. However, if a function is constructed from simpler functions, then the derivative of the more complicated function can be constructed from the derivatives of the simpler functions. Ultimately, we need to know only the derivatives of a few basic functions and ways to assemble derivatives of constructed functions from the derivatives of their components. For example, if functions f and g have derivatives f' and g', respectively, then $f + g$ has a derivative given by $(f + g)' = f' + g'$. However, some *rules* are less intuitive. For example, if $f \cdot g$ denotes the function whose value at x is given by $(f \cdot g)(x) = f(x) \cdot g(x)$, then $(f \cdot g)' = f' \cdot g + f \cdot g'$. In this chapter we study most such combining rules and some basic rules for calculating derivatives of certain basic functions.

We begin by showing that the derivative of a constant function is zero. Recall that the graph of the constant function $f(x) = c$ is a horizontal line (see Figure 11.11), which has a slope of zero at each point. This means that $f'(x) = 0$ regardless of x. As a formal proof of this result, we apply the definition of the derivative to $f(x) = c$:

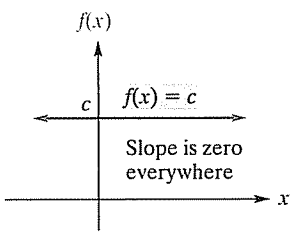


FIGURE 11.11 The slope of a constant function is 0.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

Thus, we have our first rule:

BASIC RULE 0 Derivative of a Constant

If c is a constant, then

$$\frac{d}{dx}(c) = 0$$

That is, the derivative of a constant function is zero.

EXAMPLE 1 Derivatives of Constant Functions

a. $\dfrac{d}{dx}(3) = 0$ because 3 is a constant function.

b. If $g(x) = \sqrt{5}$, then $g'(x) = 0$ because g is a constant function. For example, the derivative of g when $x = 4$ is $g'(4) = 0$.

c. If $s(t) = (1{,}938{,}623)^{807.4}$, then $ds/dt = 0$.

Now Work Problem 1 ◁

The next rule gives a formula for the derivative of "x raised to a constant power"—that is, the derivative of $f(x) = x^a$, where a is an arbitrary real number. A function of this form is called a **power function.** For example, $f(x) = x^2$ is a power function. While the rule we record is valid for all real a, we will establish it only in the case where

a is a positive integer, n. The rule is so central to differential calculus that it warrants a detailed calculation—if only in the case where a is a positive integer, n. Whether we use the $h \to 0$ form of the definition of derivative or the $z \to x$ form, the calculation of $\frac{dx^n}{dx}$ is instructive and provides good practice with summation notation, whose use is more essential in later chapters. We provide a calculation for each possibility. We must either expand $(x+h)^n$, to use the $h \to 0$ form of Equation (2) from Section 11.1, or factor $z^n - x^n$, to use the $z \to x$ form.

For the first of these we recall the *binomial theorem* of Section 9.2:

$$(x+h)^n = \sum_{i=0}^{n} {}_nC_i x^{n-i} h^i$$

where the ${}_nC_i$ are the binomial coefficients, whose precise descriptions, except for ${}_nC_0 = 1$ and ${}_nC_1 = n$, are not necessary here (but are given in Section 8.2). For the second we have

$$(z-x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) = z^n - x^n$$

which is easily verified by carrying out the multiplication using the rules for manipulating summations given in Section 1.5. In fact, we have

$$\begin{aligned}
(z-x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) &= z\sum_{i=0}^{n-1} x^i z^{n-1-i} - x\sum_{i=0}^{n-1} x^i z^{n-1-i} \\
&= \sum_{i=0}^{n-1} x^i z^{n-i} - \sum_{i=0}^{n-1} x^{i+1} z^{n-1-i} \\
&= \left(z^n + \sum_{i=1}^{n-1} x^i z^{n-i}\right) - \left(\sum_{i=0}^{n-2} x^{i+1} z^{n-1-i} + x^n\right) \\
&= z^n - x^n
\end{aligned}$$

where the reader should check that the two summations in the second to last line really do cancel as shown.

BASIC RULE 1 Derivative of x^a

If a is any real number, then

$$\frac{d}{dx}(x^a) = ax^{a-1}$$

That is, the derivative of a constant power of x is the exponent times x raised to a power one less than the given power.

CAUTION!

There is a lot more to calculus than this rule.

For n a positive integer, if $f(x) = x^n$, the definition of the derivative gives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

By our previous discussion on expanding $(x+h)^n$,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sum_{i=0}^{n} {}_nC_i x^{n-i} h^i - x^n}{h} \\
&\overset{(1)}{=} \lim_{h\to 0} \frac{\sum_{i=1}^{n} {}_nC_i x^{n-i} h^i}{h}
\end{aligned}$$

$$\overset{(2)}{=} \lim_{h \to 0} \frac{h \sum_{i=1}^{n} {}_nC_i x^{n-i} h^{i-1}}{h}$$

$$\overset{(3)}{=} \lim_{h \to 0} \sum_{i=1}^{n} {}_nC_i x^{n-i} h^{i-1}$$

$$\overset{(4)}{=} \lim_{h \to 0} \left(nx^{n-1} + \sum_{i=2}^{n} {}_nC_i x^{n-i} h^{i-1} \right)$$

$$\overset{(5)}{=} nx^{n-1}$$

where we justify the further steps as follows:

(1) The $i = 0$ term in the summation is ${}_nC_0 x^n h^0 = x^n$ so it cancels with the separate, last, term: $-x^n$.

(2) We are able to extract a common factor of h from each term in the sum.

(3) This is the crucial step. The expressions separated by the equal sign are limits as $h \to 0$ of functions of h that are equal for $h \neq 0$.

(4) The $i = 1$ term in the summation is ${}_nC_1 x^{n-1} h^0 = nx^{n-1}$. It is the only one that does not contain a factor of h, and we separated it from the other terms.

(5) Finally, in determining the limit we made use of the fact that the isolated term is independent of h; while all the others contain h as a factor and so have limit 0 as $h \to 0$.

Now, using the $z \to x$ limit for the definition of the derivative and $f(x) = x^n$, we have

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{h \to 0} \frac{z^n - x^n}{z - x}$$

By our previous discussion on factoring $z^n - x^n$, we have

$$f'(x) = \lim_{z \to x} \frac{(z - x) \left(\sum_{i=0}^{n-1} x^i z^{n-1-i} \right)}{z - x}$$

$$\overset{(1)}{=} \lim_{z \to x} \sum_{i=0}^{n-1} x^i z^{n-1-i}$$

$$\overset{(2)}{=} \sum_{i=0}^{n-1} x^i x^{n-1-i}$$

$$\overset{(3)}{=} \sum_{i=0}^{n-1} x^{n-1}$$

$$\overset{(4)}{=} nx^{n-1}$$

where this time we justify the further steps as follows:

(1) Here the crucial step comes first. The expressions separated by the equal sign are limits as $z \to x$ of functions of z that are equal for $z \neq x$.

(2) The limit is given by evaluation because the expression is a polynomial in the variable z.

(3) An obvious rule for exponents is used.

(4) Each term in the sum is x^{n-1}, independent of i, and there are n such terms.

EXAMPLE 2 Derivatives of Powers of x

a. By Basic Rule 1, $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$.

b. If $F(x) = x = x^1$, then $F'(x) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$. Thus, the derivative of x with respect to x is 1.

c. If $f(x) = x^{-10}$, then $f'(x) = -10x^{-10-1} = -10x^{-11}$.

Now Work Problem 3 ◁

When we apply a differentiation rule to a function, sometimes the function must first be rewritten so that it has the proper form for that rule. For example, to differentiate $f(x) = \frac{1}{x^{10}}$ we would first rewrite f as $f(x) = x^{-10}$ and then proceed as in Example 2(c).

EXAMPLE 3 Rewriting Functions in the Form x^a

a. To differentiate $y = \sqrt{x}$, we rewrite $\sqrt{x}$ as $x^{1/2}$ so that it has the form x^n. Thus,

$$\frac{dy}{dx} = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

which agrees with our limit calculation in Example 5 of Section 11.1.

CAUTION!

In Example 3(b), do not rewrite $\frac{1}{x\sqrt{x}}$ as $\frac{1}{x^{3/2}}$ and then merely differentiate the denominator.

b. Let $h(x) = \frac{1}{x\sqrt{x}}$. To apply Basic Rule 1, we must rewrite $h(x)$ as $h(x) = x^{-3/2}$ so that it has the form x^n. We have

$$h'(x) = \frac{d}{dx}(x^{-3/2}) = -\frac{3}{2}x^{(-3/2)-1} = -\frac{3}{2}x^{-5/2}$$

Now Work Problem 39 ◁

Now that we can say immediately that the derivative of x^3 is $3x^2$, the question arises as to what we could say about the derivative of a *multiple* of x^3, such as $5x^3$. Our next rule will handle this situation of differentiating a constant times a function.

COMBINING RULE 1 Constant Factor Rule

If f is a differentiable function and c is a constant, then $cf(x)$ is differentiable, and

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

That is, the derivative of a constant times a function is the constant times the derivative of the function.

Proof. If $g(x) = cf(x)$, applying the definition of the derivative of g gives

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= \lim_{h\to 0}\left(c\cdot\frac{f(x+h)-f(x)}{h}\right) = c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is $f'(x)$; so $g'(x) = cf'(x)$.

EXAMPLE 4 Differentiating a Constant Times a Function

Differentiate the following functions.

a. $g(x) = 5x^3$

Solution: Here g is a constant (5) times a function (x^3). So

$$\frac{d}{dx}(5x^3) = 5\frac{d}{dx}(x^3) \quad \text{Combining Rule 1}$$

$$= 5(3x^{3-1}) = 15x^2 \quad \text{Basic Rule 1}$$

b. $f(q) = \dfrac{13q}{5}$

Solution:

Strategy We first rewrite f as a constant times a function and then apply Basic Rule 1.

