Chapter 17
Simultaneous Equations Models

In any regression modeling, generally, an equation is considered to represent a relationship describing a phenomenon. Many situations involve a **set of relationships** which explain the behaviour of certain variables. For example, in analyzing the market conditions for a particular commodity, there can be a demand equation and a supply equation which explain the price and quantity of commodity exchanged in the market at market equilibrium. So there are two equations to explain the whole phenomenon - one for demand and another for supply. In such cases, it is not necessary that all the variables should appear in all the equations. So estimation of parameters under this type of situation has those features that are not present when a model involves only a single relationship. In particular, when a relationship is a part of a system, then some explanatory variables are stochastic and are correlated with the disturbances. So the basic assumption of a linear regression model that the explanatory variable and disturbance are uncorrelated or explanatory variables are fixed is violated and consequently ordinary least squares estimator becomes inconsistent.

Similar to the classification of variables as explanatory variable and study variable in linear regression model, the variables in simultaneous equation models are classified as endogenous variables and exogenous variables.

Endogenous variables (Jointly determined variables)

The variables which are explained by the functioning of system and values of which are determined by the simultaneous interaction of the relations in the model are endogenous variables or jointly determined variables.

Exogenous variables (Predetermined variables)

The variables that contribute to provide explanations for the endogenous variables and values of which are determined from outside the model are exogenous variables or predetermined variables.

Exogenous variables help is explaining the variations in endogenous variables. It is customary to include past values of endogenous variables in the predetermined group. Since exogenous variables are predetermined, so they are independent of disturbance term in the model. They satisfy those assumptions which explanatory variables satisfy in the usual regression model. Exogenous variables influence the endogenous variables but are not themselves influenced by them. One variable which is endogenous for one model can be exogenous variable for the other model.

Note that in the linear regression model, the explanatory variables influence the study variable but not vice versa. So relationship is one sided.

The classification of variables as endogenous and exogenous is important because a necessary condition for uniquely estimating all the parameters is that the number of endogenous variables is equal to the number of independent equations in the system. Moreover, the main distinction of predetermined variable in estimation of parameters is that they are uncorrelated with disturbance term in the equations in which they appear.

Simultaneous equation systems:

A model constitutes a system of simultaneous equations if all the relationships involved are needed for determining the value of at least one of the endogenous variables included in the model. This implies that at least one of the relationships includes more them one endogenous variable.

Example 1:

Now we consider the following example in detail and introduce various concepts and terminologies used in describing the simultaneous equations models.

Consider a situation of an ideal market where transaction of only one commodity, say wheat, takes place. Assume that the number of buyers and sellers is large so that the market is a perfectly competitive market. It is also assumed that the amount of wheat that comes into the market in a day is completely sold out on the same day. No seller takes it back. Now we develop a model for such mechanism.

Let

d_t denotes the demand of the commodity, say wheat, at time t,

s_t denotes the supply of the commodity, say wheat, at time t, and

q_t denotes the quantity of the commodity, say wheat, transacted at time t.

By economic theory about the ideal market, we have the following condition:

$$d_t = s_t,\ t = 1, 2, ..., n .$$

Observe that

- the demand of wheat depends on
  - price of wheat (p_t) at time t.
  - income of buyer (i_t) at time t.
- the supply of wheat depends on
  - price of wheat (p_t) at time t.
  - rainfall (r_t) at time t.

From market conditions, we have

$$q_t = d_t = s_t.$$

Demand, supply and price are determined from each other.

Note that

- income can influence demand and supply, but demand and supply cannot influence the income.
- supply is influenced by rainfall, but rainfall is not influenced by the supply of wheat.

Our aim is to study the behaviour of s_t, p_t and r_t which are determined by the simultaneous equation model.

Since endogenous variables are influenced by exogenous variables but not vice versa, so

- s_t, p_t and r_t are endogenous variables.
- i_t and r_t are exogenous variables.

Now consider an additional variable for the model as lagged value of price p_t, denoted as p_{t-1}. In a market, generally the price of the commodity depends on the price of the commodity on previous day. If the price of commodity today is less than the previous day, then buyer would like to buy more. For seller also, today's price of commodity depends on previous day's price and based on which he decides the quantity of commodity (wheat) to be brought in the market.

So the lagged price affects the demand and supply equations both. Updating both the models, we can now write that

- demand depends on p_t, i_t and p_{t-1}.
- supply depends on p_t, r_t and p_{t-1}.

Note that the lagged variables are considered as exogenous variable. The updated list of endogenous and exogenous variables is as follows:

- Endogenous variables: p_t, d_t, s_t
- Exogenous variables: p_{t-1}, i_t, r_t.

The mechanism of the market is now described by the following set of equations.

- demand $d_t = \alpha_1 + \beta_1 p_t + \varepsilon_{1t}$
- supply $s_t = \alpha_2 + \beta_2 p_t + \varepsilon_{2t}$
- equilibrium condition $d_t = s_t = q_t$

where $\alpha's$ denote the intercept terms, $\beta's$ denote the regression coefficients and $\varepsilon's$ denote the disturbance terms.

These equations are called **structural equations**. The error terms ε_1 and ε_2 are called **structural disturbances.** The coefficients $\alpha_1, \alpha_2, \beta_1$ and β_2 are called the **structural coefficients.**

The system of equations is called the **structural form of the model**.

Since $q_t = d_t = s_t$, so the demand and supply equations can be expressed as

$$q_t = \alpha_1 + \beta_1 p_t + \varepsilon_{1t} \quad \text{(I)}$$
$$q_t = \alpha_2 + \beta_2 p_t + \varepsilon_{2t} \quad \text{(II)}$$

So there are only two structural relationships. The price is determined by the mechanism of market and not by the buyer or supplier. Thus q_t and p_t are the endogenous variables. Without loss of generality, we can assume that the variables associated with α_1 and α_2 are X_1 and X_2 respectively such that $X_1 = 1$ and $X_2 = 1$. So $X_1 = 1$ and $X_2 = 1$ are predetermined and so they can be regarded as exogenous variables.

From the statistical point of view, we would like to write the model in such a form so that the OLS can be directly applied. So writing equations (I) and (II) as

$$\alpha_1 + \beta_1 p_t + \varepsilon_{1t} = \alpha_2 + \beta_2 p_t + \varepsilon_{2t}$$

or $$p_t = \frac{\alpha_1 - \alpha_2}{\beta_2 - \beta_1} + \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\beta_2 - \beta_1}$$

$$= \pi_{11} + v_{1t} \qquad \text{(III)}$$

$$q_t = \frac{\beta_2\alpha_1 - \beta_1\alpha_2}{\beta_2 - \beta_1} + \frac{\beta_2\varepsilon_{1t} - \beta_1\varepsilon_{2t}}{\beta_2 - \beta_1}$$

$$= \pi_{21} + v_{2t} \qquad \text{(IV)}$$

where

$$\pi_{11} = \frac{\alpha_1 - \alpha_2}{\beta_2 - \beta_1}, \quad \pi_{21} = \frac{\beta_2\alpha_1 - \beta_1\alpha_2}{\beta_2 - \beta_1}$$

$$v_{1t} = \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\beta_2 - \beta_1}, \quad v_{2t} = \frac{\beta_2\varepsilon_{1t} - \beta_1\varepsilon_{2t}}{\beta_2 - \beta_1}.$$

Each endogenous variable is expressed as a function of the exogenous variable. Note that the exogenous variable 1 (from $X_1 = 1, \text{ or } X_2 = 1)$ is not clearly identifiable.

The equations (III) and (IV) are called the **reduced form relationships** and in general, called the **reduced form of the model**.

The coefficients π_{11} and π_{21} are called **reduced form coefficients** and errors v_{1t} and v_{2t} are called the **reduced form disturbances.** The reduced from essentially express every endogenous variable as a function of exogenous variable. This presents a clear relationship between reduced form coefficients and structural coefficients as well as between structural disturbances and reduced form disturbances. The reduced form is ready for the application of OLS technique. The reduced form of the model satisfies all the assumptions needed for the application of OLS.

