Electrodynamics II

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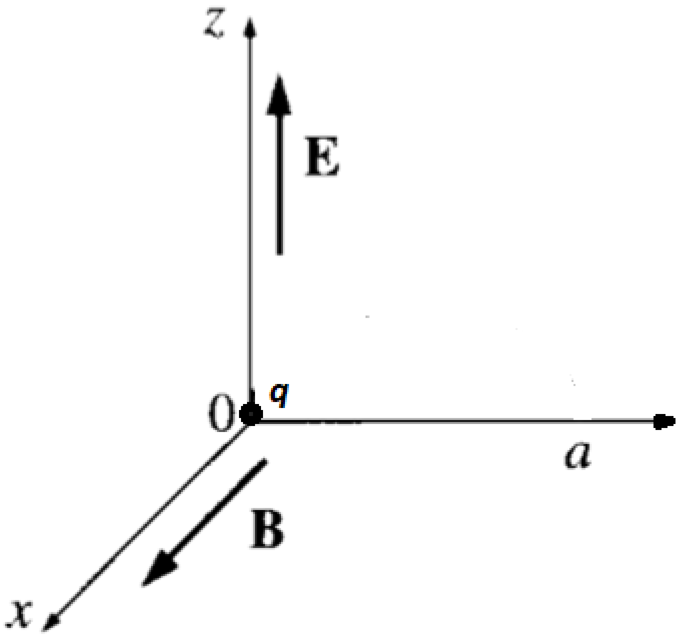
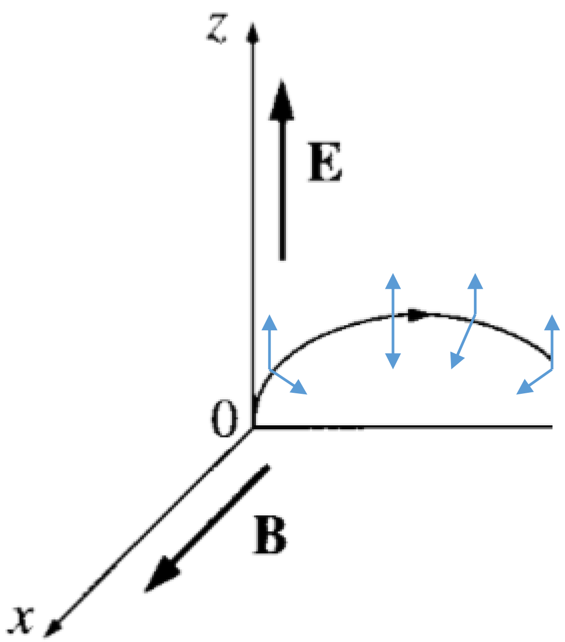
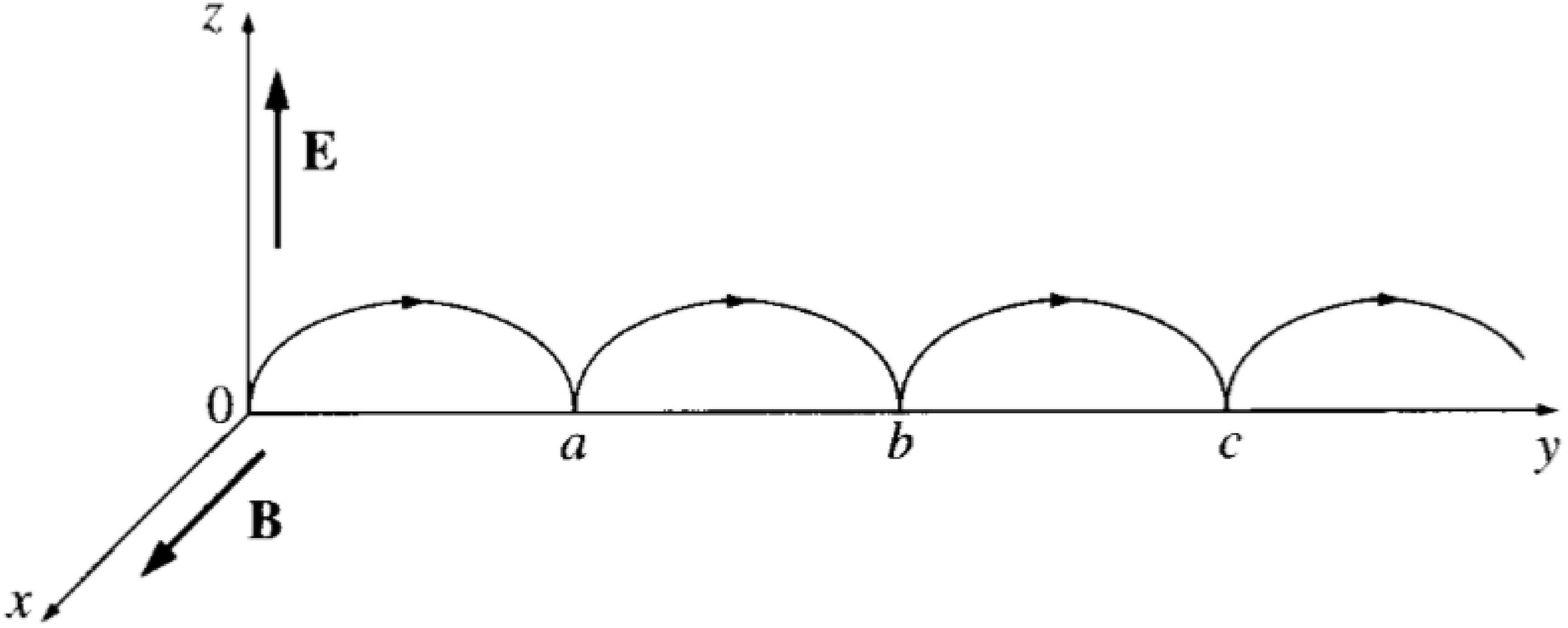
Cycloid Motion

