6.3 Euler's Equation

To determine the result of the condition expressed by Equation 6.4, we perform the indicated differentiation in Equation 6.3:

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f\{y, y'; x\}\, dx \tag{6.11}$$

Because the limits of integration are fixed, the differential operation affects only the integrand. Hence,

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx \tag{6.12}$$

From Equation 6.2, we have

$$\frac{\partial y}{\partial \alpha} = \eta(x); \qquad \frac{\partial y'}{\partial \alpha} = \frac{d\eta}{dx} \tag{6.13}$$

Equation 6.12 becomes

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right) dx \tag{6.14}$$

The second term in the integrand can be integrated by parts:

$$\int u\, dv = uv - \int v\, du \tag{6.15}$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\, dx = \frac{\partial f}{\partial y'}\eta(x)\Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\, dx \tag{6.16}$$

The integrated term vanishes because $\eta(x_1) = \eta(x_2) = 0$. Therefore, Equation 6.12 becomes

$$\begin{aligned}\frac{\partial J}{\partial \alpha} &= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \eta(x) \right] dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \eta(x)\, dx \end{aligned} \tag{6.17}$$

The integral in Equation 6.17 now appears to be independent of α. But the functions y and y' with respect to which the derivatives of f are taken are still functions of α. Because $(\partial J/\partial \alpha)|_{\alpha=0}$ must vanish for the extremum value and because $\eta(x)$ is an arbitrary function (subject to the conditions already stated), the integrand in Equation 6.17 must itself vanish for $\alpha = 0$:

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \qquad \text{Euler's equation} \tag{6.18}$$

where now y and y' are the original functions, independent of α. This result is known as **Euler's equation,*** which is a necessary condition for J to have an extremum value.

EXAMPLE 6.2

We can use the calculus of variations to solve a classic problem in the history of physics: the *brachistochrone.*† Consider a particle moving in a constant force field starting at rest from some point (x_1, y_1) to some lower point (x_2, y_2). Find the path that allows the particle to accomplish the transit in the least possible time.

Solution. The coordinate system may be chosen so that the point (x_1, y_1) is at the origin. Further, let the force field be directed along the positive x-axis as in Figure 6-3. Because the force on the particle is constant—and if we ignore the possibility of friction—the field is conservative, and the total energy of the particle is $T + U =$ const. If we measure the potential from the point $x = 0$ [i.e., $U(x = 0) = 0$], then, because the particle starts from rest, $T + U = 0$. The kinetic energy is $T = \frac{1}{2}mv^2$, and the potential energy is $U = -Fx = -mgx$, where g is the acceleration imparted by the force. Thus

$$v = \sqrt{2gx} \tag{6.19}$$

The time required for the particle to make the transit from the origin to (x_2, y_2) is

$$t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \int \frac{(dx^2 + dy^2)^{1/2}}{(2gx)^{1/2}}$$
$$= \int_{x_1=0}^{x_2} \left(\frac{1 + y'^2}{2gx}\right)^{1/2} dx \tag{6.20}$$