Suppose we apply OLS technique to equations (III) and (IV) and obtained the OLS estimates of π_{11} and π_{12} as $\hat{\pi}_{11}$ and $\hat{\pi}_{12}$ respectively which are given by

$$\hat{\pi}_{11} = \frac{\alpha_1 - \alpha_2}{\beta_2 - \beta_1}$$

$$\hat{\pi}_{21} = \frac{\beta_2\alpha_1 - \alpha_2\beta_1}{\beta_2 - \beta_1}.$$

Note that $\hat{\pi}_{11}$ and $\hat{\pi}_{21}$ are the numerical values of the estimates. So now there are two equations and four unknown parameters $\alpha_1, \alpha_2, \beta_1$ and β_2. So it is not possible to derive the unique estimates of parameters of the model by applying OLS technique to reduced form. This is known as **problem of identifications.**

By this example, the following have been described upto now:

- Structural form relationship.
- Reduced form relationship.
- Need for reducing the structural form into reduced form.
- Reason for the problem of identification.

Now we describe the problem of identification in more detail.

The identification problem:

Consider the model in earlier Example 1, which describes the behaviour of a perfectly competitive market in which only one commodity, say wheat, is transacted. The models describing the behaviour of consumer and supplier are prescribed by demand and supply conditions given as

$$\begin{aligned} &\text{Demand}: \quad d_t = \alpha_1 + \beta_1 p_t + \varepsilon_{1t},\; t = 1, 2, ..., n \\ &\text{Supply}: \quad s_t = \alpha_2 + \beta_2 p_t + \varepsilon_{2t} \\ &\text{Equilibrium condition}: \; d_t = s_t. \end{aligned}$$

If quantity q_t is transacted at time t then

$$d_t = s_t = q_t.$$

So we have two structural equations model in two endogenous variables $(q_t$ and $p_t)$ and one exogenous variable (value is 1 given by $X_1 = 1, X_2 = 1$). The set of three equations is reduced to a set of two equations as follows:

$$\begin{aligned} &\text{Demand}: \quad q_t = \alpha_1 + \beta_1 p_t + \varepsilon_{1t} \qquad (1) \\ &\text{Supply}: \quad q_t = \alpha_2 + \beta_2 p_t + \varepsilon_{2t} \qquad (2) \end{aligned}$$

Before analysis, we would like to check whether it is possible to estimate the parameters $\alpha_1, \alpha_2, \beta_1$ and β_2 or not.

Multiplying equations (1) by λ and (2) by $(1-\lambda)$ and then adding them together gives

$$\lambda q_t + (1-\lambda) q_t = \left[\lambda \alpha_1 + (1-\lambda)\alpha_2\right] + \left[\lambda \beta_1 + (1-\lambda)\beta_2\right] p_t + \left[\lambda \varepsilon_{1t} + (1-\lambda)\varepsilon_{2t}\right]$$

or $q_t = \alpha + \beta p_t + \varepsilon_t \qquad (3)$

where $\alpha = \lambda\alpha_1 + (1-\lambda)\alpha_2$, $\beta = \lambda\beta_1 + (1-\lambda)\beta_2$, $\varepsilon_t = \lambda\varepsilon_{1t} + (1-\lambda)\varepsilon_{2t}$ and λ is any scalar lying between 0 and 1.

Comparing equation (3) with equations (1) and (2), we notice that they have same form. So it is difficult to say that which is supply equation and which is demand equation. To find this, let equation (3) be demand equation. Then there is no way to identify the true demand equation (1) and pretended demand equation (3).

A similar exercise can be done for the supply equation, and we find that there is no way to identify the true supply equation (2) and pretended supply equation (3).

Suppose we apply OLS technique to these models. Applying OLS to equation (1) yields

$$\hat{\beta}_1 = \frac{\sum_{t=1}^{n}(p_t - \bar{p})(q_t - \bar{q})}{\sum_{t=1}^{n}(p_t - \bar{p})^2} = 0.6, \text{ say}$$

where $\bar{p} = \frac{1}{n}\sum_{t=1}^{n} p_t,\ \bar{q} = \frac{1}{n}\sum_{t=1}^{n} q_t.$

Applying OLS to equation (3) yields

$$\hat{\beta} = \frac{\sum_{t=1}^{n}(p_t - \bar{p})(q_t - \bar{q})}{\sum_{t=1}^{n}(p_t - \bar{p})^2} = 0.6.$$

Note that $\hat{\beta}_1$ and $\hat{\beta}$ have same analytical expressions, so they will also have same numerical values, say 0.6. Looking at the value 0.6, it is difficult to say that the value 0.6 determines equation (1) or (3).

Applying OLS to equation (3) fields

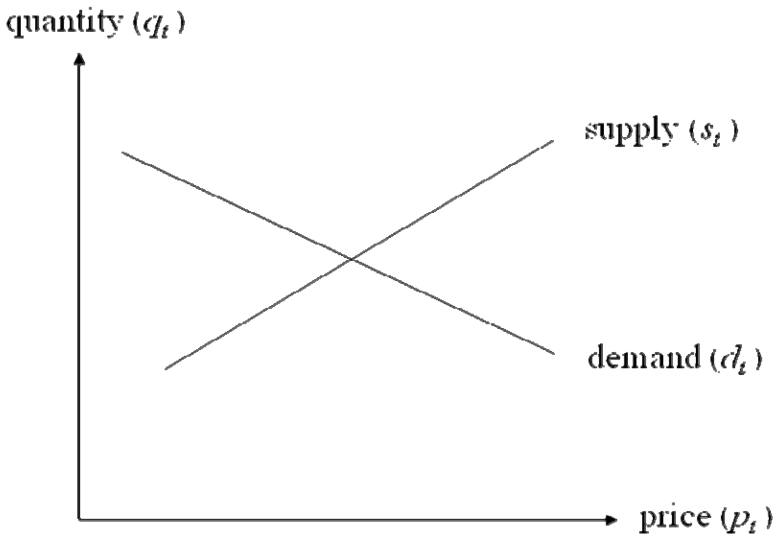
$$\hat{\beta}_2 = \frac{\sum_{t=1}^{n}(p_t - \bar{p})(q_t - \bar{q})}{\sum_{t=1}^{n}(p_t - \bar{p})^2} = 0.6$$

because $\hat{\beta}_2$ has the same analytical expression as of $\hat{\beta}_1$ and $\hat{\beta}$, so

$$\hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3.$$

Thus it is difficult to decide and identify whether $\hat{\beta}_1$ is determined by the value 0.6 or $\hat{\beta}_2$ is determined by the value 0.6. Increasing the number of observations also does not helps in the identification of these equations. So we are not able to identify the parameters. So we take the help of economic theory to identify the parameters.

The economic theory suggests that when price increases then supply increases but demand decreases. So the plot will look like



and this implies $\beta_1 < 0$ and $\beta_2 > 0$. Thus since $0.6 > 0$, so we can say that the value 0.6 represents $\hat{\beta}_2 > 0$ and so $\hat{\beta}_2 = 0.6$. But one can always choose a value of λ such that pretended equation does not violate the sign of coefficients, say $\beta > 0$. So it again becomes difficult to see whether equation (3) represents supply equation (2) or not. So none of the parameters is identifiable.

Now we obtain the reduced form of the model as

$$p_t = \frac{\alpha_1 - \alpha_2}{\beta_2 - \beta_1} + \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\beta_2 - \beta_1}$$

$$\text{or } p_t = \pi_{11} + v_{1t} \qquad (4)$$

$$q_t = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\beta_2 - \beta_1} + \frac{\beta_2\varepsilon_{1t} - \beta_1\varepsilon_{2t}}{\beta_2 - \beta_1}$$

$$\text{or } q_t = \pi_{21} + v_{2t}. \qquad (5)$$

Applying OLS to equations (4) and (5) and obtain OLSEs $\hat{\pi}_{11}$ and $\hat{\pi}_{21}$ which are given by

$$\hat{\pi}_{11} = \frac{\alpha_1 - \alpha_2}{\beta_2 - \beta_1}$$
$$\hat{\pi}_{21} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\beta_2 - \beta_1}.$$

There are two equations and four unknowns. So the unique estimates of the parameters $\alpha_1, \alpha_2, \beta_1$ and β_2 cannot be obtained. Thus the equations (1) and (2) can not be identified. Thus the model is not identifiable and estimation of parameters is not possible.

