6.1 Introduction

Many problems in Newtonian mechanics are more easily analyzed by means of alternative statements of the laws, including **Lagrange's equation** and **Hamilton's principle.*** As a prelude to these techniques, we consider in this chapter some general principles of the techniques of the calculus of variations.

Emphasis will be placed on those aspects of the theory of variations that have a direct bearing on classical systems, omitting some existence proofs. Our primary interest here is in determining the path that gives extremum solutions, for example, the shortest distance (or time) between two points. A well-known example of the use of the theory of variations is **Fermat's principle:** Light travels by the path that takes the least amount of time (see Problem 6-7).

6.2 Statement of the Problem

The basic problem of the calculus of variations is to determine the function $y(x)$ such that the integral

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\}\, dx \tag{6.1}$$

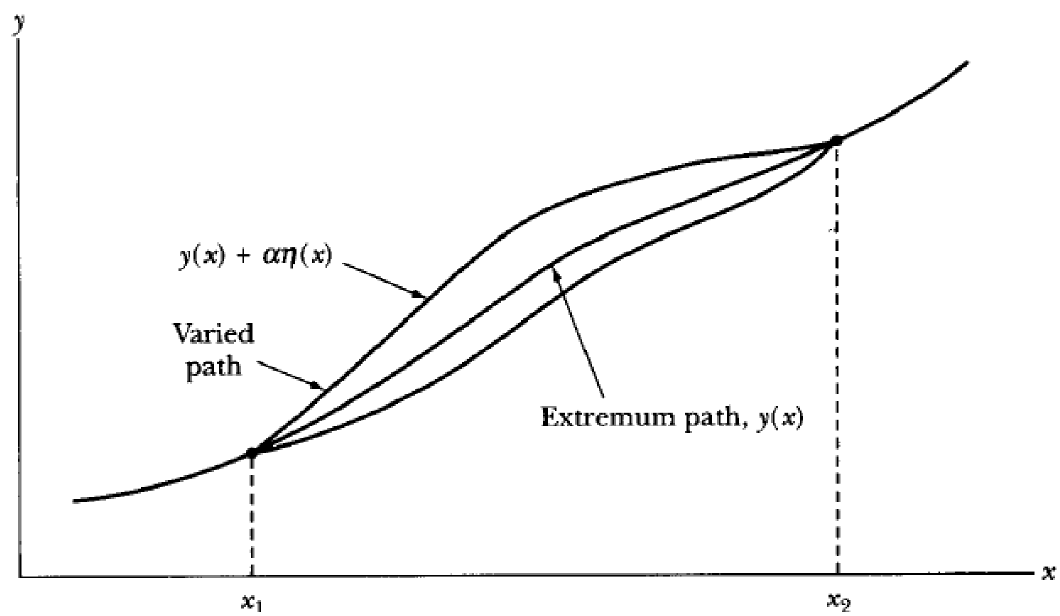

FIGURE 6-1 The function $y(x)$ is the path that makes the functional J an extremum. The neighboring functions $y(x) + \alpha\eta(x)$ vanish at the endpoints and may be close to $y(x)$, but are not the extremum.

If functions of the type given by Equation 6.2 are considered, the integral J becomes a functional of the parameter α:

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\}\, dx \tag{6.3}$$

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is an *extremum* (i.e., either a maximum or a minimum). In Equation 6.1, $y'(x) \equiv dy/dx$, and the semicolon in f separates the independent variable x from the dependent variable $y(x)$ and its derivative $y'(x)$. The functional* J depends on the function $y(x)$, and the limits of integration are fixed.† The function $y(x)$ is then to be varied until an extreme value of J is found. By this we mean that if a function $y = y(x)$ gives the integral J a minimum value, then any *neighboring function,* no matter how close to $y(x)$, must make J increase. The definition of a neighboring function may be made as follows. We give all possible functions y a parametric representation $y = y(\alpha, x)$ such that, for $\alpha = 0$, $y = y(0, x) = y(x)$ is the function that yields an extremum for J. We can then write

$$y(\alpha, x) = y(0, x) + \alpha\eta(x) \tag{6.2}$$

where $\eta(x)$ is some function of x that has a continuous first derivative and that vanishes at x_1 and x_2, because the varied function $y(\alpha, x)$ must be identical with $y(x)$ at the endpoints of the path: $\eta(x_1) = \eta(x_2) = 0$. The situation is depicted schematically in Figure 6-1.