Because $\frac{13q}{5} = \frac{13}{5}q$, f is the constant $\frac{13}{5}$ times the function q. Thus,

$$f'(q) = \frac{13}{5}\frac{d}{dq}(q) \quad \text{Combining Rule 1}$$

$$= \frac{13}{5} \cdot 1 = \frac{13}{5} \quad \text{Basic Rule 1}$$

c. $y = \dfrac{0.25}{\sqrt[5]{x^2}}$

Solution: We can express y as a constant times a function:

$$y = 0.25 \cdot \frac{1}{\sqrt[5]{x^2}} = 0.25x^{-2/5}$$

Hence,

$$y' = 0.25\frac{d}{dx}(x^{-2/5}) \quad \text{Combining Rule 1}$$

$$= 0.25\left(-\frac{2}{5}x^{-7/5}\right) = -0.1x^{-7/5} \quad \text{Basic Rule 1}$$

Now Work Problem 7 ◁

CAUTION!

In differentiating $f(x) = (4x)^3$, Basic Rule 1 cannot be applied directly. It applies to a power of the variable x, *not* to a power of an expression involving x, such as $4x$. To apply our rules, write $f(x) = (4x)^3 = 4^3x^3 = 64x^3$. Thus,

$$f'(x) = 64\frac{d}{dx}(x^3) = 64(3x^2) = 192x^2.$$

The next rule involves derivatives of sums and differences of functions.

COMBINING RULE 2 Sum or Difference Rule

If f and g are differentiable functions, then $f + g$ and $f - g$ are differentiable, and

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

and

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

That is, the derivative of the sum (difference) of two functions is the sum (difference) of their derivatives.

Proof. For the case of a sum, if $F(x) = f(x) + g(x)$, applying the definition of the derivative of F gives

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h} \qquad \text{regrouping}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right)$$

Because the limit of a sum is the sum of the limits,

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

But these two limits are $f'(x)$ and $g'(x)$. Thus,

$$F'(x) = f'(x) + g'(x)$$

The proof for the derivative of a difference of two functions is similar.

Combining Rule 2 can be extended to the derivative of any number of sums and differences of functions. For example,

$$\frac{d}{dx}[f(x) - g(x) + h(x) + k(x)] = f'(x) - g'(x) + h'(x) + k'(x)$$

APPLY IT ▷

2. If the revenue function for a certain product is $r(q) = 50q - 0.3q^2$, find the derivative of this function, also known as the marginal revenue.

EXAMPLE 5 Differentiating Sums and Differences of Functions

Differentiate the following functions.

a. $F(x) = 3x^5 + \sqrt{x}$

Solution: Here F is the sum of two functions, $3x^5$ and $\sqrt{x}$. Therefore,

$$F'(x) = \frac{d}{dx}(3x^5) + \frac{d}{dx}(x^{1/2}) \qquad \text{Combining Rule 2}$$

$$= 3\frac{d}{dx}(x^5) + \frac{d}{dx}(x^{1/2}) \qquad \text{Combining Rule 1}$$

$$= 3(5x^4) + \frac{1}{2}x^{-1/2} = 15x^4 + \frac{1}{2\sqrt{x}} \qquad \text{Basic Rule 1}$$

b. $f(z) = \dfrac{z^4}{4} - \dfrac{5}{z^{1/3}}$

Solution: To apply our rules, we will rewrite $f(z)$ in the form $f(z) = \frac{1}{4}z^4 - 5z^{-1/3}$. Since f is the difference of two functions,

$$f'(z) = \frac{d}{dz}\left(\frac{1}{4}z^4\right) - \frac{d}{dz}(5z^{-1/3}) \qquad \text{Combining Rule 2}$$

$$= \frac{1}{4}\frac{d}{dz}(z^4) - 5\frac{d}{dz}(z^{-1/3}) \qquad \text{Combining Rule 1}$$

$$= \frac{1}{4}(4z^3) - 5\left(-\frac{1}{3}z^{-4/3}\right) \qquad \text{Basic Rule 1}$$

$$= z^3 + \frac{5}{3}z^{-4/3}$$

c. $y = 6x^3 - 2x^2 + 7x - 8$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(6x^3) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(8)$$

$$= 6\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) - \frac{d}{dx}(8)$$

$$= 6(3x^2) - 2(2x) + 7(1) - 0$$

$$= 18x^2 - 4x + 7$$

Now Work Problem 47 ◁

In Examples 6 and 7, we need to rewrite the given function in a form to which our rules apply.

EXAMPLE 6 Finding a Derivative

Find the derivative of $f(x) = 2x(x^2 - 5x + 2)$ when $x = 2$.

Solution: We multiply and then differentiate each term:

$$f(x) = 2x^3 - 10x^2 + 4x$$
$$f'(x) = 2(3x^2) - 10(2x) + 4(1)$$
$$= 6x^2 - 20x + 4$$
$$f'(2) = 6(2)^2 - 20(2) + 4 = -12$$

Now Work Problem 75 ◁

EXAMPLE 7 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the curve

$$y = \frac{3x^2 - 2}{x}$$

when $x = 1$.

Solution:

Strategy First we find $\frac{dy}{dx}$, which gives the slope of the tangent line at any point. Evaluating $\frac{dy}{dx}$ when $x = 1$ gives the slope of the required tangent line. We then determine the y-coordinate of the point on the curve when $x = 1$. Finally, we substitute the slope and both of the coordinates of the point in point-slope form to obtain an equation of the tangent line.

Rewriting y as a difference of two functions, we have

$$y = \frac{3x^2}{x} - \frac{2}{x} = 3x - 2x^{-1}$$

Thus,

$$\frac{dy}{dx} = 3(1) - 2((-1)x^{-2}) = 3 + \frac{2}{x^2}$$

The slope of the tangent line to the curve when $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 + \frac{2}{1^2} = 5$$

To find the y-coordinate of the point on the curve where $x = 1$, we evaluate $y = \frac{3x^2 - 2}{x}$ at $x = 1$. This gives

$$y = \frac{3(1)^2 - 2}{1} = 1$$

CAUTION!

To obtain the y-value of the point on the curve when $x = 1$, evaluate the *original* function at $x = 1$.

Hence, the point $(1, 1)$ lies on both the curve and the tangent line. Therefore, an equation of the tangent line is

$$y - 1 = 5(x - 1)$$

In slope-intercept form, we have

$$y = 5x - 4$$

Now Work Problem 81 ◁

PROBLEMS 11.2

In Problems 1–74, differentiate the functions.

1. $f(x) = \pi$

2. $f(x) = \left(\frac{6}{7}\right)^{2/3}$

3. $y = x^6$

4. $f(x) = x^{21}$

5. $y = x^{80}$

6. $y = x^{2.1}$

7. $f(x) = 9x^2$

8. $y = 4x^3$

9. $g(w) = 8w^7$

10. $v(x) = x^e$

11. $y = \frac{3}{5}x^6$

12. $f(p) = \sqrt{3}p^4$

13. $f(t) = \dfrac{t^7}{25}$

14. $y = \dfrac{x^7}{7}$

15. $f(x) = x + 3$

16. $f(x) = 5x - e$

17. $f(x) = 4x^2 - 2x + 3$

18. $F(x) = 5x^2 - 9x$

19. $g(p) = p^4 - 3p^3 - 1$

20. $f(t) = -13t^2 + 14t + 1$

21. $y = x^4 - \sqrt[3]{x}$

22. $y = -8x^4 + \ln 2$

23. $y = -13x^3 + 14x^2 - 2x + 3$

24. $V(r) = r^8 - 7r^6 + 3r^2 + 1$

25. $f(x) = 2(13 - x^4)$

26. $\psi(t) = e(t^7 - 5^3)$

27. $g(x) = \dfrac{13 - x^4}{3}$

28. $f(x) = \dfrac{5(x^4 - 6)}{2}$

29. $h(x) = 4x^4 + x^3 - \dfrac{9x^2}{2} + 8x$

30. $k(x) = -2x^2 + \dfrac{5}{3}x + 11$

31. $f(x) = \dfrac{5}{7}x^9 + \dfrac{3}{5}x^7$

32. $p(x) = \dfrac{x^7}{7} + \dfrac{2x}{3}$

33. $f(x) = x^{3/5}$

34. $f(x) = 2x^{-14/5}$

35. $y = x^{3/4} + 2x^{5/3}$

36. $y = 4x^2 - x^{-3/5}$

37. $y = 11\sqrt{x}$

38. $y = \sqrt{x^7}$

39. $f(r) = 6\sqrt[3]{r}$

40. $y = 4\sqrt[8]{x^2}$

41. $f(x) = x^{-6}$

42. $f(s) = 2s^{-3}$

43. $f(x) = x^{-3} + x^{-5} - 2x^{-6}$

44. $f(x) = 100x^{-3} + 10x^{1/2}$

45. $y = \dfrac{1}{x}$

46. $f(x) = \dfrac{3}{x^4}$

47. $y = \dfrac{8}{x^5}$

48. $y = \dfrac{1}{4x^5}$

49. $g(x) = \dfrac{4}{3x^3}$

50. $y = \dfrac{1}{x^2}$

51. $f(t) = \dfrac{3}{5t^3}$

52. $g(x) = \dfrac{7}{9x}$

53. $f(x) = \dfrac{x}{7} + \dfrac{7}{x}$

54. $\Phi(x) = \dfrac{x^3}{3} - \dfrac{3}{x^3}$

55. $f(x) = -9x^{1/3} + 5x^{-2/5}$

56. $f(z) = 5z^{3/4} - 6^2 - 8z^{1/4}$

57. $q(x) = \dfrac{1}{\sqrt[3]{8x^2}}$

58. $f(x) = \dfrac{3}{\sqrt[4]{x^3}}$

59. $y = \dfrac{2}{\sqrt{x}}$

60. $y = \dfrac{1}{2\sqrt{x}}$

61. $y = x^3\sqrt[3]{x}$

62. $f(x) = (2x^3)(4x^2)$

63. $f(x) = x(3x^2 - 10x + 7)$

64. $f(x) = x^3(3x^6 - 5x^2 + 4)$

65. $f(x) = x^3(3x)^2$

66. $s(x) = \sqrt{x}(\sqrt[5]{x} + 7x + 2)$

67. $v(x) = x^{-2/3}(x + 5)$

68. $f(x) = x^{3/5}(x^2 + 7x + 11)$

69. $f(q) = \dfrac{3q^2 + 4q - 2}{q}$

70. $f(w) = \dfrac{w - 5}{w^5}$

71. $f(x) = (x - 1)(x + 2)$

72. $f(x) = x^2(x - 2)(x + 4)$

73. $w(x) = \dfrac{x^2 + x^3}{x^2}$

74. $f(x) = \dfrac{7x^3 + x}{6\sqrt{x}}$

For each curve in Problems 75–78, find the slopes at the indicated points.