A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that **B** points in the x-direction, and **E** in the z-direction, as shown in Fig. 5.7. A particle at rest is released from the origin; what path will it follow?







z
E
0
a
b
c
y
B
x

[Position of partic$\dot{y}$le at any time= $(0, y(t), z(t))$

Velocity of particle at any time= $(0, \dot{y}, \dot{z})$ $\quad where\ \ \dot{y} = \frac{dy}{dt}$

where dot represents the time derivatives

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = \mathbf{0}\hat{x} + B\dot{z}\hat{y} - B\dot{y}\hat{z}$$

$$\vec{E} = E\hat{z}$$

$$F = Q(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}) = Q(E\hat{z}+B\dot{z}\hat{y} - B\dot{y}\hat{z})$$

$$\vec{F} = QB\dot{z}\hat{y} + Q(E - B\dot{y})\hat{z}$$

and hence, applying Newton's second law $\vec{F} = m\vec{a} = m(\ddot{y}\hat{y} + \ddot{z}\hat{z})$

$$m(\ddot{y}\hat{y} + \ddot{z}\hat{z}) = QB\dot{z}\hat{y} + Q(E - B\dot{y})\hat{z}$$

$$m\ddot{y} = QB\dot{z} \qquad m\ddot{z} = Q(E - B\dot{y})$$

- For convenience we let

$$\omega = \frac{QB}{m}$$

$$m\ddot{y} = QB\dot{z} \quad \Rightarrow \quad \ddot{y} = \omega\dot{z}$$

$$m\ddot{z} = Q(E - B\dot{y}) \quad \Rightarrow \quad \ddot{z} = \frac{QB}{m}(E/B - \dot{y})$$

$$\ddot{z} = \omega(E/B - \dot{y})$$

$$\ddot{y} = \omega \dot{z}$$ $\Rightarrow$

$$\ddot{y} = \frac{d^2 y}{dt^2} = \omega \frac{\dot{d}y}{dt} \qquad (1)$$

$$\ddot{z} = \omega(E/B - \dot{y})$$ $\Rightarrow$

$$\ddot{z} = \frac{d^2 z}{dt^2} = \omega(E/B - \frac{\dot{d}y}{dt}) \qquad (2)$$

The general solution of equation (1) and (2) is

$$y(t) = C_1 cos\omega t + C_2 sin\omega t + \left(\frac{E}{B}\right) t + C_3 \dots\dots\dots\dots\dots\dots\dots..(3)$$

$$z(t) = C_2 cos\omega t - C_1 sin\omega t + C_4 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.(4)$$

Boundary Conditions of The Problem

(i) Particle starts from rest.