Suppose a new exogenous variable income i_t is introduced in the model which represents the income of buyer at time t. Note that the demand of commodity is influenced by income. On the other hand, the supply of commodity is not influenced by the income, so this variable is introduced only in the demand equation. The structural equations (1) and (2) now become

$$\text{Demand}: \quad q_t = \alpha_1 + \beta_1 p_t + \gamma_1 i_t + \varepsilon_{1t} \qquad (6)$$
$$\text{Supply}: \quad q_t = \alpha_2 + \beta_2 p_t + \varepsilon_{2t} \qquad (7)$$

where γ_1 is the structural coefficient associate with income. The pretended equation is obtained by multiplying the equations (6) and (7) by λ and $(1-\lambda)$, respectively, and then adding them together. This is obtained as follows:

$$\lambda q_t + (1-\lambda) q_t = \left[\lambda\alpha_1 + (1-\lambda)\alpha_2\right] + \left[\lambda\beta_1 + (1-\lambda)\beta_2\right] p_t + \lambda\gamma_1 i_t + \left[\lambda\varepsilon_{1t} + (1-\lambda)\varepsilon_{2t}\right]$$
$$\text{or} \quad q_t = \alpha + \beta p_t + \gamma i_t + \varepsilon_t \qquad (8)$$

where $\alpha = \lambda\alpha_1 + (1-\lambda)\beta_2$, $\beta = \lambda\beta_1 + (1-\lambda)\beta_2$, $\gamma = \lambda\gamma_1, \varepsilon_t = \lambda\varepsilon_{1t} + (1-\lambda)\varepsilon_{2t}$, $0 \le \lambda \le 1$ is a scalar.

Suppose now if we claim that equation (8) is true demand equation because it contains p_t and i_t which influence the demand. But we note that it is difficult to decide that between the two equations (6) or (8), which one is the true demand equation.

Suppose now if we claim that equation (8) is the true supply equation. This claim is wrong because income does not affect the supply. So equation (6) is the supply equation.

Thus the supply equation is now identifiable but demand equation is not identifiable. Such situation is termed as **partial identification.**

Now we find the reduced form of structural equations (6) and (7). This is achieved by first solving equation (6) for p_t and substituting it in equation (7) to obtain an equation in q_t. Such an exercise yields the reduced form equations of the following form:

$$p_t = \pi_{11} + \pi_{12} i_t + v_{1t} \qquad (9)$$

$$q_t = \pi_{21} + \pi_{22} i_t + v_{2t}. \qquad (10)$$

Applying OLS to equations (9) and (10), we get OLSEs $\hat{\pi}_{11}, \hat{\pi}_{22}, \hat{\pi}_{12}, \hat{\pi}_{21}$. Now we have four equations ((6),(7),(9),(10)) and there are five unknowns $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1)$. So the parameters are not determined uniquely. Thus the model is not identifiable.

However, here

$$\beta_2 = \frac{\pi_{22}}{\pi_{11}}$$

$$\alpha_2 = \pi_{21} - \pi_{22}\frac{\pi_{11}}{\pi_{22}}.$$

If $\hat{\pi}_{11}, \hat{\pi}_{22}, \hat{\pi}_{12}$ and $\hat{\pi}_{21}$ are available, then $\hat{\alpha}_2$ and $\hat{\beta}_2$ can be obtained by substituting $\hat{\pi}$'s in place of π's. So α_2 and β_2 are determined uniquely which are the parameters of supply equation. So supply equation is identified but demand equation is still not identifiable.

Now, as done earlier, we introduce another exogenous variable- rainfall, denoted as r_t which denotes the amount of rainfall at time t. The rainfall influences the supply because better rainfall produces better yield of wheat. On the other hand, the demand of wheat is not influenced by the rainfall. So the updated set of structural equations is

$$\text{Demand:} \quad q_t = \alpha_1 + \beta_1 p_t + \gamma_1 i_t + \varepsilon_{1t} \qquad (11)$$

$$\text{Supply:} \quad q_t = \alpha_2 + \beta_2 p_t + \delta_2 r_t + \varepsilon_{2t}. \qquad (12)$$

The pretended equation is obtained by adding together the equations obtained after multiplying equation (11) by λ and equation (12) by $(1-\lambda)$ as follows:

$$\lambda q_t + (1-\lambda) q_t = \left[\lambda\alpha_1 + (1-\lambda)\alpha_2\right] + \left[\lambda\beta_1 + (1-\lambda)\beta_2\right] p_t + \lambda\gamma_1 i_t + (1-\lambda)\delta_2 r_t + \left[\lambda\varepsilon_{1t} + (1-\lambda)\varepsilon_{2t}\right]$$

$$\text{or} \quad q_t = \alpha + \beta p_t + \gamma i_t + \delta r_t + \varepsilon_t \qquad (13)$$

where $\alpha = \lambda\alpha_1 + (1-\lambda)\alpha_2,\ \beta = \lambda\beta_1 + (1-\lambda)\beta_2,\ \gamma = \lambda\gamma_1, \delta = (1-\lambda)\delta_2, \varepsilon_t = \lambda\varepsilon_{1t} + (1-\lambda)\varepsilon_{2t}$ and $0 \le \lambda \le 1$ is a scalar.

Now we claim that equation (13) is a demand equation. The demand does not depend on rainfall. So unless $\lambda = 1$ so that r_t is absent in the model, the equation (13) cannot be a demand equation. Thus equation (11) is a demand equation. So demand equation is identified.

Now we claim that equation (13) is the supply equation. The supply is not influenced by the income of the buyer, so (13) cannot be a supply equation. Thus equation (12) is the supply equation. So now the supply equation is also identified.

The reduced form model from structural equations (11) and (12) can be obtained which have the following forms:

$$p_t = \pi_{11} + \pi_{12} i_t + \pi_{13} r_t + v_{1t} \qquad (14)$$

$$q_t = \pi_{21} + \pi_{22} i_t + \pi_{23} r_t + v_{2t}. \qquad (15)$$

Application of OLS technique to equations (14) and (15) yields the OLSEs $\hat{\pi}_{11}, \hat{\pi}_{12}, \hat{\pi}_{13}, \hat{\pi}_{21}, \hat{\pi}_{22}$ and $\hat{\pi}_{23}$. So now there are six such equations and six unknowns $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and δ_2. So all the estimates are uniquely determined. Thus the equations (11) and (12) are **exactly identifiable.**

Finally, we introduce a lagged endogenous variable p_{t-1} which denotes the price of the commodity on the previous day. Since only the supply of wheat is affected by the price on the previous day, so it is introduced in the supply equation only as

$$\text{Demand}: \quad q_t = \alpha_1 + \beta_1 p_t + \gamma_1 i_t + \varepsilon_{1t} \qquad (16)$$

$$\text{Supply}: \quad q_t = \alpha_2 + \beta_2 p_t + \delta_2 r_t + \theta_2 p_{t-1} + \varepsilon_{2t} \qquad (17)$$

where θ_2 is the structural coefficient associated with p_{t-1}.

The pretended equation is obtained by first multiplying equations (16) by λ and (17) by $(1-\lambda)$ and then adding them together as follows:

$$\lambda q_t + (1-\lambda) q_t = \alpha + \beta p_t + \gamma i_t + \delta r_t + (1-\lambda)\theta_2 p_{t-1} + \varepsilon_t$$

$$\text{or } q_t = \alpha + \beta p_t + \gamma i_t + \delta r_t + \theta p_{t-1} + \varepsilon_t. \qquad (18)$$

Now we claim that equation (18) represents the demand equation. Since rainfall and lagged price do not affect the demand, so equation (18) cannot be demand equation. Thus equation (16) is a demand equation and the demand equation is identified.