75. $y = 3x^2 + 4x - 8$; $(0, -8), (2, 12), (-3, 7)$

76. $y = 3 + 5x - 3x^3$; $(0, 3), (\frac{1}{2}, \frac{41}{8}), (2, -11)$

77. $y = 4$; when $x = -4, x = 7, x = 22$

78. $y = 3x - 4\sqrt{x}$; when $x = 4, x = 9, x = 25$

In Problems 79–82, find an equation of the tangent line to the curve at the indicated point.

79. $y = 4x^2 + 5x + 6$; $(1, 15)$

80. $y = \dfrac{1 - x^2}{5}$; $(4, -3)$

81. $y = \dfrac{1}{x^2}$; $(2, \frac{1}{4})$

82. $y = -\sqrt[3]{x}$; $(8, -2)$

83. Find an equation of the tangent line to the curve

$$y = 3 + x - 5x^2 + x^4$$

when $x = 0$.

84. Repeat Problem 83 for the curve

$$y = \frac{\sqrt{x}(2 - x^2)}{x}$$

when $x = 4$.

85. Find all points on the curve

$$y = \frac{5}{2}x^2 - x^3$$

where the tangent line is horizontal.

86. Repeat Problem 85 for the curve

$$y = \frac{x^6}{6} - \frac{x^2}{2} + 1$$

87. Find all points on the curve

$$y = x^2 - 5x + 3$$

where the slope is 1.

88. Repeat Problem 87 for the curve

$$y = x^4 - 31x + 11$$

89. If $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, evaluate the expression

$$\frac{x - 1}{2x\sqrt{x}} - f'(x)$$

90. Economics Eswaran and Kotwal[2] consider agrarian economies in which there are two types of workers, permanent and casual. Permanent workers are employed on long-term contracts and may receive benefits such as holiday gifts and emergency aid. Casual workers are hired on a daily basis and perform routine and menial tasks such as weeding, harvesting, and threshing. The difference z in the present-value cost of hiring a permanent worker over that of hiring a casual worker is given by

$$z = (1+b)w_p - bw_c$$

where w_p and w_c are wage rates for permanent labor and casual labor, respectively, b is a constant, and w_p is a function of w_c.

Eswaran and Kotwal claim that

$$\frac{dz}{dw_c} = (1+b)\left[\frac{dw_p}{dw_c} - \frac{b}{1+b}\right]$$

Verify this.

91. Find an equation of the tangent line to the graph of $y = x^3 - 2x + 1$ at the point $(1, 0)$. Graph both the function and the tangent line on the same screen.

92. Find an equation of the tangent line to the graph of $y = \sqrt[3]{x}$, at the point $(-8, -2)$. Graph both the function and the tangent line on the same screen. Notice that the line passes through $(-8, -2)$ and the line appears to be tangent to the curve.

Objective

To motivate the instantaneous rate of change of a function by means of velocity and to interpret the derivative as an instantaneous rate of change. To develop the "marginal" concept, which is frequently used in business and economics.

11.3 The Derivative as a Rate of Change

We have given a geometric interpretation of the derivative as being the slope of the tangent line to a curve at a point. Historically, an important application of the derivative involves the motion of an object traveling in a straight line. This gives us a convenient way to interpret the derivative as a *rate of change.*

To denote the change in a variable such as x, the symbol Δx (read "delta x") is commonly used. For example, if x changes from 1 to 3, then the change in x is $\Delta x = 3 - 1 = 2$. The new value of $x(=3)$ is the old value plus the change, which is $1 + \Delta x$. Similarly, if t increases by Δt, the new value is $t + \Delta t$. We will use Δ-notation in the discussion that follows.

Suppose an object moves along the number line in Figure 11.12 according to the equation

$$s = f(t) = t^2$$

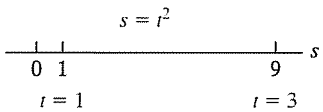


FIGURE 11.12 Motion along a number line.

where s is the position of the object at time t. This equation is called an **equation of motion,** and f is called a ***position function.*** Assume that t is in seconds and s is in meters. At $t = 1$ the position is $s = f(1) = 1^2 = 1$, and at $t = 3$ the position is $s = f(3) = 3^2 = 9$. Over this two-second time interval, the object has a change in position, or a *displacement,* of $9 - 1 = 8$ m, and the *average velocity* of the object is defined as

$$v_{\text{ave}} = \frac{\text{displacement}}{\text{length of time interval}} \tag{1}$$

$$= \frac{8}{2} = 4 \text{ m/s}$$

To say that the average velocity is 4 m/s from $t = 1$ to $t = 3$ means that, *on the average,* the position of the object changed by 4 m to the right each second during that time interval. Let us denote the changes in s-values and t-values by Δs and Δt, respectively. Then the average velocity is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = 4 \text{ m/s} \quad \text{(for the interval } t = 1 \text{ to } t = 3)$$

The ratio $\Delta s/\Delta t$ is also called the **average rate of change of s with respect to t** over the interval from $t = 1$ to $t = 3$.

Now, let the time interval be only 1 second long (that is, $\Delta t = 1$). Then, for the *shorter* interval from $t = 1$ to $t = 1 + \Delta t = 2$, we have $f(2) = 2^2 = 4$, so

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(2) - f(1)}{\Delta t} = \frac{4-1}{1} = 3 \text{ m/s}$$

[2]M. Eswaran and A. Kotwal, "A Theory of Two-Tier Labor Markets in Agrarian Economies," *The American Economic Review,* 75, no. 1 (1985), 162–77.

Table 11.2

Length of Time Interval Δt	Time Interval $t = 1$ to $t = 1 + \Delta t$	Average Velocity $\frac{\Delta s}{\Delta t} = \frac{f(1+\Delta t) - f(1)}{\Delta t}$
0.1	$t = 1$ to $t = 1.1$	2.1 m/s
0.07	$t = 1$ to $t = 1.07$	2.07 m/s
0.05	$t = 1$ to $t = 1.05$	2.05 m/s
0.03	$t = 1$ to $t = 1.03$	2.03 m/s
0.01	$t = 1$ to $t = 1.01$	2.01 m/s
0.001	$t = 1$ to $t = 1.001$	2.001 m/s

More generally, over the time interval from $t = 1$ to $t = 1 + \Delta t$, the object moves from position $f(1)$ to position $f(1 + \Delta t)$. Thus, its displacement is

$$\Delta s = f(1 + \Delta t) - f(1)$$

Since the time interval has length Δt, the object's average velocity is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(1 + \Delta t) - f(1)}{\Delta t}$$

If Δt were to become smaller and smaller, the average velocity over the interval from $t = 1$ to $t = 1 + \Delta t$ would be close to what we might call the *instantaneous velocity* at time $t = 1$; that is, the velocity at a *point* in time ($t = 1$) as opposed to the velocity over an *interval* of time. For some typical values of Δt between 0.1 and 0.001, we get the average velocities in Table 11.2, which the reader can verify.

The table suggests that as the length of the time interval approaches zero, the average velocity approaches the value 2 m/s. In other words, as Δt approaches 0, $\Delta s/\Delta t$ approaches 2 m/s. We define the limit of the average velocity as $\Delta t \to 0$ to be the **instantaneous velocity** (or simply the **velocity**), v, at time $t = 1$. This limit is also called the **instantaneous rate of change** of s with respect to t at $t = 1$:

$$v = \lim_{\Delta t \to 0} v_{\text{ave}} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(1 + \Delta t) - f(1)}{\Delta t}$$

If we think of Δt as h, then the limit on the right is simply the derivative of s with respect to t at $t = 1$. Thus, the instantaneous velocity of the object at $t = 1$ is just ds/dt at $t = 1$. Because $s = t^2$ and

$$\frac{ds}{dt} = 2t$$

the velocity at $t = 1$ is

$$v = \left.\frac{ds}{dt}\right|_{t=1} = 2(1) = 2 \text{ m/s}$$

which confirms our previous conclusion.

In summary, if $s = f(t)$ is the position function of an object moving in a straight line, then the average velocity of the object over the time interval $[t, t + \Delta t]$ is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

and the velocity at time t is given by

$$v = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{ds}{dt}$$

Selectively combining equations for v, we have

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

which provides motivation for the otherwise bizarre Leibniz notation. (After all, Δ is the [uppercase] Greek letter corresponding to d.)

EXAMPLE 1 **Finding Average Velocity and Velocity**

Suppose the position function of an object moving along a number line is given by $s = f(t) = 3t^2 + 5$, where t is in seconds and s is in meters.

a. Find the average velocity over the interval [10, 10.1].
b. Find the velocity when $t = 10$.

Solution:

a. Here $t = 10$ and $\Delta t = 10.1 - 10 = 0.1$. So we have

$$\begin{aligned} v_{\text{ave}} &= \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} \\ &= \frac{f(10+0.1) - f(10)}{0.1} \\ &= \frac{f(10.1) - f(10)}{0.1} \\ &= \frac{311.03 - 305}{0.1} = \frac{6.03}{0.1} = 60.3 \text{ m/s} \end{aligned}$$

b. The velocity at time t is given by

$$v = \frac{ds}{dt} = 6t$$

When $t = 10$, the velocity is

$$\left.\frac{ds}{dt}\right|_{t=10} = 6(10) = 60 \text{ m/s}$$

Notice that the average velocity over the interval [10, 10.1] is close to the velocity at $t = 10$. This is to be expected because the length of the interval is small.

