Topic:- Properties of transpose

⇒ Definition:-

The new matrix obtainted by interchanging the rows and columns of the original matrix is called as the transpose of the matrix. It is denoted by A^t. In other words, if $A = [a_{ij}]_{m \times n}$ then $A^t = [a_{ji}]_{n \times m}$.

For example:-

If $U = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ then $U^t = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 3 \end{bmatrix}$

⇒ Properties:-

- Transpose of transpose of a matrix:-
  The transpose of the transpose of a matrix is the matrix itself: $(A^T)^T = A$. e.g

  $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$

  Therefore $(A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

  Hence $\boxed{(A^T)^T = A}$

- Transpose of a product:-
  The transpose of

a product is the product of the transpose in the reverse order. $(AB)^T = B^T A^T$. The same is true for the product of multiple matrices: $(ABC)^T = C^T B^T A^T$.

For example: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$

L.H.S $= (AB)^t$

$$= \left(\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \right)^T = \begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix}$$

R.H.S $= B^T A^T$, $B^T = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$, $A^t = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

$$B^T A^T = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 0+2 \\ 6+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix} = \text{R.H.S}$$

Thus $\boxed{(AB)^T = B^T A^T}$

- **Transpose of a sum:-**

The transpose of a sum is the sum of transposes. E.g $(A+B)^T = A^T + B^T$. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

L.H.S: $(A+B)^T = \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \right)^T = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}^T$

$$(A+B)^T = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$$

R.H.S: $A^T + B^T$

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

Then $A^T + B^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$

Hence $\boxed{(A+B)^T = A^T + B^T}$

- **Transpose of subtraction:-**

The subtraction of two matrices is equal to the subtraction of their transpose. $(A-B)^T = A^T - B^T$

E.g Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

L.H.S $= (A-B)^T$

$$= \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\right)^T = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

R.H.S $= A^T - B^T$ where $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

$$= \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

Hence $\boxed{(A-B)^T = A^T - B^T}$

- **Transpose of a scalar product:-**

The transpose of a matrix times a scalar (c) is equal to the constant times the transpose of the matrix. $(cA)^T = c(A^T)$

E.g:- Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $c = 3$

L.H.S $= (cA)^T = \left(3\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\right)^T = \begin{bmatrix} 3 & 3 \\ 6 & 9 \end{bmatrix}^T$

$$(cA)^T = \begin{bmatrix} 3 & 6 \\ 3 & 9 \end{bmatrix}$$

<u>R.H.S:</u> $c(A^T) = 3\left(\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\right)^T = 3\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 9 \end{bmatrix}$

Hence $(cA)^T = c(A^T)$

⇒ Inverse of a matrix:-

In matrix, the inverse of a square matrix A, which is shown by A^{-1} (read A inverse), is the matrix of the same order such that:

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix of the same order.

⇒ Properties:-

1 $(A^{-1})^{-1} = A$

If A is non-singular matrix then

$$(A^{-1})^{-1} = A$$

E.g Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4-3 = 1 \quad , adj(A) = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}}{1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$(A^{-1})^{-1} = \frac{adj(A^{-1})}{|A^{-1}|}$, $|A^{-1}| = \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} = 4 - 3 = 1$

$Adj(A^{-1}) = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $(A^{-1})^{-1} = \frac{adj(A^{-1})^{-1}}{|A^{-1}|}$

$(A^{-1})^{-1} = \frac{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}{1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = A$

Hence prove that $(A^{-1})^{-1} = A$.

2. $(AB)^{-1} = B^{-1}A^{-1}$

If A and B are non-singular matrices, then AB is non-singular.

$$(AB)^{-1} = B^{-1}A^{-1}$$

Example: Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$

L.H.S:- $(AB)^{-1}$

$AB = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$

$|AB| = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3 + 0 = 3$

$(AB)^{-1} = \frac{adj(AB)}{|AB|} = \frac{\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}}{3} = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1 \end{bmatrix}$

R.H.S:- $B^{-1}A^{-1}$

where $B^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$, $A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \cdot \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 0+1 & -2+3 \\ 0 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1 \end{bmatrix}$

Thus $(AB)^{-1} = B^{-1}A^{-1}$ is verified.

3. $(A^t)^{-1} = (A^{-1})^t$

If A is non-singular matrix then
$$(A^t)^{-1} = (A^{-1})^t$$

Example: Let $A = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}$ then

L.H.S. $(A^t)^{-1}$

$A^t = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$, $|A| = \begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -6+1 = -5 \neq 0$

$(A^t)^{-1} = \dfrac{adj(A^t)}{|A^t|} = \dfrac{\begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix}}{-5} = \begin{bmatrix} -3/5 & -1/5 \\ -1/5 & -2/5 \end{bmatrix}$

R.H.S. $(A^{-1})^t$, $|A| = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -6+1 = -5$

$A^{-1} = \dfrac{adj(A)}{|A|} = \dfrac{\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}}{-5} = \begin{bmatrix} -3/5 & -1/5 \\ -1/5 & -2/5 \end{bmatrix}$.

$(A^{-1})^t = \begin{bmatrix} 3/5 & 1/5 \\ -1/5 & -2/5 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ 1/5 & -2/5 \end{bmatrix} = (A^t)^{-1}$

Hence prove that $(A^t)^{-1} = (A^{-1})^t$.

4. $(kA)^{-1} = \frac{1}{k} A^{-1}$

If A be any non-singular matrix then $(kA)^{-1} = \frac{1}{k} A^{-1}$

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $k = 2$ then

L.H.S. $(kA)^{-1} = \left(2\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right)^{-1} \Rightarrow \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}^{-1}$

$(kA)^{-1} = \dfrac{adj(kA)}{|kA|}$, $|kA| = \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} = 4$

$Adj(kA) = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix}$ then $(kA)^{-1} = \dfrac{adj(kA)}{|kA|}$

$$= \frac{\begin{bmatrix}2 & -4\\0 & 2\end{bmatrix}}{4} = \begin{bmatrix}1/2 & -1\\0 & 1/2\end{bmatrix}$$

R.H.S:- $\frac{1}{k}(A)^{-1}$, $|A| = \begin{vmatrix}1 & 2\\0 & 1\end{vmatrix} = 1$

Adj (A) $= \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}}{1} = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$$

Now $\frac{1}{k}(A^{-1}) = \frac{1}{2}\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix} = \begin{bmatrix}1/2 & -1\\0 & 1/2\end{bmatrix} = (kA)^{-1}$

Hence proved that $(kA)^{-1} = \frac{1}{k}A^{-1}$.