Now finally, we claim that equation (18) is the supply equation. Since income does not affect supply, so equation (18) cannot be a supply equation. Thus equation (17) is supply equation and the supply equation is identified.

The reduced form equations from equations (16) and (17) can be obtained as of the following form:

$$p_t = \pi_{11} + \pi_{12} i_t + \pi_{13} r_t + \pi_{14} p_{t-1} + v_{1t} \qquad (19)$$

$$q_t = \pi_{21} + \pi_{22} i_t + \pi_{23} r_t + r_{24} p_{t-1} + v_{2t}. \qquad (20)$$

Applying the OLS technique to equations (19) and (20) gives the OLSEs as $\hat{\pi}_{11}, \hat{\pi}_{12}, \hat{\pi}_{13}, \hat{\pi}_{14}, \hat{\pi}_{21}, \hat{\pi}_{22}, \hat{\pi}_{23}$ and $\hat{\pi}_{24}$. So there are eight equations in seven parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \delta_2$ and θ_2. So unique estimates of all the parameters are not available. In fact, in this case, the supply equation (17) is identifiable and demand equation (16) is **overly identified** (in terms of multiple solutions).

The whole analysis in this example can be classified into three categories –

(i) Under identifiable case

The estimation of parameters is not at all possible in this case. No enough estimates are available for structural parameters.

(2) Exactly identifiable case :

The estimation of parameters is possible in this case. The OLSE of reduced form coefficients leads to unique estimates of structural coefficients.

(3) Over identifiable case :

The estimation of parameters, in this case, is possible. The OLSE of reduced form coefficients leads to multiple estimates of structural coefficients.

Analysis:

Suppose there are G jointly dependent (endogenous) variables $y_1, y_2, ..., y_G$ and K predetermined (exogenous) variables $x_1, x_2, ..., x_K$. Let there are n observations available on each of the variable and there are G structural equations connecting both the variables which describe the complete model as follows:

$$\begin{aligned}
&\beta_{11}y_{1t} + \beta_{12}y_{2t} + ... + \beta_{1G}y_{Gt} + \gamma_{11}x_{1t} + \gamma_{12}x_{2t} + ... + \gamma_{1K}x_{Kt} = \varepsilon_{1t} \\
&\beta_{21}y_{1t} + \beta_{22}y_{2t} + ... + \beta_{2G}y_{Gt} + \gamma_{21}x_{2t} + \gamma_{22}x_{2t} + ... + \gamma_{2k}x_{K2} = \varepsilon_{2t} \\
&\vdots \\
&\beta_{G1}y_{1t} + \beta_{G2}y_{2t} + ... + \beta_{GG}y_{Gt} + \gamma_{G1}x_{1t} + \gamma_{G2}x_{2t} + ... + \gamma_{Gk}x_{Kt} = \varepsilon_{Gt}.
\end{aligned}$$

These equations can be expressed in matrix form as

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1G} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2G} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{G1} & \beta_{G2} & \cdots & \beta_{GG} \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{Gt} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1K} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{K1} & \gamma_{K2} & \cdots & \gamma_{KK} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{Kt} \end{bmatrix} = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{Gt} \end{bmatrix}$$

or

$$S : By_t + \Gamma x_t = \varepsilon_t, \; t = 1, 2, ..., n$$

where B is a $G \times G$ matrix of unknown coefficients of predetermined variables, y_t is a $(n \times 1)$ vector of observations on G jointly dependent variables, Γ is $(G \times K)$ matrix of structural coefficients, x_t *is* $(K \times 1)$ vector of observations on K predetermined variables and ε_t is $(G \times 1)$ vector of **structural disturbances**. The **structural form** S describes the functioning of model at time t.

Assuming B is nonsingular, premultiplying the structural form equations by B^{-1}, we get

$$B^{-1}By_t + B^{-1}\Gamma x_t = B^{-1}\varepsilon_t$$

$$\text{or } y_t = \pi x_t + v_t, \; t = 1, 2, ..., n.$$

This is the reduced form equation of the model where $\pi = B^{-1}\Gamma$ is the matrix of **reduced form coefficients** and $v_t = B^{-1}\varepsilon_t$ is the **reduced form disturbance** vectors.

If B is singular, then one or more structural relations would be a linear combination of other structural relations. If B is non-singular, such identities are eliminated.

The structural form relationship describes the interaction taking place inside the model. In reduced form relationship, the jointly dependent (endogenous) variables are expressed as linear combination of predetermined (exogenous) variables. This is the difference between structural and reduced form relationships.

Assume that ε_t's are identically and independently distributed following $N(0,\Sigma)$ and V_t's are identically and independently distributed following $N(0,\Omega)$ where $\Omega = B^{-1}\Sigma B'^{-1}$ with

$$E(\varepsilon_t)=0,\ E(\varepsilon_t,\varepsilon_t^{'})=\Sigma,\ E(\varepsilon_t\varepsilon_{t*}^{'})=0 \text{ for all } t\neq t*.$$
$$E(v_t)=0,\ E(v_t,v_t^{'})=\Omega,\ E(v_t v_{t*}^{'})=0 \text{ for all } t\neq t*.$$

The joint probability density function of y_t given x_t is

$$\begin{aligned} p(y_t \mid x_t) &= p(v_t) \\ &= p(\varepsilon_t)\left|\frac{\partial \varepsilon_t}{\partial v_t}\right| \\ &= p(\varepsilon_t)\left|\det(B)\right| \end{aligned}$$

where $\left|\frac{\partial \varepsilon_t}{\partial v_t}\right|$ is the related Jacobian of transformation and $\left|\det(B)\right|$ is the absolute value of the determinant of B.

The likelihood function is

$$\begin{aligned} L &= p(y_1,y_2,...,y_n \left| x_1,x_2,...,x_n\right.) \\ &= \prod_{t=1}^{n} p(y_t \left| x_t\right.) \\ &= \left|\det(B)\right|^n \prod_{t=1}^{n} p(\varepsilon_t). \end{aligned}$$

Applying a nonsingular linear transformation on structural equation S with a nonsingular matrix D, we get

$$DBy_t + D\Gamma x_t = D\varepsilon_t$$
$$\text{or} \quad S*: B*y_t + \Gamma * x_t = \varepsilon_t^*,\ t=1,2,...,n$$

where $B* = DB, \Gamma* = D\Gamma,\ \varepsilon_t^* = D\varepsilon_t$ and structural model $S*$ describes the functioning of this model at time t. Now find $p(y_t \left| x_t\right.)$ with $S*$ as follows:

$$p(y_t \mid x_t) = p(\varepsilon_t^*)\left|\frac{\partial \varepsilon_t^*}{\partial v_t}\right|$$
$$= p(\varepsilon_t^*)\left|\det(B^*)\right|.$$

Also

$$\varepsilon_t^* = D\varepsilon_t$$
$$\Rightarrow \frac{\partial \varepsilon_t^*}{\partial \varepsilon_t} = D.$$

Thus

$$p(y_t | x_t) = p(\varepsilon_t)\left|\frac{\partial \varepsilon_t}{\partial \varepsilon_t^*}\right|\left|\det(B^*)\right|$$
$$= p(\varepsilon_t)\left|\det(D^{-1})\right|\left|\det(DB)\right|$$
$$= p(\varepsilon_t)\left|\det(D^{-1})\right|\left|\det(D)\right|\left|\det(B)\right|$$
$$= p(\varepsilon_t)\left|\det(B)\right|.$$

The likelihood function corresponding to S^* is

$$L^* = \left|\det(B^*)\right|^n \prod_{t=1}^{n} p(\varepsilon_t^*)$$
$$= \left|\det(D)\right|^n \left|\det(B)\right|^n \prod_{t=1}^{n} p(\varepsilon_t)\left|\frac{\partial \varepsilon_t}{\partial \varepsilon_t^*}\right|$$
$$= \left|\det(D)\right|^n \left|\det(B)\right|^n \prod_{t=1}^{n} p(\varepsilon_t)\left|\det(D^{-1})\right|$$
$$= L.$$

Thus both the structural forms S and S^* have the same likelihood functions. Since the likelihood functions form the basis of statistical analysis, so both S and S^* have same implications. Moreover, it is difficult to identify whether the likelihood function corresponds to S and S^*. Any attempt to estimate the parameters will result into failure in the sense that we cannot know whether we are estimating S and S^*. Thus S and S^* are observationally equivalent. So the model is not identifiable.