Now Work Problem 1 ◁

Our discussion of the rate of change of s with respect to t applies equally well to *any* function $y = f(x)$. This means that we have the following:

If $y = f(x)$, then

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \begin{cases} \text{average rate of change} \\ \text{of } y \text{ with respect to } x \\ \text{over the interval from} \\ x \text{ to } x + \Delta x \end{cases}$$

and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \begin{cases} \text{instantaneous rate of change} \\ \text{of } y \text{ with respect to } x \end{cases} \qquad (2)$$

Because the instantaneous rate of change of $y = f(x)$ at a point is a derivative, it is also the *slope of the tangent line* to the graph of $y = f(x)$ at that point. For convenience, we usually refer to the instantaneous rate of change simply as the **rate of change.** The interpretation of a derivative as a rate of change is extremely important.

Let us now consider the significance of the rate of change of y with respect to x. From Equation (2), if Δx (a change in x) is close to 0, then $\Delta y/\Delta x$ is close to dy/dx. That is,

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

Therefore,

$$\Delta y \approx \frac{dy}{dx}\Delta x \tag{3}$$

That is, if x changes by Δx, then the change in y, Δy, is approximately dy/dx times the change in x. In particular,

$$\text{if } x \text{ changes by 1, an estimate of the change in } y \text{ is } \frac{dy}{dx}$$

APPLY IT ▷

3. Suppose that the profit P made by selling a certain product at a price of p per unit is given by $P = f(p)$ and the rate of change of that profit with respect to change in price is $\frac{dP}{dp} = 5$ at $p = 25$. Estimate the change in the profit P if the price changes from 25 to 25.5.

EXAMPLE 2 Estimating Δy by Using dy/dx

Suppose that $y = f(x)$ and $\frac{dy}{dx} = 8$ when $x = 3$. Estimate the change in y if x changes from 3 to 3.5.

Solution: We have $dy/dx = 8$ and $\Delta x = 3.5 - 3 = 0.5$. The change in y is given by Δy, and, from Equation (3),

$$\Delta y \approx \frac{dy}{dx}\Delta x = 8(0.5) = 4$$

We remark that, since $\Delta y = f(3.5) - f(3)$, we have $f(3.5) = f(3) + \Delta y$. For example, if $f(3) = 5$, then $f(3.5)$ can be estimated by $5 + 4 = 9$.

◁

APPLY IT ▷

4. The position of an object thrown upward at a speed of 16 feet/s from a height of 0 feet is given by $y(t) = 16t - 16t^2$. Find the rate of change of y with respect to t, and evaluate it when $t = 0.5$. Use your graphing calculator to graph $y(t)$. Use the graph to interpret the behavior of the object when $t = 0.5$.

EXAMPLE 3 Finding a Rate of Change

Find the rate of change of $y = x^4$ with respect to x, and evaluate it when $x = 2$ and when $x = -1$. Interpret your results.

Solution: The rate of change is

$$\frac{dy}{dx} = 4x^3$$

When $x = 2, dy/dx = 4(2)^3 = 32$. This means that if x increases, from 2, by a small amount, then y increases approximately 32 times as much. More simply, we say that, when $x = 2$, y is increasing 32 times as fast as x does. When $x = -1$, $dy/dx = 4(-1)^3 = -4$. The significance of the minus sign on -4 is that, when $x = -1$, y is *decreasing* 4 times as fast as x increases.

Now Work Problem 11 ◁

EXAMPLE 4 Rate of Change of Price with Respect to Quantity

Let $p = 100 - q^2$ be the demand function for a manufacturer's product. Find the rate of change of price p per unit with respect to quantity q. How fast is the price changing with respect to q when $q = 5$? Assume that p is in dollars.

Solution: The rate of change of p with respect to q is

$$\frac{dp}{dq} = \frac{d}{dq}(100 - q^2) = -2q$$

Thus,

$$\left.\frac{dp}{dq}\right|_{q=5} = -2(5) = -10$$

This means that when five units are demanded, an *increase* of one extra unit demanded corresponds to a decrease of approximately \$10 in the price per unit that consumers are willing to pay.

◁

EXAMPLE 5 Rate of Change of Volume

A spherical balloon is being filled with air. Find the rate of change of the volume of air in the balloon with respect to its radius. Evaluate this rate of change when the radius is 2 ft.

Solution: The formula for the volume V of a ball of radius r is $V = \frac{4}{3}\pi r^3$. The rate of change of V with respect to r is

$$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$$

When $r = 2$ ft, the rate of change is

$$\left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = 16\pi\frac{\text{ft}^3}{\text{ft}}$$

This means that when the radius is 2 ft, changing the radius by 1 ft will change the volume by approximately 16π ft^3.

◁

EXAMPLE 6 Rate of Change of Enrollment

A sociologist is studying various suggested programs that can aid in the education of preschool-age children in a certain city. The sociologist believes that x years after the beginning of a particular program, $f(x)$ thousand preschoolers will be enrolled, where

$$f(x) = \frac{10}{9}(12x - x^2) \quad 0 \leq x \leq 12$$

At what rate would enrollment change (a) after three years from the start of this program and (b) after nine years?

Solution: The rate of change of $f(x)$ is

$$f'(x) = \frac{10}{9}(12 - 2x)$$

a. After three years, the rate of change is

$$f'(3) = \frac{10}{9}(12 - 2(3)) = \frac{10}{9} \cdot 6 = \frac{20}{3} = 6\frac{2}{3}$$

Thus, enrollment would be increasing at the rate of $6\frac{2}{3}$ thousand preschoolers per year.

b. After nine years, the rate is

$$f'(9) = \frac{10}{9}(12 - 2(9)) = \frac{10}{9}(-6) = -\frac{20}{3} = -6\frac{2}{3}$$

Thus, enrollment would be *decreasing* at the rate of $6\frac{2}{3}$ thousand preschoolers per year.

Now Work Problem 9 ◁

Applications of Rate of Change to Economics

A manufacturer's **total-cost function,** $c = f(q)$, gives the total cost c of producing and marketing q units of a product. The rate of change of c with respect to q is called the **marginal cost.** Thus,

$$\text{marginal cost} = \frac{dc}{dq}$$

For example, suppose $c = f(q) = 0.1q^2 + 3$ is a cost function, where c is in dollars and q is in pounds. Then

$$\frac{dc}{dq} = 0.2q$$

The marginal cost when 4 lb are produced is dc/dq, evaluated when $q = 4$:

$$\left.\frac{dc}{dq}\right|_{q=4} = 0.2(4) = 0.80$$

This means that if production is increased by 1 lb, from 4 lb to 5 lb, then the change in cost is approximately \$0.80. That is, the additional pound costs about \$0.80. In general, *we interpret marginal cost as the approximate cost of one additional unit of output.* After all, the difference $f(q+1) - f(q)$ can be seen as a difference quotient

$$\frac{f(q+1) - f(q)}{1}$$

(the case where $h = 1$). Any difference quotient can be regarded as an approximation of the corresponding derivative and, conversely, any derivative can be regarded as an approximation of any of its corresponding difference quotients. Thus, for any function f of q we can always regard $f'(q)$ and $f(q+1) - f(q)$ as approximations of each other. In economics, the latter can usually be regarded as the exact value of the cost, or profit depending upon the function, of the $(q+1)$th item when q are produced. The derivative is often easier to compute than the exact value. [In the case at hand, the actual cost of producing one more pound beyond 4 lb is $f(5) - f(4) = 5.5 - 4.6 = \$0.90$.]

If c is the total cost of producing q units of a product, then the **average cost per unit,** $\bar{c}$, is

$$\bar{c} = \frac{c}{q} \tag{4}$$

For example, if the total cost of 20 units is \$100, then the average cost per unit is $\bar{c} = 100/20 = \$5$. By multiplying both sides of Equation (4) by q, we have

$$c = q\bar{c}$$

That is, total cost is the product of the number of units produced and the average cost per unit.

EXAMPLE 7 Marginal Cost

If a manufacturer's average-cost equation is

$$\bar{c} = 0.0001q^2 - 0.02q + 5 + \frac{5000}{q}$$

find the marginal-cost function. What is the marginal cost when 50 units are produced?

Solution:

Strategy The marginal-cost function is the derivative of the total-cost function c. Thus, we first find c by multiplying $\bar{c}$ by q. We have

$$\begin{aligned} c &= q\bar{c} \\ &= q\left(0.0001q^2 - 0.02q + 5 + \frac{5000}{q}\right) \\ c &= 0.0001q^3 - 0.02q^2 + 5q + 5000 \end{aligned}$$

Differentiating c, we have the marginal-cost function:

$$\frac{dc}{dq} = 0.0001(3q^2) - 0.02(2q) + 5(1) + 0$$
$$= 0.0003q^2 - 0.04q + 5$$

The marginal cost when 50 units are produced is

$$\left.\frac{dc}{dq}\right|_{q=50} = 0.0003(50)^2 - 0.04(50) + 5 = 3.75$$

If c is in dollars and production is increased by one unit, from $q = 50$ to $q = 51$, then the cost of the additional unit is approximately \$3.75. If production is increased by $\frac{1}{3}$ unit, from $q = 50$, then the cost of the additional output is approximately $\left(\frac{1}{3}\right)(3.75) = \$1.25$.

Now Work Problem 21 ◁

Suppose $r = f(q)$ is the **total-revenue function** for a manufacturer. The equation $r = f(q)$ states that the total dollar value received for selling q units of a product is r. The **marginal revenue** is defined as the rate of change of the total dollar value received with respect to the total number of units sold. Hence, marginal revenue is merely the derivative of r with respect to q:

$$\text{marginal revenue} = \frac{dr}{dq}$$

Marginal revenue indicates the rate at which revenue changes with respect to units sold. We interpret it as *the approximate revenue received from selling one additional unit of output.*

EXAMPLE 8 **Marginal Revenue**

Suppose a manufacturer sells a product at \$2 per unit. If q units are sold, the total revenue is given by

$$r = 2q$$

The marginal-revenue function is

$$\frac{dr}{dq} = \frac{d}{dq}(2q) = 2$$

which is a constant function. Thus, the marginal revenue is 2 regardless of the number of units sold. This is what we would expect, because the manufacturer receives \$2 for each unit sold.