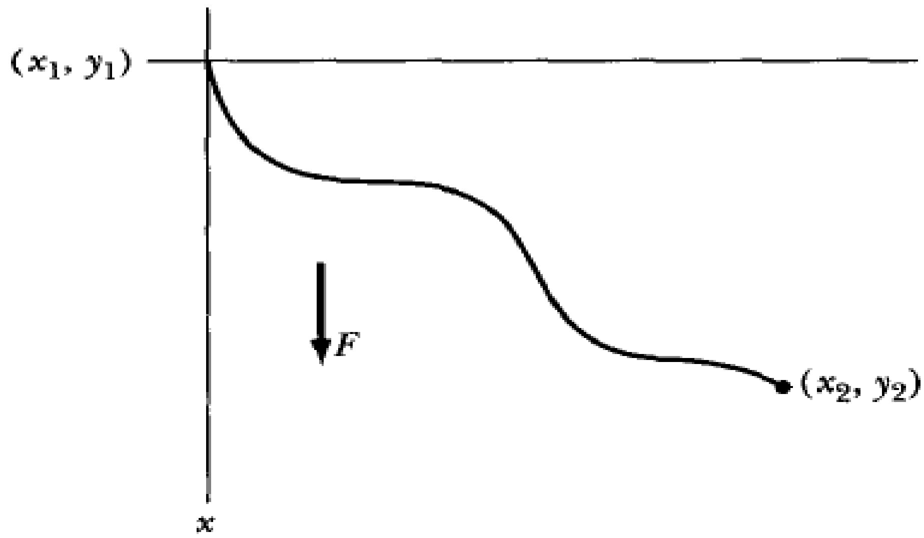

FIGURE 6-3 Example 6.2. The *brachistochrone* problem is to find the path of a particle moving from (x_1, y_1) to (x_2, y_2) that occurs in the least possible time. The force field acting on the particle is *F*, which is down and constant.

The time of transit is the quantity for which a minimum is desired. Because the constant $(2g)^{-1/2}$ does not affect the final equation, the function f may be identified as

$$f = \left(\frac{1 + y'^2}{x}\right)^{1/2} \tag{6.21}$$

And, because $\partial f/\partial y = 0$, the Euler equation (Equation 6.18) becomes

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y'} = \text{constant} \equiv (2a)^{-1/2}$$

where a is a new constant.

Performing the differentiation $\partial f/\partial y'$ on Equation 6.21 and squaring the result, we have

$$\frac{y'^2}{x(1 + y'^2)} = \frac{1}{2a} \tag{6.22}$$

This may be put in the form

$$y = \int \frac{x\,dx}{(2ax - x^2)^{1/2}} \tag{6.23}$$

We now make the following change of variable:

$$\begin{aligned} x &= a(1 - \cos\theta) \\ dx &= a \sin\theta \, d\theta \end{aligned} \tag{6.24}$$

The integral in Equation 6.23 then becomes

$$y = \int a(1 - \cos\theta)\,d\theta$$

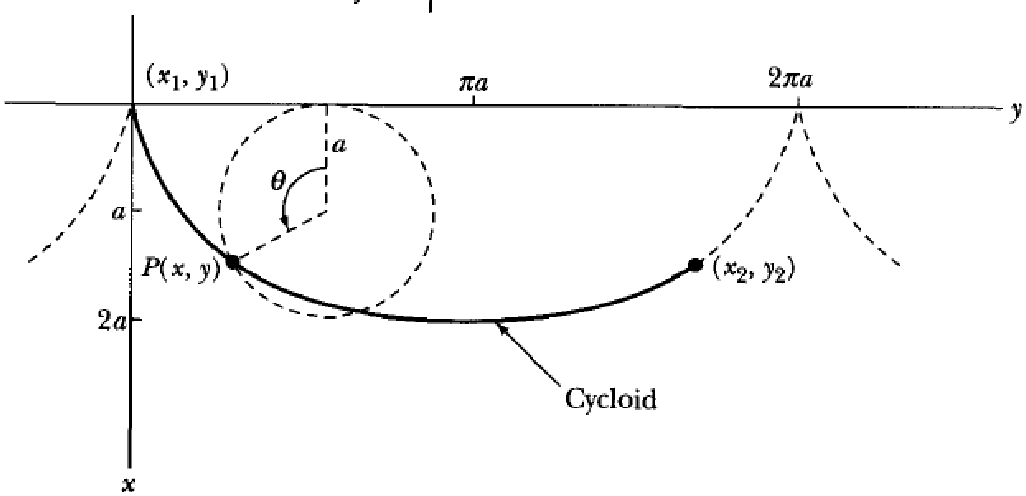

FIGURE 6-4 Example 6.2. The solution of the brachistochrone problem is a cycloid.

and

$$y = a(\theta - \sin\theta) + \text{constant} \tag{6.25}$$

The parametric equations for a *cycloid** passing through the origin are

$$\left.\begin{aligned} x &= a(1 - \cos\theta) \\ y &= a(\theta - \sin\theta) \end{aligned}\right\} \tag{6.26}$$

which is just the solution found, with the constant of integration set equal to zero to conform with the requirement that (0, 0) is the starting point of the motion. The path is then as shown in Figure 6-4, and the constant a must be adjusted to allow the cycloid to pass through the specified point (x_2, y_2).