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$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\}\, dx \tag{6.3}$$

The condition that the integral have a *stationary value* (i.e., that an extremum results) is that J be independent of α in first order along the path giving the extremum ($\alpha = 0$), or, equivalently, that

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{6.4}$$

for all functions $\eta(x)$. This is only a *necessary* condition; it is not sufficient.

EXAMPLE 6.1

Consider the function $f = (dy/dx)^2$, where $y(x) = x$. Add to $y(x)$ the function $\eta(x) = \sin x$, and find $J(\alpha)$ between the limits of $x = 0$ and $x = 2\pi$. Show that the stationary value of $J(\alpha)$ occurs for $\alpha = 0$.

Solution. We may construct neighboring varied paths by adding to $y(x)$,

$$y(x) = x \tag{6.5}$$

the sinusoidal variation $\alpha \sin x$,

$$y(\alpha, x) = x + \alpha \sin x \tag{6.6}$$

These paths are illustrated in Figure 6-2 for $\alpha = 0$ and for two different nonvanishing values of α. Clearly, the function $\eta(x) = \sin x$ obeys the endpoint conditions, that is, $\eta(0) = 0 = \eta(2\pi)$. To determine $f(y, y'; x)$ we first determine

$$\frac{dy(\alpha, x)}{dx} = 1 + \alpha \cos x \tag{6.7}$$

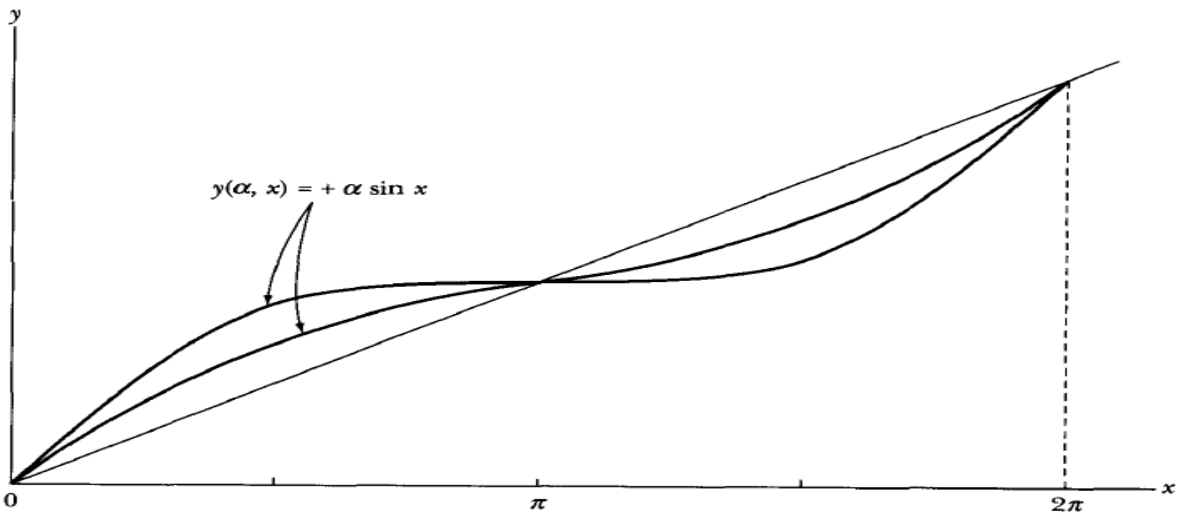

FIGURE 6-2 Example 6.1. The various paths $y(\alpha, x) = x + \alpha \sin x$. The extremum path occurs for $\alpha = 0$.

then

$$f = \left(\frac{dy(\alpha, x)}{dx}\right)^2 = 1 + 2\alpha \cos x + \alpha^2 \cos^2 x \tag{6.8}$$

Equation 6.3 now becomes

$$J(\alpha) = \int_0^{2\pi} (1 + 2\alpha \cos x + \alpha^2 \cos^2 x)\, dx \tag{6.9}$$

$$= 2\pi + \alpha^2 \pi \tag{6.10}$$

Thus we see the value of $J(\alpha)$ is always greater than $J(0)$, no matter what value (positive or negative) we choose for α. The condition of Equation 6.4 is also satisfied.