A parameter is said to be identifiable within the model if the parameter has the same value for all equivalent structures contained in the model.

If all the parameters in the structural equation are identifiable, then the structural equation is said to be identifiable.

Given a structure, we can thus find many observationally equivalent structures by non-singular linear transformation.

The apriori restrictions on B and Γ may help in the identification of parameters. The derived structures may not satisfy these restrictions and may therefore not be admissible.

The presence and/or absence of certain variables helps in the identifiability. So we use and apply some apriori restrictions. These apriori restrictions may arise from various sources like economic theory, e.g. it is known that the price and income affect the demand of wheat but rainfall does not affect it. Similarly, supply of wheat depends on income, price and rainfall. There are many types of restrictions available which can solve the problem of identification. We consider zero-one type restrictions.

Zero-one type restrictions:

Suppose the apriori restrictions are of zero-one type, i.e., some of the coefficients are one and others are zero. Without loss of generality, consider S as

$$By_t + \Gamma x_t = \varepsilon_t,\ t = 1, 2, ..., n.$$

When the zero-one type restrictions are incorporated in the model, suppose there are G_Δ jointly dependent and K_* predetermined variables in S having nonzero coefficients. Rest $(G - G_\Delta)$ jointly dependent and $(K - K_*)$ predetermined variables are having coefficients zero.

Without loss of generality, let β_Δ and γ_* be the row vectors formed by the nonzero elements in the first row of B and Γ respectively. Thus the first row of B can be expressed as $(\beta_\Delta \quad 0)$. So B has G_Δ coefficients which are one and $(G - G_\Delta)$ coefficients which are zero.

Similarly, the first row of Γ can be written as $(\gamma_* \quad 0)$. So in Γ, there are K_* elements present (those take value one) and $(K - K_*)$ elements absent (those take value zero).

The first equation of the model can be rewritten as

$$\beta_{11}y_{1t} + \beta_{12}y_{2t} + ... + \beta_{1G_\Delta}y_{G_\Delta t} + \gamma_{11}x_{1t} + \gamma_{12}x_{2t} + ... + \gamma_{1K_*}x_{K_*t} = \varepsilon_{1t}$$

or $\quad \begin{pmatrix}\beta_\Delta & 0\end{pmatrix} y_t + \begin{pmatrix}\gamma_* & 0\end{pmatrix} x_t = \varepsilon_{1t},\ t = 1,2,...,n.$

Assume every equation describes the behaviour of a particular variable, so that we can take $\beta_{11} = 1.$

If $\beta_{11} \neq 1,$ then divide the whole equation by β_{11} so that the coefficient of y_{1t} is one.

So the first equation of the model becomes

$$y_{1t} + \beta_{12}y_{2t} + ... + \beta_{1G_\Delta}y_{G_\Delta t} + \gamma_{11}x_{1t} + \gamma_{12}x_{2t} + ... + \gamma_{1K_*}x_{K^*t} = \varepsilon_{1t}.$$

Now the reduced form coefficient relationship is

$\pi = -B^{-1}\Gamma$

or $B\pi = -\Gamma$

or $\begin{pmatrix}\beta_\Delta & \quad 0\end{pmatrix}\pi = -\begin{pmatrix}\gamma_* & \quad 0\end{pmatrix}.$

$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
G_Δ elements	$(G - G_\Delta)$ elements	K_* elements	$K - K_*$ elements

Partition

$$\pi = \begin{pmatrix}\pi_{\Delta*} & \pi_{\Delta**} \\ \pi_{\Delta\Delta*} & \pi_{\Delta\Delta**}\end{pmatrix}$$

where the orders of $\pi_{\Delta*}$ is $(G_\Delta \times K_*)$, $\pi_{\Delta**}$ is $(G_\Delta \times K_{**})$, $\pi_{\Delta\Delta*}$ is $\left(G_{\Delta\Delta} \times K_*\right)$ and $\pi_{\Delta\Delta**}$ is $\left(G_{\Delta\Delta} \times K_{**}\right)$

where $G_{\Delta\Delta} = G - G_\Delta$ and $K_{**} = K - K_*$.

We can re-express

$$\begin{pmatrix}\beta_\Delta & 0\end{pmatrix}\pi = -\begin{pmatrix}\gamma_* & 0\end{pmatrix}$$

$$\begin{pmatrix}\beta_\Delta & 0_{\Delta\Delta}\end{pmatrix}\pi = -\begin{pmatrix}\gamma_* & 0_{**}\end{pmatrix}$$

or $$\begin{pmatrix}\beta_\Delta & 0_{\Delta\Delta}\end{pmatrix}\begin{pmatrix}\pi_{\Delta*} & \pi_{\Delta**} \\ \pi_{\Delta\Delta*} & \pi_{\Delta\Delta**}\end{pmatrix} = -\begin{pmatrix}\gamma_* & 0_{**}\end{pmatrix}$$

$$\Rightarrow \beta_\Delta \pi_{\Delta*} = -\gamma_* \qquad (i)$$

$$\beta_\Delta \pi_{\Delta**} = 0_{**}. \qquad (ii)$$

Assume π is known. Then (i) gives a unique solution for γ_* if β_Δ is uniquely found from (ii). Thus identifiability of S lies upon the unique determination of β_Δ. Out of G_Δ elements in β_Δ, one has coefficient 1, so there are $(G_\Delta - 1)$ unknown elements in β_Δ that are unknown.

Note that

$$\beta_\Delta = \left(1, \beta_{12}, ..., \beta_{1G_\Delta}\right).$$

As

$$\beta_\Delta \pi_{\Delta**} = 0_{**}$$

or $(1 \quad \beta)\pi_{\Delta**} = 0_{**}$.

So $(G_\Delta - 1)$ elements of β_Δ are uniquely determined as non-trivial solution when

$$rank\left(\pi_{\Delta**}\right) = G_\Delta - 1.$$

Thus β_0 is identifiable when

$$rank\left(\pi_{\Delta**}\right) = G_\Delta - 1.$$

This is known as a **rank condition** for the identifiability of parameters in S. This condition is **necessary and sufficient.**

Another condition, known as **order condition** is only necessary and not sufficient. The order condition is derived as follows:

We now use the result that for any matrix A of order $m \times n$, the $rank(A) = Min(m,n)$. For identifiability, if $rank(A) = m$ then obviously $n > m.$

Since $\beta_\Delta \pi_{\Delta**} = 0$ and β_Δ has only $(G_\Delta - 1)$ elements which are identifiable when

$$rank\left(\pi_{\Delta**}\right) = G_\Delta - 1$$
$$\Rightarrow K - K_* \geq G_\Delta - 1.$$

This is known as the **order condition** for identifiability.

There are various ways in which these conditions can be represented to have meaningful interpretations. We discuss them as follows:

1. $\left(G-G_{\Delta}\right)+(K-K_{*})\geq\left(G-G_{\Delta}\right)+\left(G_{\Delta}-1\right)$

or $\left(G-G_{\Delta}\right)+(K-K_{*})\geq G-1$

Here

$G-G_{\Delta}$: Number of jointly dependent variables left out in the equation

$$y_{1t}+\beta_{12}y_{2t}+...+\beta_{1G_{\Delta}}y_{G_{\Delta}}+\gamma_{11}x_{1t}+...+\lambda_{1K}x_{Kt}=\varepsilon_{1t}$$

$K-K_{*}$: Number of predetermined variables left out in the equation

$$y_{1t}+\beta_{12}y_{2t}+...+\beta_{1G_{\Delta}}y_{G_{\Delta}}+\gamma_{11}x_{1t}+...+\lambda_{1K}x_{Kt}=\varepsilon_{1t}$$

$G-1$: Number of total equations - 1.