Solving the problem of the brachistochrone does indeed yield a path the particle traverses in a *minimum* time. But the procedures of variational calculus are designed only to produce an extremum—either a minimum or a maximum. It is almost always the case in dynamics that we desire (and find) a minimum for the problem.

EXAMPLE 6.3

Consider the surface generated by revolving a line connecting two fixed points (x_1, y_1) and (x_2, y_2) about an axis coplanar with the two points. Find the equation of the line connecting the points such that the surface area generated by the revolution (i.e., the area of the surface of revolution) is a minimum.

Solution. We assume that the curve passing through (x_1, y_1) and (x_2, y_2) is revolved about the y-axis, coplanar with the two points. To calculate the total area

$$dA = 2\pi x\, ds = 2\pi x(dx^2 + dy^2)^{1/2} \tag{6.27}$$

$$A = 2\pi \int_{x_1}^{x_2} x(1 + y'^2)^{1/2}\, dx \tag{6.28}$$

where $y' = dy/dx$. To find the extremum value we let

$$f = x(1 + y'^2)^{1/2} \tag{6.29}$$

and insert into Equation 6.18:

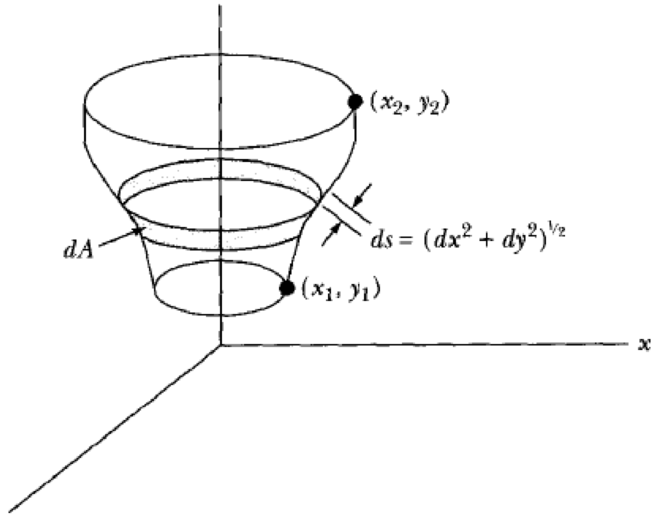

FIGURE 6-5 Example 6.3. The geometry of the problem and area dA are indicated to minimize the surface of revolution around the y-axis.

$$dA = 2\pi x\, ds = 2\pi x(dx^2 + dy^2)^{1/2} \tag{6.27}$$

$$A = 2\pi \int_{x_1}^{x_2} x(1 + y'^2)^{1/2}\, dx \tag{6.28}$$

where $y' = dy/dx$. To find the extremum value we let

$$f = x(1 + y'^2)^{1/2} \tag{6.29}$$

and insert into Equation 6.18:

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{xy'}{(1 + y'^2)^{1/2}}$$

therefore,

$$\frac{d}{dx}\left[\frac{xy'}{(1 + y'^2)^{1/2}}\right] = 0$$

$$\frac{xy'}{(1 + y'^2)^{1/2}} = \text{constant} \equiv a \tag{6.30}$$

From Equation 6.30, we determine

$$y' = \frac{a}{(x^2 - a^2)^{1/2}} \tag{6.31}$$

$$y = \int \frac{a\,dx}{(x^2 - a^2)^{1/2}} \qquad \textbf{(6.32)}$$

The solution of this integration is

$$y = a \cos \mathrm{h}^{-1}\left(\frac{x}{a}\right) + b \qquad \textbf{(6.33)}$$

where a and b are constants of integration determined by requiring the curve to pass through the points (x_1, y_1) and (x_2, y_2). Equation 6.33 can also be written as

$$x = a \cos \mathrm{h}\left(\frac{y - b}{a}\right) \qquad \textbf{(6.34)}$$

which is more easily recognized as the equation of a *catenary*, the curve of a flexible cord hanging freely between two points of support.