6.4 The "Second Form" of the Euler Equation

Choose two points located at (x_1, y_1) and (x_2, y_2) joined by a curve $y(x)$. We want to find $y(x)$ such that if we revolve the curve around the x-axis, the surface area of the revolution is a minimum. This is the "soap film" problem, because a soap film suspended between two wire circular rings takes this shape (Figure 6-6). We want to minimize the integral of the area $dA = 2\pi y\ ds$ where $ds = \sqrt{1 + y'^2}\, dx$ and $y' = dy/dx$.

$$A = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + y'^2}dx \tag{6.35}$$

We find the extremum by setting $f = y\sqrt{1 + y'^2}$ and inserting into Equation 6.18. The derivatives we need are

$$\frac{\partial f}{\partial y} = \sqrt{1 + y'^2}$$

$$\frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}}$$

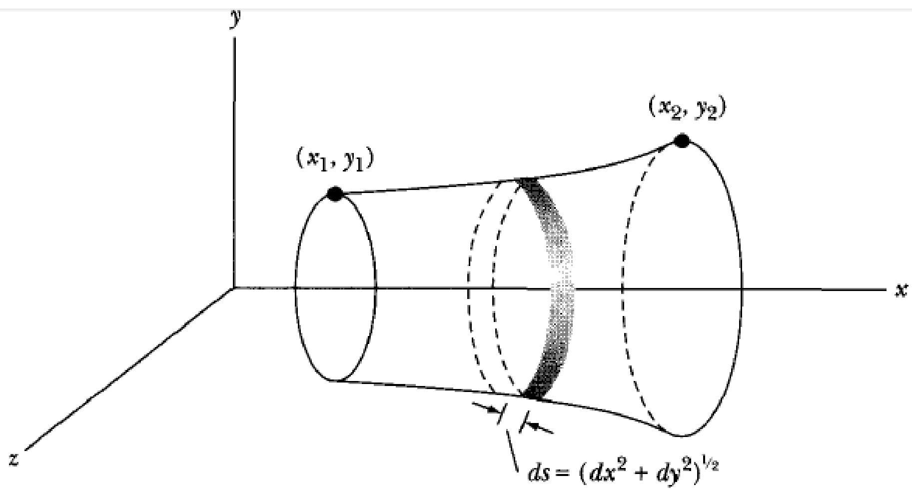

FIGURE 6-6 The "soap film" problem in which we want to minimize the surface area of revolution around the x-axis.

Equation 6.18 becomes

$$\sqrt{1 + y'^2} = \frac{d}{dx}\left[\frac{yy'}{\sqrt{1 + y'^2}}\right] \tag{6.36}$$

Equation 6.36 does not appear to be a simple equation to solve for $y(x)$. Let's stop and think about whether there might be an easier method of solution. You may have noticed that this problem is just like Example 6.3, but in that case we were minimizing a surface of revolution about the y-axis rather than around the x-axis. The solution to the soap film problem should be identical to Equation 6.34 if we interchange x and y. But how did we end up with such a complicated equation as Equation 6.36? We blindly chose x as the independent variable and decided to find the function $y(x)$. In fact, in general, we can choose the independent variable to be anything we want: x, θ, t, or even y. If we choose y as the independent variable, we would need to interchange x and y in many of the previous equations that led up to Euler's equation (Equation 6.18). It might be easier in the beginning to just interchange the variables that we started with (i.e., call the horizontal axis y in Figure 6-6 and let the independent variable be x). (In a right-handed coordinate system, the x-direction would be down, but that presents no difficulty in this case because of symmetry.) No matter what we do, the solution of our present problem would just parallel Example 6.3. Unfortunately, it is not always possible to look ahead to make the best choice of independent variable. Sometimes we just have to proceed by trial and error.

A second equation may be derived from Euler's equation that is convenient for functions that do not explicitly depend on x: $\partial f/\partial x = 0$. We first note that for any function $f(y, y'; x)$ the derivative is a sum of terms

$$\begin{aligned}\frac{df}{dx} = \frac{d}{dx}f\{y, y'; x\} &= \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x}\\ &= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}\end{aligned} \tag{6.37}$$

Also

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'}$$

or, substituting from Equation 6.37 for $y''(\partial f/\partial y')$,

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \tag{6.38}$$

The last two terms in Equation 6.38 may be written as

$$y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right)$$

which vanishes in view of the Euler equation (Equation 6.18). Therefore,

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0} \tag{6.39}$$

We can use this so-called "second form" of the Euler equation in cases in which f does not depend explicitly on x, and $\partial f/\partial x = 0$. Then,

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} \qquad \left(\text{for } \frac{\partial f}{\partial x} = 0\right) \tag{6.40}$$

Assignment!

EXAMPLE 6.4

A *geodesic* is a line that represents the shortest path between any two points when the path is restricted to a particular surface. Find the geodesic on a sphere.

6.5 Functions with Several Dependent Variables

The Euler equation derived in the preceding section is the solution of the variational problem in which it was desired to find the single function $y(x)$ such that the integral of the functional f was an extremum. The case more commonly encountered in mechanics is that in which f is a functional of several dependent variables:

$$f = f\{y_1(x), y_1'(x), y_2(x), y_2'(x), \cdots; x\} \tag{6.53}$$

or simply

$$f = f\{y_i(x), y_i'(x); x\}, \quad i = 1, 2, \cdots, n \tag{6.54}$$

In analogy with Equation 6.2, we write

$$y_i(\alpha, x) = y_i(0, x) + \alpha\eta_i(x) \tag{6.55}$$

The development proceeds analogously (cf. Equation 6.17), resulting in

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y'_i} \right) \eta_i(x) \, dx \tag{6.56}$$

Because the individual variations—the $\eta_i(x)$—are all independent, the vanishing of Equation 6.56 when evaluated at $\alpha = 0$ requires the separate vanishing of *each* expression in the brackets:

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y'_i} = 0, \qquad i = 1, 2, \cdots, n} \tag{6.57}$$