So left-hand side of the condition denotes the total number of variables excluded is this equation.

Thus if the total number of variables excluded in this equation exceeds $(G-1)$, then the model is identifiable.

2. $K\geq K_{*}+G_{\Delta}$ -1

Here

K_{*} : The number of predetermined variables present in the equation.

G_{Δ}: The number of jointly dependent variables present is the equation.

3. Define $L=K-K_{*}-(G_{\Delta}-1)$

L measures the **degree of overidentification.**

If $L=0$ then the equation is said to be exactly identified.

$L>0$, then the equation is said to be over identified.

$L<0\Rightarrow$ the parameters cannot be estimated.

$L=0\Rightarrow$ the unique estimates of parameters are obtained.

$L>0\Rightarrow$ the parameters are estimated with multiple estimators.

So by looking at the value of L, we can check the identifiability of the equation.

Rank condition tells whether the equation is identified or not. If identified, then the order condition tells whether the equation is exactly identified or over identified.

Note

We have illustrated the various aspects of estimation and conditions for identification in the simultaneous equation system using the first equation of the system. It may be noted that this can be done for any equation of the system and it is not necessary to consider only the first equation. Now onwards, we do not restrict to first equation but we consider any, say i^h equation (i = 1, 2, …,G).

Working rule for rank condition

The checking of the rank condition sometimes, in practice, can be a difficult task. An equivalent form of rank condition based on the partitioning of structural coefficients is as follows.

Suppose

$$B = \begin{pmatrix} \beta_\Delta & 0_{\Delta\Delta} \\ B_\Delta & B_{\Delta\Delta} \end{pmatrix}, \ \Gamma = \begin{pmatrix} \gamma_* & 0_{**} \\ \Gamma_* & \Gamma_{**} \end{pmatrix},$$

where $\beta_\Delta, \gamma_*, 0_{\Delta\Delta}$ and 0_{**} are row vectors consisting of $G_\Delta, K_*, G_{\Delta\Delta}$ and K_{**} elements respectively in them. Similarly the orders of B_Δ is $((G-1)\times G_\Delta)$, $B_{\Delta\Delta}$ is $((G-1)\times G_{\Delta\Delta})$, Γ_* is $((G-1)\times K_*)$ and Γ^{**} is $((G-1)\times K_{**})$. Note that $B_{\Delta\Delta}$ and Γ_{**} are the matrices of structural coefficients for the variable omitted from the i^{th} equation (i = 1,2,…,G) but included in other structural equations.

Form a new matrix

$$\begin{aligned}
\Delta &= \begin{pmatrix} B & \Gamma \end{pmatrix} \\
&= \begin{pmatrix} \beta_\Delta & 0_{\Delta\Delta} & \gamma_* & 0_{**} \\ B_\Delta & B_{\Delta\Delta} & \Gamma_* & \Gamma_{**} \end{pmatrix} \\
B^{-1}\Delta &= \begin{pmatrix} B^{-1}B & B^{-1}\Gamma \end{pmatrix} \\
&= \begin{pmatrix} I & -\pi \end{pmatrix} \\
&= \begin{bmatrix} I_{\Delta\Delta} & 0 & -\pi_{\Delta,*} & -\pi_{\Delta,**} \\ 0 & I_{\Delta\Delta,\Delta\Delta} & -\pi_{\Delta\Delta,*} & -\pi_{\Delta\Delta,**} \end{bmatrix}.
\end{aligned}$$

If

$$\Delta_* = \begin{bmatrix} 0_{\Delta\Delta} & 0_{**} \\ B_{\Delta\Delta} & \Gamma_{**} \end{bmatrix}$$

then clearly the rank of Δ_* is same as the rank of Δ since the rank of a matrix is not affected by enlarging the matrix by a rows of zeros or switching any columns.

Now using the result that if a matrix A is multiplied by a nonsingular matrix, then the product has the same rank as of A, we can write

$$rank(\Delta_*) = rank(B^{-1}\Delta_*)$$

$$\begin{aligned} rank(B^{-1}\Delta_*) &= rank\begin{pmatrix} 0 & -\pi_{\Delta,**} \\ I_{\Delta\Delta,\Delta\Delta} & -\pi_{\Delta\Delta,**} \end{pmatrix} \\ &= rank(I_{\Delta\Delta,\Delta\Delta}) + rank(\pi_{\Delta,**}) \\ &= \left(G - G_\Delta\right) + \left(G_\Delta - 1\right) \\ &= G - 1 \end{aligned}$$

$$rank\left(B^{-1}\Delta_*\right) = rank\left(\Delta_*\right) = rank\left(B_{\Delta\Delta} \quad \Gamma_{**}\right).$$

So

$$rank(B_{\Delta\Delta} \quad \Gamma_{**}) = G - 1$$

and then the equation is identifiable.

Note that $\left(B_{\Delta\Delta} \quad \Gamma_{**}\right)$ is a matrix constructed from the coefficients of variables excluded from that particular equation but included in other equations of the model. If $rank\left(\beta_{\Delta\Delta} \quad \Gamma_{**}\right) = G - 1,$ then the equation is identifiable and this is a necessary and sufficient condition. An advantage of this term is that it avoids the inversion of matrix. A working rule is proposed like following.

Working rule:

1. Write the model in tabular form by putting ‘X’ if the variable is present and ‘0’ if the variable is absent.
2. For the equation under study, mark the 0’s (zeros) and pick up the corresponding columns suppressing that row.
3. If we can choose $\left(G - 1\right)$ rows and $(G - 1)$ columns that are not all zero, then it can be identified.

Example 2:

Now we discuss the identifiability of following simultaneous equations model with three structural equations

(1) $y_1 = \beta_{13} y_3 + \gamma_{11} x_1 + \gamma_{13} x_3 + \varepsilon_1$

(2) $y_1 = \gamma_{21} x_1 + \gamma_{23} x_3 + \varepsilon_2$

(3) $y_2 = \beta_{33} y_3 + \gamma_{31} x_1 + \gamma_{32} x_2 + \varepsilon_3$.

First, we represent the equation (1)-(3) in tabular form as follows

Equation number	$y_1\, y_2\, y_3$	$x_1\, x_2\, x_3$	$G_\Delta - 1$	K_*	L
1	X 0 X	X 0 X	2 – 1 = 1	2	3 – 3 = 0
2	X 0 0	X 0 X	1 – 1 = 0	2	3 – 2 = 1
3	0 X X	X X 0	2 – 1 = 1	2	3 – 3 = 0

- $G =$ Number of equations = 3.
- `X' denotes the presence and '0' denotes the absence of variables in an equation.
- $G_\Delta =$ Numbers of 'X' in $(y_1 \quad y_2 \quad y_3)$.
- K_* = Number of 'X' in $(x_1 \quad x_2 \quad x_3)$.

Consider equation (1):

❖ Write columns corresponding to '0' which are columns corresponding to y_2 and x_2 and write as follows

	y_2	x_2
Equation (2)	0	0
Equation (3)	X	X

❖ Check if any row/column is present with all elements '0'. If we can pick up a block of order $(G-1)$ is which all rows/columns are all '0' , then the equation is identifiable.

Here $G-1=3-1=2$, $B_{\Delta\Delta} = \begin{pmatrix} 0 \\ \text{X} \end{pmatrix}$, $\Gamma_{**} = \begin{pmatrix} 0 \\ \text{X} \end{pmatrix}$. So we need (2×2) matrix in which no row and no column has all elements '0' . In case of equation (1), the first row has all '0', so it is not identifiable.

Notice that from order condition, we have $L=0$ which indicates that the equation is identified and this conclusion is misleading. This is happening because order condition is just necessary but not sufficient.

Consider equation (2):

- Identify '0' and then

	$y_2 \quad y_3$	x_2
Equation (1)	0 X	0
Equation (3)	X X	X

- $B_{\Delta\Delta} = \begin{pmatrix} 0 & \text{X} \\ \text{X} & \text{X} \end{pmatrix}, \quad \Gamma_{**} = \begin{pmatrix} 0 \\ \text{X} \end{pmatrix}.$
- $G-1=3-1=2.$
- We see that there is atleast one block in which no row is '0' and no column is '0'. So the equation (2) is identified.
- Also, $L=1>0 \Rightarrow$ Equation (2) is over identified.

