6.6 Euler's Equations When Auxiliary Conditions Are Imposed

Suppose we want to find, for example, the shortest path between two points on a surface. Then, in addition to the conditions already discussed, there is the condition that the path must satisfy the equation of the surface, say, $g\{y_i; x\} = 0$. Such an equation was implicit in the solution of Example 6.4 for the geodesic on a sphere where the condition was

$$g = \sum_i x_i^2 - \rho^2 = 0 \tag{6.58}$$

that is,

$$r = \rho = \text{constant} \tag{6.59}$$

But in the general case, we must make explicit use of the auxiliary equation or equations. These equations are also called **equations of constraint.** Consider the case in which

$$f = f\{y_i, y_i'; x\} = f\{y, y', z, z'; x\} \tag{6.60}$$

The equation corresponding to Equation 6.17 for the case of *two* variables is

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} \right) \frac{\partial z}{\partial \alpha} \right] dx \tag{6.61}$$

But now there also exists an equation of constraint of the form

$$g\{y_i; x\} = g\{y, z; x\} = 0 \tag{6.62}$$

and the variations $\partial y/\partial \alpha$ and $\partial z/\partial \alpha$ are no longer independent, so the expressions in parentheses in Equation 6.61 do not separately vanish at $\alpha = 0$.

Differentiating g from Equation 6.62, we have

$$dg = \left(\frac{\partial g}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial \alpha} \right) d\alpha = 0 \tag{6.63}$$

where no term in x appears since $\partial x/\partial\alpha = 0$. Now

$$\left.\begin{aligned} y(\alpha, x) &= y(x) + \alpha\eta_1(x) \\ z(\alpha, x) &= z(x) + \alpha\eta_2(x) \end{aligned}\right\} \tag{6.64}$$

Therefore, by determining $\partial y/\partial\alpha$ and $\partial z/\partial\alpha$ from Equation 6.64 and inserting into the term in parentheses of Equation 6.63, which, in general, must be zero, we obtain

$$\frac{\partial g}{\partial y}\eta_1(x) = -\frac{\partial g}{\partial z}\eta_2(x) \tag{6.65}$$

Equation 6.61 becomes

$$\frac{\partial J}{\partial\alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\eta_2(x)\right]dx$$

Factoring $\eta_1(x)$ out of the square brackets and writing Equation 6.65 as

$$\frac{\eta_2(x)}{\eta_1(x)} = -\frac{\partial g/\partial y}{\partial g/\partial z}$$

we have

$$\frac{\partial J}{\partial\alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g/\partial y}{\partial g/\partial z}\right)\right]\eta_1(x)\,dx \tag{6.66}$$

This latter equation now contains the single arbitrary function $\eta_1(x)$, which is not in any way restricted by Equation 6.64, and on requiring the condition of Equation 6.4, the expression in the brackets must vanish. Thus we have

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1} \tag{6.67}$$

The left-hand side of this equation involves only derivatives of f and g with respect to y and y', and the right-hand side involves only derivatives with respect to z and z'. Because y and z are both functions of x, the two sides of Equation 6.67 may be set equal to a function of x, which we write as $-\lambda(x)$:

$$\left.\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} &= 0 \\ \frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} &= 0 \end{aligned}\right\} \tag{6.68}$$

The complete solution to the problem now depends on finding *three* functions: $y(x)$, $z(x)$, and $\lambda(x)$. But there are *three* relations that may be used: the two equations (Equation 6.68) and the equation of constraint (Equation 6.62). Thus, there is a sufficient number of relations to allow a complete solution. Note that here $\lambda(x)$ is considered to be *undetermined** and is obtained as a part of the solution. The function $\lambda(x)$ is known as a **Lagrange undetermined multiplier.**

For the general case of several dependent variables and several auxiliary conditions, we have the following set of equations:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_j \lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0 \tag{6.69}$$

$$g_j\{y_i; x\} = 0 \tag{6.70}$$

If $i = 1, 2, \cdots, m$, and $j = 1, 2, \cdots, n$, Equation 6.69 represents m equations in $m + n$ unknowns, but there are also the n equations of constraint (Equation 6.70). Thus, there are $m + n$ equations in $m + n$ unknowns, and the system is soluble.

Equation 6.70 is equivalent to the set of n differential equations

$$\sum_i \frac{\partial g_j}{\partial y_i}\, dy_i = 0, \qquad \begin{cases} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, n \end{cases} \tag{6.71}$$

In problems in mechanics, the constraint equations are frequently differential equations rather than algebraic equations. Therefore, equations such as Equation 6.71 are sometimes more useful than the equations represented by Equation 6.70. (See Section 7.5 for an amplification of this point.)

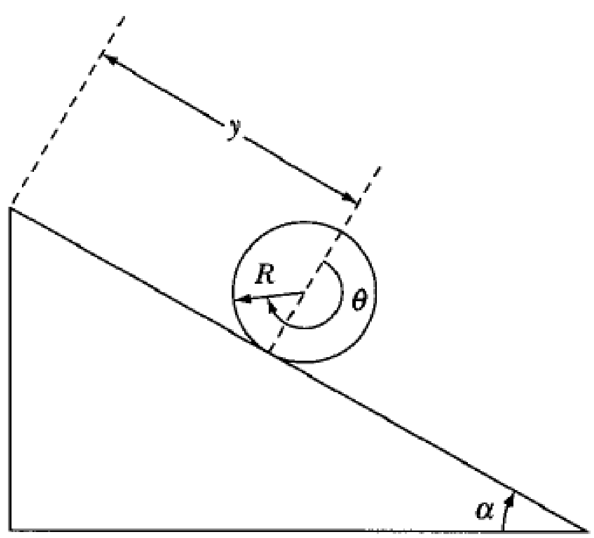

FIGURE 6-7 Example 6.5. A disk rolls down an inclined plane without slipping.

and

$$\frac{\partial g}{\partial y} = 1, \qquad \frac{\partial g}{\partial \theta} = -R \tag{6.74}$$

are the quantities associated with λ, the single undetermined multiplier for this case.