Now Work Problem 23 ◁

Relative and Percentage Rates of Change

For the total-revenue function in Example 8, namely, $r = f(q) = 2q$, we have

$$\frac{dr}{dq} = 2$$

This means that revenue is changing at the rate of \$2 per unit, regardless of the number of units sold. Although this is valuable information, it may be more significant when compared to r itself. For example, if $q = 50$, then $r = 2(50) = 100$. Thus, the rate of change of revenue is $2/100 = 0.02$ *of* r. On the other hand, if $q = 5000$, then $r = 2(5000) = \$10{,}000$, so the rate of change of r is $2/10{,}000 = 0.0002$ *of* r.

Although r changes at the same rate at each level, compared to r itself, this rate is relatively smaller when $r = 10{,}000$ than when $r = 100$. By considering the ratio

$$\frac{dr/dq}{r}$$

we have a means of comparing the rate of change of r with r itself. This ratio is called the *relative rate of change* of r. We have shown that the relative rate of change when $q = 50$ is

$$\frac{dr/dq}{r} = \frac{2}{100} = 0.02$$

and when $q = 5000$, it is

$$\frac{dr/dq}{r} = \frac{2}{10{,}000} = 0.0002$$

By multiplying relative rates by 100%, we obtain the so-called *percentage rates of change*. The percentage rate of change when $q = 50$ is $(0.02)(100\%) = 2\%$; when $q = 5000$, it is $(0.0002)(100\%) = 0.02\%$. For example, if an additional unit beyond 50 is sold, then revenue increases by approximately 2%.

In general, for any function f, we have the following definition:

CAUTION!

Percentages can be confusing! Remember that *percent* means "per hundred." Thus $100\% = \frac{100}{100} = 1$, $2\% = \frac{2}{100} = 0.02$, and so on.

Definition

The ***relative rate of change*** of $f(x)$ is

$$\frac{f'(x)}{f(x)}$$

The ***percentage rate of change*** of $f(x)$ is

$$\frac{f'(x)}{f(x)} \cdot 100\%$$

APPLY IT ▷

5. The volume V enclosed by a capsule-shaped container with a cylindrical height of 4 feet and radius r is given by

$$V(r) = \frac{4}{3}\pi r^3 + 4\pi r^2$$

Determine the relative and percentage rates of change of volume with respect to the radius when the radius is 2 feet.

EXAMPLE 9 Relative and Percentage Rates of Change

Determine the relative and percentage rates of change of

$$y = f(x) = 3x^2 - 5x + 25$$

when $x = 5$.

Solution: Here

$$f'(x) = 6x - 5$$

Since $f'(5) = 6(5) - 5 = 25$ and $f(5) = 3(5)^2 - 5(5) + 25 = 75$, the relative rate of change of y when $x = 5$ is

$$\frac{f'(5)}{f(5)} = \frac{25}{75} \approx 0.333$$

Multiplying 0.333 by 100% gives the percentage rate of change: $(0.333)(100) = 33.3\%$.

Now Work Problem 35 ◁

PROBLEMS 11.3

1. Suppose that the position function of an object moving along a straight line is $s = f(t) = 2t^2 + 3t$, where t is in seconds and s is in meters. Find the average velocity $\Delta s/\Delta t$ over the interval $[1, 1 + \Delta t]$, where Δt is given in the following table:

Δt	1	0.5	0.2	0.1	0.01	0.001
$\Delta s/\Delta t$						

From your results, estimate the velocity when $t = 1$. Verify your estimate by using differentiation.

2. If $y = f(x) = \sqrt{2x+5}$, find the average rate of change of y with respect to x over the interval $[3, 3+\Delta x]$, where Δx is given in the following table:

Δx	1	0.5	0.2	0.1	0.01	0.001
$\Delta y/\Delta x$						

From your result, estimate the rate of change of y with respect to x when $x = 3$.

In each of Problems 3–8, a position function is given, where t is in seconds and s is in meters.
(a) *Find the position at the given t-value.*
(b) *Find the average velocity over the given interval.*
(c) *Find the velocity at the given t-value.*

3. $s = 2t^2 - 4t; [7, 7.5]; t = 7$

4. $s = \frac{1}{2}t + 1; [2, 2.1]; t = 2$

5. $s = 5t^3 + 3t + 24; [1, 1.01]; t = 1$

6. $s = -3t^2 + 2t + 1; [1, 1.25]; t = 1$

7. $s = t^4 - 2t^3 + t; [2, 2.1]; t = 2$

8. $s = 3t^4 - t^{7/2}; [0, \frac{1}{4}]; t = 0$

9. Income–Education Sociologists studied the relation between income and number of years of education for members of a particular urban group. They found that a person with x years of education before seeking regular employment can expect to receive an average yearly income of y dollars per year, where

$$y = 5x^{5/2} + 5900 \qquad 4 \le x \le 16$$

Find the rate of change of income with respect to number of years of education. Evaluation the expression when $x = 9$.

10. Find the rate of change of the volume V of a ball, with respect to its radius r, when $r = 1.5$ m. The volume V of a ball as a function of its radius r is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

11. Skin Temperature The approximate temperature T of the skin in terms of the temperature T_e of the environment is given by

$$T = 32.8 + 0.27(T_e - 20)$$

where T and T_e are in degrees Celsius.[3] Find the rate of change of T with respect to T_e.

12. Biology The volume V of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Find the rate of change of volume with respect to the radius when $r = 6.3 \times 10^{-4}$ cm.

In Problems 13–18, cost functions are given, where c is the cost of producing q units of a product. In each case, find the marginal-cost function. What is the marginal cost at the given value(s) of q?

13. $c = 500 + 10q; q = 100$

14. $c = 5000 + 6q; q = 36$

15. $c = 0.2q^2 + 4q + 50; q = 10$

16. $c = 0.1q^2 + 3q + 2; q = 3$

17. $c = q^2 + 50q + 1000; q = 15, q = 16, q = 17$

18. $c = 0.04q^3 - 0.5q^2 + 4.4q + 7500; q = 5, q = 25, q = 1000$

In Problems 19–22, $\bar{c}$ represents average cost per unit, which is a function of the number q of units produced. Find the marginal-cost function and the marginal cost for the indicated values of q.

19. $\bar{c} = 0.01q + 5 + \frac{500}{q}; q = 50, q = 100$

20. $\bar{c} = 5 + \frac{2000}{q}; q = 25, q = 250$

21. $\bar{c} = 0.00002q^2 - 0.01q + 6 + \frac{20{,}000}{q}; q = 100, q = 500$

22. $\bar{c} = 0.002q^2 - 0.5q + 60 + \frac{7000}{q}; q = 15, q = 25$

In Problems 23–26, r represents total revenue and is a function of the number q of units sold. Find the marginal-revenue function and the marginal revenue for the indicated values of q.

23. $r = 0.8q; q = 9, q = 300, q = 500$

24. $r = q\left(15 - \frac{1}{30}q\right); q = 5, q = 15, q = 150$

25. $r = 240q + 40q^2 - 2q^3; q = 10; q = 15; q = 20$

26. $r = 2q(30 - 0.1q); q = 10, q = 20$

27. Hosiery Mill The total-cost function for a hosiery mill is estimated by Dean[4] to be

$$c = -10{,}484.69 + 6.750q - 0.000328q^2$$

where q is output in dozens of pairs and c is total cost in dollars. Find the marginal-cost function and the average cost function and evaluate each when $q = 2000$.

28. Light and Power Plant The total-cost function for an electric light and power plant is estimated by Nordin[5] to be

$$c = 32.07 - 0.79q + 0.02142q^2 - 0.0001q^3 \qquad 20 \le q \le 90$$

where q is the eight-hour total output (as a percentage of capacity) and c is the total fuel cost in dollars. Find the marginal-cost function and evaluate it when $q = 70$.

29. Urban Concentration Suppose the 100 largest cities in the United States in 1920 are ranked according to magnitude (areas of cities). From Lotka,[6] the following relation holds approximately:

$$PR^{0.93} = 5{,}000{,}000$$

Here, P is the population of the city having respective rank R. This relation is called the *law of urban concentration* for 1920. Solve for P in terms of R, and then find how fast the population is changing with respect to rank.

30. Depreciation Under the straight-line method of depreciation, the value v of a certain machine after t years have elapsed is given by

$$v = 120{,}000 - 15{,}500t$$

where $0 \le t \le 6$. How fast is v changing with respect to t when $t = 2$? $t = 4$? at any time?

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[3] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).

[4] J. Dean, "Statistical Cost Functions of a Hosiery Mill," *Studies in Business Administration*, XI, no. 4 (Chicago: University of Chicago Press, 1941).

[5] J. A. Nordin, "Note on a Light Plant's Cost Curves," *Econometrica*, 15 (1947), 231–35.

[6] A. J. Lotka, *Elements of Mathematical Biology* (New York: Dover Publications, Inc., 1956).

31. Winter Moth A study of the winter moth was made in Nova Scotia (adapted from Embree).[7] The prepupae of the moth fall onto the ground from host trees. At a distance of x ft from the base of a host tree, the prepupal density (number of prepupae per square foot of soil) was y, where

$$y = 59.3 - 1.5x - 0.5x^2 \quad 1 \le x \le 9$$

(a) At what rate is the prepupal density changing with respect to distance from the base of the tree when $x = 6$?
(b) For what value of x is the prepupal density decreasing at the rate of 6 prepupae per square foot per foot?

32. Cost Function For the cost function

$$c = 0.4q^2 + 4q + 5$$

find the rate of change of c with respect to q when $q = 2$. Also, what is $\Delta c/\Delta q$ over the interval [2, 3]?

In Problems 33–38, find **(a)** *the rate of change of y with respect to x and* **(b)** *the relative rate of change of y. At the given value of x, find* **(c)** *the rate of change of y,* **(d)** *the relative rate of change of y, and* **(e)** *the percentage rate of change of y.*

33. $y = f(x) = x + 4; x = 5$ **34.** $y = f(x) = 7 - 3x; x = 6$

35. $y = 2x^2 + 5; x = 10$ **36.** $y = 5 - 3x^3; x = 1$

37. $y = 8 - x^3; x = 1$ **38.** $y = x^2 + 3x - 4; x = -1$

39. Cost Function For the cost function

$$c = 0.3q^2 + 3.5q + 9$$

how fast does c change with respect to q when $q = 10$? Determine the percentage rate of change of c with respect to q when $q = 10$.