$$\dot{y}(0) = \dot{z}(0) = 0$$

(ii) Particles starts from origin.

$$y(0) = z(0) = 0$$

To apply the boundary conditions we take the derivatives of equation (1) and (2)

$$\dot{y}(t) = -C_1\omega sin\omega t + C_2\omega cos\omega t + \left(\frac{E}{B}\right) \dots\dots\dots\dots\dots\dots\dots..(5)$$

$$\dot{z}(t) = -C_2\omega sin\omega t - C_1\omega cos\omega t \dots\dots\dots\dots\dots\dots\dots..(6)$$

Apply these B.Cs $y(0) = z(0) = 0$ to the equations

$y(t) = C_1 cos\omega t + C_2 sin\omega t + \left(\frac{E}{B}\right) t + C_3$ and

z$(t) = C_2 cos\omega t - C_1 sin\omega t + C_4$

$$y(0) = C_1 cos\omega(0) + C_2 sin\omega(0) + \left(\frac{E}{B}\right)(0) + C_3$$

And $z(0) = C_2 cos\omega(0) - C_1 sin\omega(0) + C_4$

We get

0$= C_1 + C_3$

0$= C_2 + C_4$

Apply these B.Cs $\dot{y}(0) = \dot{z}(0) = 0$ to the equations

$$\dot{y}(t) = -C_1\omega sin\omega t + C_2\omega cos\omega t + \left(\frac{E}{B}\right)$$ and

$$\dot{z}(t) = -C_2\omega sin\omega t - C_1\omega cos\omega t$$

$$0 = -C_1\omega sin\omega(0) + C_2\omega cos\omega(0) + \left(\frac{E}{B}\right)$$

And $$0 = -C_2\omega sin\omega(0) - C_1\omega cos\omega(0)$$

$$0 = \omega C_2 + \left(\frac{E}{B}\right)$$

$$C_2 = -\left(\frac{E}{\omega B}\right)$$

$$\boxed{C_1 = 0}$$

The equations of constants after applying boundary conditions are

$$C_1 + C_3 = 0$$

$$\boxed{C_2 = -\left(\frac{E}{\omega B}\right)}$$

$$C_2 + C_4 = 0$$

$$\boxed{C_1 = 0}$$

$$\boxed{C_3 = 0}$$

$$\boxed{C_4 = \frac{E}{\omega B}}$$

$$y(t) = (0)cos\omega t - \left(\frac{E}{\omega B}\right) sin\omega t + \left(\frac{E}{B}\right) t + 0$$

$$y(t) = \left(\frac{E}{B}\right) t - \left(\frac{E}{\omega B}\right) sin\omega t + +0$$

$$y(t) = \left(\frac{E}{\omega B}\right) \omega t - \left(\frac{E}{\omega B}\right) sin\omega t$$

$$y(t) = \left(\frac{E}{\omega B}\right) (\omega t - sin\omega t)$$

$$\mathrm{z}(t) = -\left(\frac{E}{\omega B}\right) cos\omega t - (0)sin\omega t + \frac{E}{\omega B}$$

$$\mathrm{z}(t) = \frac{E}{\omega B} - \left(\frac{E}{\omega B}\right) cos\omega t$$

$$z(t) = \frac{E}{\omega B} (1 - cos\omega t)$$

$$y(t) = \left(\frac{E}{\omega B}\right)(\omega t - sin\omega t)$$

$$z(t) = \frac{E}{\omega B} - \left(\frac{E}{\omega B}\right) cos\omega t$$

$$y(t) = \left(\frac{E}{\omega B}\right)(\omega t - sin\omega t)$$

$$z(t) = \frac{E}{\omega B}(1 - cos\omega t)$$

Let $R = \frac{E}{\omega B}$

$$y(t) = R\ (\omega t - sin\omega t)$$

$$z(t) = R\ (1 - cos\omega t)$$

$$(y(t) - R\,\omega t) = Rsin\omega t$$

$$(y(t) - R\,\omega t)^2 = (Rsin\omega t)^2 \qquad (R - z(t))^2 = (Rcos\omega t)^2$$

Adding these two

$$(z(t) - R)^2 + (y(t) - R\,\omega t)^2 = R^2$$

Which is the equation of a circle of radius R with center $(0, R\omega t, R)$ whose center moves with the constant speed

$$v = R\omega = \frac{E}{B}$$