Consider equation (3)

- identify '0'.

	y_1	x_3
Equation (1)	X	X
Equation (2)	X	X

- So $B_{\Delta\Delta} = \begin{pmatrix} \text{X} \\ \text{X} \end{pmatrix}, \ \Gamma_{**} = \begin{pmatrix} \text{X} \\ \text{X} \end{pmatrix}.$
- We see that there is no '0' present in the block. So equation (3) is identified.
- Also, $L=0 \Rightarrow$ Equation (3) is exactly identified.

Estimation of parameters

To estimate the parameters of the structural equation, we assume that the equations are identifiable.

Consider the first equation of the model

$$\beta_{11}y_{1t} + \beta_{12}y_{2t} + ... + \beta_{1G}y_{Gt} + \gamma_{11}x_{1t} + \gamma_{12}x_{2t} + ... + \gamma_{1K}x_{Kt} = \varepsilon_{1t}, t = 1,2,..,n.$$

Assume $\beta_{11} = 1$ and incorporate the zero-one type restrictions. Then we have

$$y_{1t} = \beta_{12}y_{2t} - ... - \beta_{1G_\Delta}y_{G\Delta} - \gamma_{11}x_{1t} - \gamma_{12}x_{2t} - ... - \gamma_{1K}x_{K_*t} + \varepsilon_{1t},\ t = 1,2,..,n.$$

Writing this equation in vector and matrix notations by collecting all n observations, we can write

$$y_1 = Y_1\beta + X_1\gamma + \varepsilon_1$$

where

$$y_1 = \begin{pmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1n} \end{pmatrix}, \quad Y_1 = \begin{pmatrix} y_{21} & y_{31} & \cdots & y_{G\Delta 1} \\ y_{22} & y_{32} & \cdots & y_{G_\Delta 2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{2n} & y_{3n} & \cdots & y_{G\Delta n} \end{pmatrix}$$

$$X_1 = \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{K_*1} \\ x_{12} & x_{22} & \cdots & x_{K_*2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{K_*n} \end{pmatrix}, \quad \beta = \begin{pmatrix} -\beta_{12} \\ -\beta_{13} \\ \vdots \\ -\beta_{1G\Delta} \end{pmatrix}, \quad \gamma = \begin{pmatrix} -\gamma_{11} \\ -\gamma_{12} \\ \vdots \\ -\gamma_{1K_*} \end{pmatrix}.$$

The order of y_1 is $(n \times 1)$, Y_1 is $(n \times (G_\Delta - 1))$, X_1 is $(n \times K_*)$, β is $((G_\Delta - 1) \times 1)$ and γ is $(K_* \text{ x } 1)$.

This describes one equation of the structural model.

Now we describe the general notations in this model.

Consider the model with incorporation of zero-one type restrictions as

$$y_1 = Y_1\beta + X_1\gamma + \varepsilon$$

where y_1 is a $(n \times 1)$ vector of jointly dependent variables, Y_1 is $(n \times (G_\Delta - 1))$ matrix of jointly dependent variables where G_Δ denote the number of jointly dependent variables present on the right hand side of the equation, β is a $((G_\Delta - 1) \times 1)$ vector of associated structural coefficients, X_1 is a $(n \times K_*)$ matrix of n observations on each of the K_* predetermined variables and ε is a $(n \times 1)$ vector of structural disturbances. This equation is one of the equations of complete simultaneous equation model.

Stacking all the observations according to the variance, rather than time, the complete model consisting of G equations describing the structural equation model can be written as

$$YB + X\Gamma = \Phi$$

where Y is a $(n \times G)$ matrix of observation on jointly dependent variables, B is $(G \times G)$ matrix of associated structural coefficients, X is $(n \times K)$ matrix of observations on predetermined variables, Γ is $(K \times G)$ matrix of associated structural coefficients and Φ is a $(n \times G)$ matrix of structural disturbances.

Assume $E(\Phi) = 0, \frac{1}{n}E(\Phi\Phi') = \Sigma$ where Σ is positive definite symmetric matrix.

The reduced form of the model is obtained from the structural equation model by post multiplying by B^{-1} as

$$YBB^{-1} + X\Gamma B^{-1} = \Phi B^{-1}$$
$$Y = X\pi + V$$

where $\pi = -\Gamma B^{-1}$ and $V = \Phi B^{-1}$ are the matrices of reduced-form coefficients and reduced form disturbances respectively.

The structural equation is expressed as

$$\begin{aligned} y &= Y_1\beta + X_1\gamma + \varepsilon_1 \\ &= \begin{pmatrix} Y_1 & X_1 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} + \varepsilon_1 \\ &= A\delta + \varepsilon \end{aligned}$$

where $A = (Y_1, \quad X_1), \delta = (\beta \quad \gamma)'$ and $\varepsilon = \varepsilon_1$. This model looks like a multiple linear regression model. Since X_1 is a submatrix of X, so it can be expressed as

$$X_1 = XJ_1$$

where J_1 is called as a **select matrix** and consists of two types of elements, viz., 0 and 1. If the corresponding variable is present, its value is 1 and if absent, its value is 0.

Method of consistent estimation of parameters

1. Indirect least squares (ILS) method

This method for the consistent estimation of parameters is available only for exactly identified equations.

If equations are exactly identifiable, then

$$K = G_{\Delta} + K_{*} - 1.$$

Step 1: Apply ordinary least squares to each of the reduced form equations and estimate the reduced form coefficient matrix.

Step 2: Find algebraic relations between structural and reduced-form coefficients. Then find the structural coefficients.

The structural model at a time t is

$$\beta y_t + \Gamma x_t = \varepsilon_t;\ t = 1, 2, ..., n$$

where $y_t = \left(y_{1t}, y_{2t}, ..., y_{Gt}\right)',\ x_t = \left(x_{1t}, x_{2t}, ..., x_{Kt}\right)'.$

Stacking all n such equations, the structural model is obtained as

$$BY + \Gamma X = \Phi$$

where Y is $\left(n \times G\right)$ matrix, X is $\left(n \times K\right)$ matrix and Φ is $(n \times K)$ matrix.

The reduced form equation is obtained by premultiplication of B^{-1} as

$$B^{-1}BY + B^{-1}\Gamma X = B^{-1}\Phi$$
$$Y = X\pi + V$$

where $\pi = -B^{-1}\Gamma$ and $B^{-1}\Phi.$

Applying OLS to reduced form equation yields the OLSE of π as

$$\hat{\pi} = \left(X'X\right)^{-1} X'Y.$$

This is the first step of ILS procedure and yields the set of estimated reduced form coefficients.

Suppose we are interested in the estimation of following structural equation

$$y = Y_1\beta + X_1\gamma + \varepsilon$$

where y is $(n \times 1)$ vector of n observation on dependent (endogenous) variable, Y_1 is $(n \times (G_\Delta - 1)$ matrix of observations on G_1 current endogenous variables, X_1 is $(n \times K_*)$ matrix of observations on K_* predetermined (exogenous) variables in the equation and ε is $(n \times 1)$ vector of structural disturbances. Write this model as

$$\begin{pmatrix} y_1 & Y_1 & X_1 \end{pmatrix} \begin{pmatrix} 1 \\ -\beta \\ -\gamma \end{pmatrix} = \varepsilon$$

or more general

$$\begin{pmatrix} y_1 & Y_1 & Y_2 & X_1 & X_2 \end{pmatrix} \begin{pmatrix} 1 \\ -\beta \\ 0 \\ -\gamma \\ 0 \end{pmatrix} = \varepsilon$$

where Y_2 and X_2 are the matrices of observations on $(G - G_\Delta + 1)$ endogenous and $(K - K_\Delta)$ predetermined variables which are excluded from the model due to zero-one type restrictions.