40. Organic Matters/Species Diversity In a discussion of contemporary waters of shallows seas, Odum[8] claims that in such waters the total organic matter y (in milligrams per liter) is a function of species diversity x (in number of species per thousand individuals). If $y = 100/x$, at what rate is the total organic matter changing with respect to species diversity when $x = 10$? What is the percentage rate of change when $x = 10$?

41. Revenue For a certain manufacturer, the revenue obtained from the sale of q units of a product is given by

$$r = 30q - 0.3q^2$$

(a) How fast does r change with respect to q? When $q = 10$, (b) find the relative rate of change of r, and (c) to the nearest percent, find the percentage rate of change of r.

42. Revenue Repeat Problem 43 for the revenue function given by $r = 10q - 0.2q^2$ and $q = 25$.

43. Weight of Limb The weight of a limb of a tree is given by $W = 2t^{0.432}$, where t is time. Find the relative rate of change of W with respect to t.

44. Response to Shock A psychological experiment[9] was conducted to analyze human responses to electrical shocks (stimuli). The subjects received shocks of various intensities. The response R to a shock of intensity I (in microamperes) was to be a number that indicated the perceived magnitude relative to that of a "standard" shock. The standard shock was assigned a magnitude of 10. Two groups of subjects were tested under slightly different conditions. The responses R_1 and R_2 of the first and second groups to a shock of intensity I were given by

$$R_1 = \frac{I^{1.3}}{1855.24} \quad 800 \le I \le 3500$$

and

$$R_2 = \frac{I^{1.3}}{1101.29} \quad 800 \le I \le 3500$$

(a) For each group, determine the relative rate of change of response with respect to intensity.
(b) How do these changes compare with each other?
(c) In general, if $f(x) = C_1x^n$ and $g(x) = C_2x^n$, where C_1 and C_2 are constants, how do the relative rates of change of f and g compare?

45. Cost A manufacturer of mountain bikes has found that when 20 bikes are produced per day, the average cost is \$200 and the marginal cost is \$150. Based on that information, approximate the total cost of producing 21 bikes per day.

46. Marginal and Average Costs Suppose that the cost function for a certain product is $c = f(q)$. If the relative rate of change of c (with respect to q) is $\frac{1}{q}$, prove that the marginal-cost function and the average-cost function are equal.

In Problems 47 and 48, use the numerical derivative feature of your graphing calculator.

47. If the total-cost function for a manufacturer is given by

$$c = \frac{5q^2}{\sqrt{q^2+3}} + 5000$$

where c is in dollars, find the marginal cost when 10 units are produced. Round your answer to the nearest cent.

48. The population of a city t years from now is given by

$$P = 250{,}000e^{0.04t}$$

Find the rate of change of population with respect to time t three years from now. Round your answer to the nearest integer.

Objective

To find derivatives by applying the product and quotient rules, and to develop the concepts of marginal propensity to consume and marginal propensity to save.

11.4 The Product Rule and the Quotient Rule

The equation $F(x) = (x^2 + 3x)(4x + 5)$ expresses $F(x)$ as a product of two functions: $x^2 + 3x$ and $4x + 5$. To find $F'(x)$ by using only our previous rules, we first multiply

[7] D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada*, no. 46 (1965).

[8] H. T. Odum, "Biological Circuits and the Marine Systems of Texas," in *Pollution and Marine Biology*, eds T. A. Olsen and F. J. Burgess (New York: Interscience Publishers, 1967).

[9] H. Babkoff, "Magnitude Estimation of Short Electrocutaneous Pulses," *Psychological Research*, 39, no. 1 (1976), 39–49.

the functions. Then we differentiate the result, term by term:

$$F(x) = (x^2 + 3x)(4x + 5) = 4x^3 + 17x^2 + 15x$$

$$F'(x) = 12x^2 + 34x + 15 \tag{1}$$

However, in many problems that involve differentiating a product of functions, the multiplication is not as simple as it is here. At times, it is not even practical to attempt it. Fortunately, there is a rule for differentiating a product, and the rule avoids such multiplications. Since the derivative of a sum of functions is the sum of their derivatives, you might expect a similar rule for products. However, the situation is rather subtle.

COMBINING RULE 3 The Product Rule

If f and g are differentiable functions, then the product fg is differentiable, and

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

That is, the derivative of the product of two functions is the derivative of the first function times the second, plus the first function times the derivative of the second.

$$\frac{d}{dx}(\text{product}) = \begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix}(\text{second}) + (\text{first})\begin{pmatrix}\text{derivative}\\ \text{of second}\end{pmatrix}$$

Proof. If $F(x) = f(x)g(x)$, then, by the definition of the derivative of F,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Now we use a "trick." Adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x)g(x+h) - f(x)g(x+h)}{h}$$

Regrouping gives

$$F'(x) = \lim_{h\to 0} \frac{(f(x+h)g(x+h) - f(x)g(x+h)) + (f(x)g(x+h) - f(x)g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Since we assumed that f and g are differentiable,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

and

$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

The differentiability of g implies that g is continuous, so, from Section 10.3,

$$\lim_{h\to 0} g(x+h) = g(x)$$

Thus,

$$F'(x) = f'(x)g(x) + f(x)g'(x)$$

EXAMPLE 1 Applying the Product Rule

If $F(x) = (x^2 + 3x)(4x + 5)$, find $F'(x)$.

Solution: We will consider F as a product of two functions:

$$F(x) = \underbrace{(x^2 + 3x)}_{f(x)}\underbrace{(4x + 5)}_{g(x)}$$

Therefore, we can apply the product rule:

$$\begin{aligned} F'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= \underbrace{\frac{d}{dx}(x^2 + 3x)}_{\substack{\text{Derivative} \\ \text{of first}}}\underbrace{(4x + 5)}_{\text{Second}} + \underbrace{(x^2 + 3x)}_{\text{First}}\underbrace{\frac{d}{dx}(4x + 5)}_{\substack{\text{Derivative} \\ \text{of second}}} \\ &= (2x + 3)(4x + 5) + (x^2 + 3x)(4) \\ &= 12x^2 + 34x + 15 \qquad \text{simplifying} \end{aligned}$$

This agrees with our previous result. [See Equation (1).] Although there doesn't seem to be much advantage to using the product rule here, there are times when it is impractical to avoid it.

Now Work Problem 1 ◁

CAUTION!

It is worthwhile to repeat that the derivative of the product of two functions is somewhat subtle. Do not be tempted to make up a simpler rule.

APPLY IT ▷

6. A taco stand usually sells 225 tacos per day at \$2 each. A business student's research tells him that for every \$0.15 decrease in the price, the stand will sell 20 more tacos per day. The revenue function for the taco stand is $R(x) = (2 - 0.15x)(225 + 20x)$, where x is the number of \$0.15 reductions in price. Find $\frac{dR}{dx}$.

EXAMPLE 2 Applying the Product Rule

If $y = (x^{2/3} + 3)(x^{-1/3} + 5x)$, find dy/dx.

Solution: Applying the product rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^{2/3} + 3)(x^{-1/3} + 5x) + (x^{2/3} + 3)\frac{d}{dx}(x^{-1/3} + 5x) \\ &= \left(\frac{2}{3}x^{-1/3}\right)(x^{-1/3} + 5x) + (x^{2/3} + 3)\left(\frac{-1}{3}x^{-4/3} + 5\right) \\ &= \frac{25}{3}x^{2/3} + \frac{1}{3}x^{-2/3} - x^{-4/3} + 15 \end{aligned}$$

Alternatively, we could have found the derivative without the product rule by first finding the product $(x^{2/3} + 3)(x^{-1/3} + 5x)$ and then differentiating the result, term by term.

Now Work Problem 15 ◁

EXAMPLE 3 Differentiating a Product of Three Factors

If $y = (x + 2)(x + 3)(x + 4)$, find y'.

Solution:

Strategy We would like to use the product rule, but as given it applies only to *two* factors. By treating the first two factors as a single factor, we can consider y to be a product of two functions:

$$y = [(x + 2)(x + 3)](x + 4)$$

The product rule gives

$$\begin{aligned} y' &= \frac{d}{dx}[(x + 2)(x + 3)](x + 4) + [(x + 2)(x + 3)]\frac{d}{dx}(x + 4) \\ &= \frac{d}{dx}[(x + 2)(x + 3)](x + 4) + [(x + 2)(x + 3)](1) \end{aligned}$$

Applying the product rule again, we have

$$y' = \left(\frac{d}{dx}(x+2)(x+3) + (x+2)\frac{d}{dx}(x+3)\right)(x+4) + (x+2)(x+3)$$
$$= [(1)(x+3) + (x+2)(1)](x+4) + (x+2)(x+3)$$

After simplifying, we obtain

$$y' = 3x^2 + 18x + 26$$

Two other ways of finding the derivative are as follows:

1. Multiply the first two factors of y to obtain

$$y = (x^2 + 5x + 6)(x + 4)$$

and then apply the product rule.

2. Multiply all three factors to obtain

$$y = x^3 + 9x^2 + 26x + 24$$

and then differentiate term by term.

Now Work Problem 19 ◁

It is sometimes helpful to remember differentiation rules in more streamlined notation. For example,

$$(fg)' = f'g + fg'$$

is a correct equality of functions that expresses the product rule. We can then calculate

$$\begin{aligned}(fgh)' &= ((fg)h)' \\ &= (fg)'h + (fg)h' \\ &= (f'g + fg')h + (fg)h' \\ &= f'gh + fg'h + fgh'\end{aligned}$$

It is not suggested that you try to commit to memory derived rules like

$$(fgh)' = f'gh + fg'h + fgh'$$

Because $f'g + fg' = gf' + fg'$, using commutativity of the product of functions, we can express the product rule with the derivatives as second factors:

$$(fg)' = gf' + fg'$$

and using commutativity of addition

$$(fg)' = fg' + gf'$$

Some people prefer these forms.

APPLY IT ▷

7. One hour after x milligrams of a particular drug are given to a person, the change in body temperature $T(x)$, in degrees Fahrenheit, is given approximately by $T(x) = x^2\left(1 - \frac{x}{3}\right)$. The rate at which T changes with respect to the size of the dosage x, $T'(x)$, is called the *sensitivity* of the body to the dosage. Find the sensitivity when the dosage is 1 milligram. Do not use the product rule.