The constraint equation can also appear in an integral form. Consider the isoperimetric problem that is stated as finding the curve $y = y(x)$ for which the functional

$$J[y] = \int_a^b f\{y, y'; x\} dx \tag{6.75}$$

has an extremum, and the curve $y(x)$ satisfies boundary conditions $y(a) = A$ and $y(b) = B$ as well as the second functional

$$K[y] = \int_a^b g\{y, y'; x\} dx \tag{6.76}$$

that has a fixed value for the length of the curve (ℓ). This second functional represents an integral constraint.

Similarly to what we have done previously,* there will be a constant λ such that $y(x)$ is the extremal solution of the functional

$$\int_a^b (f + \lambda g)\, dx. \tag{6.77}$$

The curve $y(x)$ then will satisfy the differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) = 0 \tag{6.78}$$

subject to the constraints $y(a) = A$, $y(b) = B$, and $K[y] = \ell$. We will work an example for this so-called *Dido Problem.*†

EXAMPLE 6.6

One version of the Dido Problem is to find the curve $y(x)$ of length ℓ bounded by the x-axis on the bottom that passes through the points $(-a, 0)$ and $(a, 0)$ and encloses the largest area. The value of the endpoints a is determined by the problem.

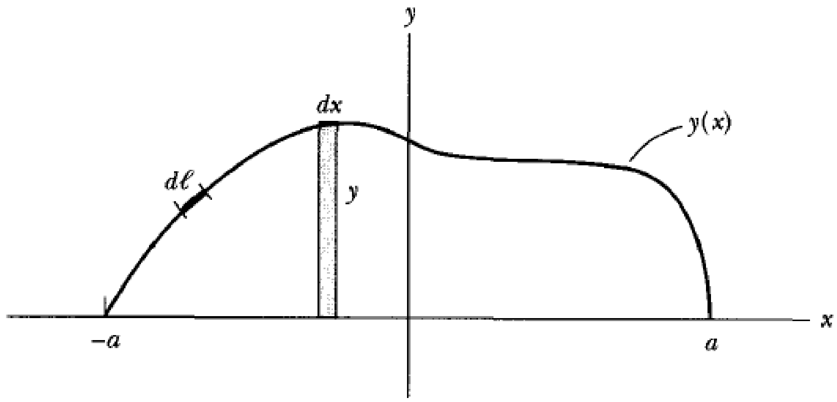

FIGURE 6-8 Example 6.6. We want to find the curve $y(x)$ that maximizes the area above the $y = 0$ line consistent with a fixed perimeter length. The curve must go through $x = -a$ and a. The differential area $dA = ydx$, and the differential length along the curve is $d\ell$.

Solution. We can use the equations just developed to solve this problem. We show in Figure 6-8 that the differential area $dA = y\,dx$. We want to maximize the area, so we want to find the extremum solution for Equation 6.75, which becomes

$$J = \int_{-a}^{a} y\,dx \tag{6.79}$$

The constraint equations are

$$y(x): y(-a) = 0, y(a) = 0 \quad \text{and} \quad K = \int d\ell = \ell. \tag{6.80}$$

The differential length along the curve $d\ell = (dx^2 + dy^2)^{1/2} = (1 + y'^2)^{1/2}\, dx$ where $y' = dy/dx$. The constraint functional becomes

$$K = \int_{-a}^{a} [1 + y'^2]^{1/2} dx = \ell. \tag{6.81}$$

We now have $y(x) = y$ and $g(x) = \sqrt{1 + y'^2}$, and we use these functions in Equation 6.78.

$$\frac{\partial f}{\partial y} = 1, \qquad \frac{\partial f}{\partial y'} = 0, \qquad \frac{\partial g}{\partial y} = 0, \qquad \frac{\partial g}{\partial y'} = \frac{y'}{(1 + y'^2)^{1/2}}$$

Equation 6.78 becomes

$$1 - \lambda \frac{d}{dx}\left[\frac{y'}{(1 + y'^2)^{1/2}}\right] = 0 \tag{6.82}$$

We manipulate Equation 6.82 to find

$$\frac{d}{dx}\left[\frac{y'}{(1 + y'^2)^{1/2}}\right] = \frac{1}{\lambda} \tag{6.83}$$

We integrate over x to find

$$\frac{\lambda y'}{\sqrt{(1 + y'^2)}} = x - C_1$$

where C_1 is an integration constant. This can be rearranged to be

$$dy = \frac{\pm(x - C_1)\, dx}{\sqrt{\lambda^2 - (x - C_1)^2}}$$

This equation is integrated to find

$$y = \mp\sqrt{\lambda^2 - (x - C_1)^2} + C_2 \tag{6.84}$$

where C_2 is another integration constant. We can rewrite this as the equation of a circle of radius λ.

$$(x - C_1)^2 + (y - C_2)^2 = \lambda^2 \tag{6.85}$$

The maximum area is a semicircle bounded by the $y = 0$ line. The semicircle must go through (x, y) points of $(-a, 0)$ and $(a, 0)$, which means the circle must be centered at the origin, so that $C_1 = 0 = C_2$, and the radius $= a = \lambda$. The perimeter of the top half of the semicircle is what we called ℓ, and the perimeter length of a half circle is πa. Therefore, we have $\pi a = \ell$, and $a = \ell/\pi$.