Write

$$\pi B = -\Gamma$$

$$\text{or } \pi \begin{pmatrix} 1 \\ -\beta \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}.$$

Substitute π as $\hat{\pi} = (X'X)^{-1} X'Y$ and solve for β and γ. This gives indirect least squares estimators b and c of β and γ respectively by solving

$$\hat{\pi}\begin{pmatrix}1\\-b\\0\end{pmatrix}=\begin{pmatrix}c\\0\end{pmatrix}$$

or $$\left(X'X\right)^{-1}X'Y\begin{pmatrix}1\\-b\\0\end{pmatrix}=\begin{pmatrix}c\\0\end{pmatrix}$$

or $$\left(X'X\right)^{-1}X'\left(y_1\ \ Y_1\ \ Y_2\right)\begin{pmatrix}1\\-b\\0\end{pmatrix}=\begin{pmatrix}c\\0\end{pmatrix}$$

$$\Rightarrow X'y_1 - X'Y_1 b = X'X\begin{pmatrix}c\\0\end{pmatrix}$$

Since $X=\left(X_1 \quad X_2\right)$

$$\Rightarrow (X_1^{'}Y_1)b+\left(X_1^{'}X_1\right)c = X_1^{'}y_1 \qquad (i)$$

$$(X_2^{'}Y_1)b+\left(X_2^{'}X_1\right)c = X_2^{'}y_1. \qquad (ii)$$

These equations (i) and (ii) are K equations is $(G_\Delta + K_* - 1)$ unknowns. Solving the equations (i) and (ii) gives ILS estimators of β and γ.

2. Two stage least squares (2SLS) or generalized classical linear (GCL) method:

This is more widely used estimation procedure as it is applicable to exactly as well as overidentified equations. The least-squares method is applied in two stages in this method.

Consider equation $y_1 = Y_1\beta + X_1\gamma + \varepsilon.$

Stage 1: Apply least squares method to estimate the reduced form parameters in the reduced form model

$$Y_1 = X\pi_1 + V_1$$
$$\Rightarrow \hat{\pi}_1 = (X'X)^{-1}X'Y_1$$
$$\Rightarrow Y_1 = X\hat{\pi}_1.$$

Stage 2: Replace Y_1 is structural equation $y_1 = Y_1\beta + X_1\gamma + \varepsilon$ by $\hat{Y}_1$ and apply OLS to thus obtained structural equation as follows:

$$\begin{aligned}
y_1 &= \hat{Y}_1\beta + X_1\gamma + \varepsilon \\
&= X\left(X'X\right)^{-1}X'Y_1\beta + X_1\gamma + \varepsilon \\
&= \left[X\left(X'X\right)^{-1}X'\hat{Y}_1 \quad X_1\right]\begin{pmatrix}\beta\\ \gamma\end{pmatrix} + \varepsilon \\
&= \hat{A}\delta + \varepsilon.
\end{aligned}$$

where $\hat{A}=\left[X\left(X'X\right)^{-1}X'\hat{Y}_1 \;\; X_1\right]$, $A=\left[X\left(X'X\right)^{-1}X'Y_1 \;\; X_1\right]=\bar{H}A$, $\bar{H}=X\left(X'X\right)^{-1}X'$ is idempotent and $\delta=(\beta \quad \gamma)'$.

Applying OLS to $Y_1=\hat{A}\delta+\varepsilon$ gives OLSE of δ as

$$\hat{\delta}=\left(\hat{A}'\hat{A}\right)^{-1}\hat{A}y_1$$
$$=\left(A'\bar{H}A\right)^{-1}A'\bar{H}y_1$$

or
$$\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix}=\begin{pmatrix}Y_1'\bar{H}Y_1 & Y_1'X_1\\ X_1'Y_1 & X_1'X_1\end{pmatrix}^{-1}\begin{pmatrix}Y_1'y_1\\ X_1'y_1\end{pmatrix}$$
$$=\begin{pmatrix}Y_1'Y_1-\hat{V}_1'\hat{V}_1 & Y_1'X_1\\ X_1'Y_1 & X_1'X_1\end{pmatrix}^{-1}\begin{pmatrix}Y_1'-\hat{V}_1'\\ X_1'\end{pmatrix}y_1.$$

where $V_1=Y_1-X\pi_1$ is estimated by $\hat{V}_1=Y_1-X\hat{\pi}_1=\left(I-\bar{H}\right)Y_1=HY_1$, $H=I-\bar{H}$. Solving these two equations, we get $\hat{\beta}$ and $\hat{\gamma}$ which are the two stage least squares estimators of β and γ respectively.

Now we see the consistency of $\hat{\delta}$.

$$\hat{\delta}=\left(\hat{A}'\hat{A}\right)^{-1}\hat{A}y_1$$
$$=\left(\hat{A}'\hat{A}\right)^{-1}\hat{A}'\left(\hat{A}\delta+\varepsilon\right)$$
$$\hat{\delta}-\delta=\left(\hat{A}'\hat{A}\right)^{-1}\hat{A}'\varepsilon$$
$$\text{plim}\left(\hat{\delta}-\delta\right)=\text{plim}\left(\frac{1}{n}\hat{A}'A\right)^{-1}\text{plim}\left(\frac{1}{n}\hat{A}'\varepsilon\right).$$

The 2SLS estimator $\hat{\delta}$ is consistent if

$$\text{plim}\left(\frac{1}{n}\hat{A}'\varepsilon\right)=\begin{bmatrix}\text{plim}\frac{1}{n}\hat{Y}_1'\varepsilon\\ \text{plim}\frac{1}{n}X_1'\varepsilon\end{bmatrix}=0.$$

Since by assumption, X variables are uncorrelated with ε in limit, so

$$\text{plim}\left(\frac{1}{n}X'\varepsilon\right)=0.$$

For $\text{plim}\left(\frac{1}{n}\hat{Y}_1\varepsilon\right)$, we observe that

$$\text{plim}\left(\frac{1}{n}\hat{Y}_1'\varepsilon\right) = \text{plim}\left(\frac{1}{n}\hat{\pi}_1 X'\varepsilon\right)$$
$$= \left(\text{plim}\hat{\pi}_1\right)\left(\text{plim}\frac{1}{n}X'\varepsilon\right)$$
$$= \pi_1 .0$$
$$= 0.$$

Thus $\text{plim}\left(\hat{\delta}\text{-}\delta\right) = 0$ and so the 2SLS estimators are consistent.

The asymptotic covariance matrix of $\hat{\delta}$ is

$$\text{Asy Var}\left(\hat{\delta}\right) = n^{-1}\,\text{plim}\left[n\left(\hat{\delta}-\delta\right)\left(\hat{\delta}-\delta\right)'\right]$$
$$= n^{-1}\,\text{plim}\left[n\left(\hat{A}'\hat{A}\right)^{-1}\hat{A}'\varepsilon\varepsilon'\hat{A}\left(\hat{A}'\hat{A}\right)^{-1}\right]$$
$$= n^{-1}\,\text{plim}\left(\frac{1}{n}\hat{A}'\hat{A}\right)^{-1}\text{plim}\left(\frac{1}{n}\hat{A}'\varepsilon\varepsilon'\hat{A}\right)\text{plim}\left(\frac{1}{n}\hat{A}'\hat{A}\right)^{-1}$$
$$= n^{-1}\sigma_\varepsilon^2\,\text{plim}\left(\frac{1}{n}\hat{A}'\hat{A}\right)^{-1}$$

where $Var(\varepsilon) = \sigma_\varepsilon^2$.

The asymptotic covariance matrix is estimated by

$$s^2\left(\hat{A}'\hat{A}\right)^{-1} = s^2\begin{bmatrix} Y_1'X(X'X)^{-1}X'Y_1 & Y_1'X_1 \\ X_1'Y_1 & X_1'X_1 \end{bmatrix}$$

where

$$s^2 = \frac{\left(y - Y_1\hat{\beta} - X_1\hat{\gamma}\right)'\left(y - Y_1\hat{\beta} - X_1\hat{\gamma}\right)}{n-G-K}$$