EXAMPLE 4 Using the Product Rule to Find Slope

Find the slope of the graph of $f(x) = (7x^3 - 5x + 2)(2x^4 + 7)$ when $x = 1$.

Solution:

Strategy We find the slope by evaluating the derivative when $x = 1$. Because f is a product of two functions, we can find the derivative by using the product rule.

We have

$$f'(x) = (7x^3 - 5x + 2)\frac{d}{dx}(2x^4 + 7) + (2x^4 + 7)\frac{d}{dx}(7x^3 - 5x + 2)$$
$$= (7x^3 - 5x + 2)(8x^3) + (2x^4 + 7)(21x^2 - 5)$$

Since we must compute $f'(x)$ when $x = 1$, *there is no need to simplify $f'(x)$ before evaluating it*. Substituting into $f'(x)$, we obtain

$$f'(1) = 4(8) + 9(16) = 176$$

Now Work Problem 49 ◁

The product rule (and quotient rule that follows) should not be applied when a more direct and efficient method is available.

Usually, we do not use the product rule when simpler ways are obvious. For example, if $f(x) = 2x(x+3)$, then it is quicker to write $f(x) = 2x^2 + 6x$, from which $f'(x) = 4x + 6$. Similarly, we do not usually use the product rule to differentiate $y = 4(x^2 - 3)$. Since the 4 is a constant factor, by the constant-factor rule we have $y' = 4(2x) = 8x$.

The next rule is used for differentiating a *quotient* of two functions.

COMBINING RULE 4 The Quotient Rule

If f and g are differentiable functions and $g(x) \neq 0$, then the quotient f/g is also differentiable, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

With the understanding about the denominator not being zero, we can write

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

That is, the derivative of the quotient of two functions is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}(\text{quotient}) = \frac{(\text{denominator})\begin{pmatrix}\text{derivative}\\ \text{of numerator}\end{pmatrix} - (\text{numerator})\begin{pmatrix}\text{derivative}\\ \text{of denominator}\end{pmatrix}}{(\text{denominator})^2}$$

Proof. If $F(x) = \dfrac{f(x)}{g(x)}$, then

$$F(x)g(x) = f(x)$$

By the product rule,

$$F(x)g'(x) + g(x)F'(x) = f'(x)$$

Solving for $F'(x)$, we have

$$F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)}$$

But $F(x) = f(x)/g(x)$. Thus,

$$F'(x) = \frac{f'(x) - \dfrac{f(x)g'(x)}{g(x)}}{g(x)}$$

Simplifying gives[10]

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

CAUTION!

The derivative of the quotient of two functions is trickier still than the product rule. We must remember where the minus sign goes!

[10] The proof given assumes the existence of $F'(x)$. However, the rule can be proved without this assumption.

EXAMPLE 5 Applying the Quotient Rule

If $F(x) = \dfrac{4x^2+3}{2x-1}$, find $F'(x)$.

Solution:

Strategy We recognize F as a quotient, so we can apply the quotient rule.

Let $f(x) = 4x^2 + 3$ and $g(x) = 2x - 1$. Then

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{\overbrace{(2x-1)}^{\text{Denominator}}\ \overbrace{\frac{d}{dx}(4x^2+3)}^{\text{Derivative of numerator}} - \overbrace{(4x^2+3)}^{\text{Numerator}}\ \overbrace{\frac{d}{dx}(2x-1)}^{\text{Derivative of numerator}}}{\underbrace{(2x-1)^2}_{\text{Square of denominator}}}$$

$$= \frac{(2x-1)(8x) - (4x^2+3)(2)}{(2x-1)^2}$$

$$= \frac{8x^2 - 8x - 6}{(2x-1)^2} = \frac{2(2x+1)(2x-3)}{(2x-1)^2}$$

Now Work Problem 21 ◁

EXAMPLE 6 Rewriting before Differentiating

Differentiate $y = \dfrac{1}{x + \dfrac{1}{x+1}}$.

Solution:

Strategy To simplify the differentiation, we will rewrite the function so that no fraction appears in the denominator.

We have

$$y = \frac{1}{x + \dfrac{1}{x+1}} = \frac{1}{\dfrac{x(x+1)+1}{x+1}} = \frac{x+1}{x^2+x+1}$$

$$\frac{dy}{dx} = \frac{(x^2+x+1)(1) - (x+1)(2x+1)}{(x^2+x+1)^2} \qquad \text{quotient rule}$$

$$= \frac{(x^2+x+1) - (2x^2+3x+1)}{(x^2+x+1)^2}$$

$$= \frac{-x^2 - 2x}{(x^2+x+1)^2} = -\frac{x^2+2x}{(x^2+x+1)^2}$$

Now Work Problem 45 ◁

Although a function may have the form of a quotient, this does not necessarily mean that the quotient rule must be used to find the derivative. The next example illustrates some typical situations in which, although the quotient rule can be used, a simpler and more efficient method is available.

EXAMPLE 7 Differentiating Quotients without Using the Quotient Rule

Differentiate the following functions.

a. $f(x) = \dfrac{2x^3}{5}$

Solution: Rewriting, we have $f(x) = \frac{2}{5}x^3$. By the constant-factor rule,

$$f'(x) = \frac{2}{5}(3x^2) = \frac{6x^2}{5}$$

b. $f(x) = \dfrac{4}{7x^3}$

Solution: Rewriting, we have $f(x) = \frac{4}{7}(x^{-3})$. Thus,

$$f'(x) = \frac{4}{7}(-3x^{-4}) = -\frac{12}{7x^4}$$

c. $f(x) = \dfrac{5x^2 - 3x}{4x}$

Solution: Rewriting, we have $f(x) = \frac{1}{4}\left(\frac{5x^2 - 3x}{x}\right) = \frac{1}{4}(5x - 3)$ for $x \neq 0$. Thus,

$$f'(x) = \frac{1}{4}(5) = \frac{5}{4} \qquad \text{for } x \neq 0$$

Since the function f is not defined for $x = 0$, f' is not defined for $x = 0$ either.

Now Work Problem 17 ◁

CAUTION!

To differentiate $f(x) = \dfrac{1}{x^2 - 2}$, we might be tempted first to rewrite the quotient as $(x^2 - 2)^{-1}$. Currently it would be a mistake to do this because we do not yet have a rule for differentiating that form. In short, we have no choice now but to use the quotient rule. However, in the next section we will develop a rule that allows us to differentiate $(x^2 - 2)^{-1}$ in a direct and efficient way.

EXAMPLE 8 Marginal Revenue

If the demand equation for a manufacturer's product is

$$p = \frac{1000}{q + 5}$$

where p is in dollars, find the marginal-revenue function and evaluate it when $q = 45$.

Solution:

Strategy First we must find the revenue function. The revenue r received for selling q units when the price per unit is p is given by

$$\textbf{revenue} = \textbf{(price)(quantity)}; \quad \text{that is,} \quad r = pq$$

Using the demand equation, we will express r in terms of q only. Then we will differentiate to find the marginal-revenue function, dr/dq.

The revenue function is

$$r = \left(\frac{1000}{q + 5}\right)q = \frac{1000q}{q + 5}$$

Thus, the marginal-revenue function is given by

$$\frac{dr}{dq} = \frac{(q+5)\frac{d}{dq}(1000q) - (1000q)\frac{d}{dq}(q+5)}{(q+5)^2}$$

$$= \frac{(q+5)(1000) - (1000q)(1)}{(q+5)^2} = \frac{5000}{(q+5)^2}$$

and

$$\left.\frac{dr}{dq}\right|_{q=45} = \frac{5000}{(45+5)^2} = \frac{5000}{2500} = 2$$

This means that selling one additional unit beyond 45 results in approximately \$2 more in revenue.

Now Work Problem 59 ◁

Consumption Function

A function that plays an important role in economic analysis is the **consumption function.** The consumption function $C = f(I)$ expresses a relationship between the total national income I and the total national consumption C. Usually, both I and C are expressed in billions of dollars and I is restricted to some interval. The *marginal propensity to consume* is defined as the rate of change of consumption with respect to income. It is merely the derivative of C with respect to I:

$$\text{Marginal propensity to consume} = \frac{dC}{dI}$$

If we assume that the difference between income I and consumption C is savings S, then

$$S = I - C$$

Differentiating both sides with respect to I gives

$$\frac{dS}{dI} = \frac{d}{dI}(I) - \frac{d}{dI}(C) = 1 - \frac{dC}{dI}$$

We define dS/dI as the **marginal propensity to save.** Thus, the marginal propensity to save indicates how fast savings change with respect to income, and

$$\begin{array}{c}\text{Marginal propensity}\\ \text{to save}\end{array} = 1 - \begin{array}{c}\text{Marginal propensity}\\ \text{to consume}\end{array}$$

EXAMPLE 9 Finding Marginal Propensities to Consume and to Save

If the consumption function is given by

$$C = \frac{5(2\sqrt{I^3}+3)}{I+10}$$

determine the marginal propensity to consume and the marginal propensity to save when $I = 100$.

Solution:

$$\frac{dC}{dI} = 5\left(\frac{(I+10)\frac{d}{dI}(2I^{3/2}+3) - (2\sqrt{I^3}+3)\frac{d}{dI}(I+10)}{(I+10)^2}\right)$$

$$= 5\left(\frac{(I+10)(3I^{1/2}) - (2\sqrt{I^3}+3)(1)}{(I+10)^2}\right)$$

When $I = 100$, the marginal propensity to consume is

$$\left.\frac{dC}{dI}\right|_{I=100} = 5\left(\frac{1297}{12{,}100}\right) \approx 0.536$$

The marginal propensity to save when $I = 100$ is $1 - 0.536 = 0.464$. This means that if a current income of \$100 billion increases by \$1 billion, the nation consumes approximately 53.6% (536/1000) and saves 46.4% (464/1000) of that increase.

Now Work Problem 69 ◁

PROBLEMS 11.4

In Problems 1–48, differentiate the functions.

1. $f(x) = (4x+1)(6x+3)$

2. $f(x) = (3x-1)(7x+2)$

3. $s(t) = (5-3t)(t^3-2t^2)$

4. $Q(x) = (x^2+3x)(7x^2-5)$

5. $f(r) = (3r^2-4)(r^2-5r+1)$

6. $C(I) = (2I^2-3)(3I^2-4I+1)$

7. $f(x) = x^2(2x^2-5)$

8. $f(x) = 3x^3(x^2-2x+2)$

9. $y = (x^2+5x-7)(6x^2-5x+4)$

10. $\phi(x) = (3-5x+2x^2)(2+x-4x^2)$

11. $f(w) = (w^2+3w-7)(2w^3-4)$

12. $f(x) = (3x-x^2)(3-x-x^2)$

13. $y = (x^2-1)(3x^3-6x+5)-4(4x^2+2x+1)$

14. $h(x) = 5(x^7+4)+4(5x^3-2)(4x^2+7x)$

15. $F(p) = \frac{3}{2}(5\sqrt{p}-2)(3p-1)$

16. $g(x) = (\sqrt{x}+5x-2)(\sqrt[3]{x}-3\sqrt{x})$

17. $y = 7 \cdot \frac{2}{3}$

18. $y = (x-1)(x-2)(x-3)$

19. $y = (5x+3)(2x-5)(7x+9)$

20. $y = \dfrac{2x-3}{4x+1}$

21. $f(x) = \dfrac{5x}{x-1}$

22. $H(x) = \dfrac{-5x}{5-x}$

23. $f(x) = \dfrac{-13}{3x^5}$

24. $f(x) = \dfrac{3(5x^2-7)}{4}$

25. $y = \dfrac{x+2}{x-1}$

26. $h(w) = \dfrac{3w^2+5w-1}{w-3}$

27. $h(z) = \dfrac{6-2z}{z^2-4}$

28. $z = \dfrac{2x^2+5x-2}{3x^2+5x+3}$

29. $y = \dfrac{4x^2+3x+2}{3x^2-2x+1}$

30. $f(x) = \dfrac{x^3-x^2+1}{x^2+1}$

31. $y = \dfrac{x^2-4x+3}{2x^2-3x+2}$

32. $F(z) = \dfrac{z^4+4}{3z}$

33. $g(x) = \dfrac{1}{x^{100}+7}$

34. $y = \dfrac{-8}{7x^6}$

35. $u(v) = \dfrac{v^3-8}{v}$

36. $y = \dfrac{x-5}{8\sqrt{x}}$

37. $y = \dfrac{3x^2-x-1}{\sqrt[3]{x}}$

38. $y = \dfrac{x^{0.3}-2}{2x^{2.1}+1}$

39. $y = 1 - \dfrac{5}{2x+5} + \dfrac{2x}{3x+1}$

40. $q(x) = 2x^3 + \dfrac{5x+1}{3x-5} - \dfrac{2}{x^3}$

41. $y = \dfrac{x-5}{(x+2)(x-4)}$

42. $y = \dfrac{(9x-1)(3x+2)}{4-5x}$

43. $s(t) = \dfrac{t^2+3t}{(t^2-1)(t^3+7)}$

44. $f(s) = \dfrac{17}{s(4s^3+5s-23)}$

45. $y = 3x - \dfrac{\frac{2}{x}-\frac{3}{x-1}}{x-2}$

46. $y = 3 - 12x^3 + \dfrac{1-\frac{5}{x^2+2}}{x^2+5}$

47. $f(x) = \dfrac{a+x}{a-x}$, where a is a constant

48. $f(x) = \dfrac{x^{-1}+a^{-1}}{x^{-1}-a^{-1}}$, where a is a constant

49. Find the slope of the curve $y = (2x^2-x+3)(x^3+x+1)$ at $(1, 12)$.

50. Find the slope of the curve $y = \dfrac{x^3}{x^4+1}$ at $(-1, -\frac{1}{2})$.

In Problems 51–54, find an equation of the tangent line to the curve at the given point.

51. $y = \dfrac{6}{x-1}$; $(3, 3)$

52. $y = \dfrac{x+5}{x^2}$; $(1, 6)$

53. $y = (2x+3)[2(x^4-5x^2+4)]$; $(0, 24)$

54. $y = \dfrac{x-1}{x(x^2+1)}$; $(2, \frac{1}{10})$

In Problems 55 and 56, determine the relative rate of change of y with respect to x for the given value of x.

55. $y = \dfrac{x}{2x-6}$; $x = 1$

56. $y = \dfrac{1-x}{1+x}$; $x = 5$

57. Motion The position function for an object moving in a straight line is

$$s = \frac{2}{t^3+1}$$

where t is in seconds and s is in meters. Find the position and velocity of the object at $t = 1$.

58. Motion The position function for an object moving in a straight-line path is

$$s = \frac{t+3}{t^2+7}$$

where t is in seconds and s is in meters. Find the positive value(s) of t for which the velocity of the object is 0.

In Problems 59–62, each equation represents a demand function for a certain product, where p denotes the price per unit for q units. Find the marginal-revenue function in each case. Recall that revenue = pq.

59. $p = 80 - 0.02q$

60. $p = 500/q$

61. $p = \dfrac{108}{q+2} - 3$

62. $p = \dfrac{q+750}{q+50}$

63. Consumption Function For the United States (1922–1942), the consumption function is estimated by[11]

$$C = 0.672I + 113.1$$

Find the marginal propensity to consume.

64. Consumption Function Repeat Problem 63 for $C = 0.836I + 127.2$.

In Problems 65–68, each equation represents a consumption function. Find the marginal propensity to consume and the marginal propensity to save for the given value of I.

65. $C = 3 + \sqrt{I} + 2\sqrt[3]{I}$; $I = 1$

66. $C = 6 + \dfrac{3I}{4} - \dfrac{\sqrt{I}}{3}$; $I = 25$

[11] T. Haavelmo, "Methods of Measuring the Marginal Propensity to Consume," *Journal of the American Statistical Association*, XLII (1947), 105–22.

67. $C = \dfrac{16\sqrt{I} + 0.8\sqrt{I^3} - 0.2I}{\sqrt{I} + 4}; I = 36$

68. $C = \dfrac{20\sqrt{I} + 0.5\sqrt{I^3} - 0.4I}{\sqrt{I} + 5}; I = 100$

69. Consumption Function Suppose that a country's consumption function is given by

$$C = \frac{9\sqrt{I} + 0.8\sqrt{I^3} - 0.3I}{\sqrt{I}}$$

where C and I are expressed in billions of dollars.

(a) Find the marginal propensity to save when income is \$25 billion.

(b) Determine the relative rate of change of C with respect to I when income is \$25 billion.

70. Marginal Propensities to Consume and to Save Suppose that the savings function of a country is

$$S = \frac{I - 2\sqrt{I} - 8}{\sqrt{I} + 2}$$

where the national income (I) and the national savings (S) are measured in billions of dollars. Find the country's marginal propensity to consume and its marginal propensity to save when the national income is \$150 billion. (*Hint:* It may be helpful to first factor the numerator.)

71. Marginal Cost If the total-cost function for a manufacturer is given by

$$c = \frac{6q^2}{q+2} + 6000$$

find the marginal-cost function.

72. Marginal and Average Costs Given the cost function $c = f(q)$, show that if $\dfrac{d}{dq}(\bar{c}) = 0$, then the marginal-cost function and average-cost function are equal.

73. Host–Parasite Relation For a particular host–parasite relationship, it is determined that when the host density (number of hosts per unit of area) is x, the number of hosts that are parasitized is y, where

$$y = \frac{900x}{10 + 45x}$$

At what rate is the number of hosts parasitized changing with respect to host density when $x = 2$?

74. Acoustics The persistence of sound in a room after the source of the sound is turned off is called *reverberation*. The *reverberation time* RT of the room is the time it takes for the intensity level of the sound to fall 60 decibels. In the acoustical design of an auditorium, the following formula may be used to compute the RT of the room:[12]

$$\text{RT} = \frac{0.05V}{A + xV}$$

Here V is the room volume, A is the total room absorption, and x is the air absorption coefficient. Assuming that A and x are positive constants, show that the rate of change of RT with respect to V is always positive. If the total room volume increases by one unit, does the reverberation time increase or decrease?

75. Predator–Prey In a predator–prey experiment,[13] it was statistically determined that the number of prey consumed, y, by an individual predator was a function of the prey density x (the number of prey per unit of area), where

$$y = \frac{0.7355x}{1 + 0.02744x}$$

Determine the rate of change of prey consumed with respect to prey density.

76. Social Security Benefits In a discussion of social security benefits, Feldstein[14] differentiates a function of the form

$$f(x) = \frac{a(1+x) - b(2+n)x}{a(2+n)(1+x) - b(2+n)x}$$

where a, b, and n are constants. He determines that

$$f'(x) = \frac{-1(1+n)ab}{(a(1+x) - bx)^2(2+n)}$$

Verify this. (*Hint:* For convenience, let $2 + n = c$.) Next observe that Feldstein's function f is of the form

$$g(x) = \frac{A + Bx}{C + Dx}, \quad \text{where } A, B, C, \text{ and } D \text{ are constants}$$

Show that $g'(x)$ is a constant divided by a nonnegative function of x. What does this mean?

77. Business The manufacturer of a product has found that when 20 units are produced per day, the average cost is \$150 and the marginal cost is \$125. What is the relative rate of change of average cost with respect to quantity when $q = 20$?

78. Use the result $(fgh)' = f'gh + fg'h + fgh'$ to find dy/dx if

$$y = (3x + 1)(2x - 1)(x - 4)$$

Objective

To introduce and apply the chain rule, to derive a special case of the chain rule, and to develop the concept of the marginal-revenue product as an application of the chain rule.

11.5 The Chain Rule

Our next rule, the *chain rule*, is ultimately the most important rule for finding derivatives. It involves a situation in which y is a function of the variable u, but u is a function of x,

[12] L. L. Doelle, *Environmental Acoustics* (New York: McGraw-Hill Book Company, 1972).

[13] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist*, XCI, no. 7 (1959), 385–98.

[14] M. Feldstein, "The Optimal Level of Social Security Benefits," *The Quarterly Journal of Economics*, C, no. 2 (1985), 303–20.